1 Circular Pascal Arrays and Corridor Paths

Definition 1. The Pascal array is the array whose row $n$, column $k$ entry is equal to $\binom{n}{k}$, where $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Definition 2. Fix an integer $d \geq 2$. The circular Pascal array of order $d$ is the array whose row $n$, column $k$ entry, $\sigma_{n,k}^{(d)}$ (or just $\sigma_{n,k}$ when the context is clear), is the (finite) sum:

$$\sigma_{n,k}^{(d)} = \sigma_{n,k} = \sum_{j \in \mathbb{Z}} \binom{n}{k + dj},$$

where $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Remark 3. Corridor numbers are useful in graph theory as $c_{n}^{(m)}$ counts the number of length $n$ paths in the path graph $P_{m+1}$ that start at the initial node of the graph.

Lemma 1.1. For all $n \geq 0$, $q_n = (I + R^2)^n q_0$.

Proof. Because $I + R^2$ and $D = I - L$ commute, a simple induction gives the result. □

Theorem 1.2. For fixed $d \geq 2$, and $0 \leq y_0 \leq d - 2$, the $n$th corridor number of order $d - 2$ is given by $c_{n,y_0} = p_{n,n+y_0} - p_{n,n+y_0+d}$.

Proof. Let $p_{n,k}$ resp. $q_{n,k}$ be the $k$th entry of $p_n$, resp. $q_n$. By definition, $c_{n,y_0} = \sum_{j=0}^{y_0-1} v_{n,k}$. So by Lemma 1.1 since $q_{n,n+y_0+j} = c_{n,n+y_0+j} = p_{n,n+y_0+j} - p_{n,n+y_0+j+1}$ is essentially a negative discrete derivative, it follows that $p_{n,n+y_0}$ is a maximum value of $p_n$, and $p_{n,n+y_0+d}$ is a minimum value of $p_n$, which completes the proof. □

Example. Let $d = 8$ and $y_0 = 2$. To find the corridor numbers of order $d - 2 = 6 \in \mathbb{N} \times \{0, 1, \ldots, 6\}$, but with corridor paths starting at $(0, 2)$, we take the range of the $n$th of the Pascal array mod 8, but using $c_0 = (1, 1, 1, 0, 0, 0, 0, 0) \in \mathbb{R}^8$ (extended periodically) as our initial row.

Corollary 1.3. The $(d-2)$-corridor number $c_n$ (i.e., when $y_0 = 0$) equals the range of the $n$th row of the (standard) circular Pascal array or order $d$.

Proof. $a^2 + b^2 = c^2$. □

Remark 4. Here and throughout, as an aid to the reader, we will bold-face the entry corresponding to $k = 0$ for numerical vectors in $\mathbb{R}^\infty$ (unless it is clear from context).
Figure 1: A koala.

2 Citing some references

Figure 1 shows a koala.

When the allowable moves in a corridor include remaining at the same level, the paths are often called “Motzkin.” (see, for example [3]) The structure we have already set up extends easily to such Motzkin paths, and the related structures are called Göder structures.

Upon the up-sampling, the extreme values fell nicely on diagonals of the circular Pascal arrays. Finally, with the introduction of the dual corridor, we found the strong connection between the two structures. Using the most basic properties of a few simple operators, we arrived at our main result, which easily led to a few nontrivial lattice path results - please refer to [1-3,5] for more details.

References


Index

Circular Pascal array, 1

Number
  Corridor, 1
  Motzkin, 2

Pascal array, 1

Structures
  Gödel, 2