Glava 1

Neodređeni integral

1.1 Integracija trigonometrijskih funkcija

1. Integral oblika $\int R(\sin x, \cos x)dx$ se smenom $\operatorname{tg}\frac{x}{2} = t$, $-\pi < x < \pi$, svodi na integral racionalne funkcije. Zaista, kako je

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}}{\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{2t}{1+t^2}, \tag{1.1}$$

$$\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}, \tag{1.2}$$

$$\frac{x}{2} = \operatorname{arctg} t, \quad x = 2\operatorname{arctg} t, \quad dx = \frac{2dt}{1+t^2}, \tag{1.3}$$

dobijamo

$$\int R(\sin x, \cos x)dx = 2\int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{dt}{1+t^2}.$$

Primer 1.1. Nađimo $\int \frac{dx}{\sin x}$ i $\int \frac{dx}{\cos x}$.

Domen funkcije $x \mapsto \frac{1}{\sin x}$ je unija intervala $(k\pi, (k+1)\pi)$, $k \in \mathbb{Z}$. Primenom (1.1) i (1.3) dobijamo:

$$\begin{aligned}\int \frac{dx}{\sin x} &= \left|\begin{array}{c}\operatorname{tg}\frac{x}{2} = t,\ x \in (-\pi,\pi)\setminus\{0\} \\ x = 2\operatorname{arctg} t \Longrightarrow dx = \frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2}\end{array}\right| = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int \frac{dt}{t} = \ln|t| + C \\ &= \ln|\operatorname{tg}\frac{x}{2}| + C, \text{ na intervalima } (-\pi, 0) \text{ i } (0,\pi).\end{aligned}$$

Kako je $x \mapsto \ln|\operatorname{tg}\frac{x}{2}|$ periodična funkcija sa periodom 2π, zaključujemo da je

$$\int \frac{dx}{\sin x} = \ln|\operatorname{tg}\frac{x}{2}| + C, \text{ na svakom od intervala } (k\pi, (k+1)\pi),\ k \in \mathbb{Z}.$$

Domen funkcije $x \mapsto \frac{1}{\cos x}$ je unija intervala $\left(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2}\right)$, $k \in \mathbb{Z}$.

$$\begin{aligned}
\int \frac{dx}{\cos x} &= \int \frac{dx}{\sin(x+\frac{\pi}{2})} = \left|\ x + \tfrac{\pi}{2} = t \Longrightarrow dx = dt\ \right| = \\
&= \int \frac{dt}{\sin t} = \ln|\operatorname{tg}\frac{t}{2}| + C = \ln|\operatorname{tg}\frac{x+\frac{\pi}{2}}{2}| + C \\
&= \ln|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)| + C, \text{ gde } x + \frac{\pi}{2} \in (k\pi, (k+1)\pi),\ k \in \mathbb{Z}.
\end{aligned}$$

Prema tome, $\int \frac{dx}{\cos x} = \ln|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)| + C$, na svakom od intervala $\left(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2}\right)$, $k \in \mathbb{Z}$. •

Primer 1.2. Nađimo $\int \frac{dx}{2+\cos x}$. Domen podintegralne funkcije je skup $\mathbb{R}$. Primenom (1.2) i (1.3) dobijamo:

$$\begin{aligned}
\int \frac{dx}{2+\cos x} &= \left| \begin{array}{c} \operatorname{tg}\frac{x}{2} = t,\ x \in (-\pi,\pi) \\ x = 2\operatorname{arctg} t \Longrightarrow dx = \frac{2dt}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2} \end{array} \right| = \int \frac{\frac{2dt}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} = 2\int \frac{dt}{3+t^2} \\
&= \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{t}{\sqrt{3}} + C = \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{tg\frac{x}{2}}{\sqrt{3}} + C,\ x \in (-\pi,\pi).
\end{aligned}$$

Budući da je funkcija $x \mapsto \operatorname{arctg}\frac{tg\frac{x}{2}}{\sqrt{3}}$ periodična sa periodom 2π, zaključujemo da je $\int \frac{dx}{2+\cos x} = \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{tg\frac{x}{2}}{\sqrt{3}} + C$ na svakom od intervala $((2k-1)\pi, (2k+1)\pi)$, $k \in \mathbb{Z}$. •

Smenom $\operatorname{tg}\frac{x}{2} = t$ možemo integraliti ma koju funkciju oblika $x \mapsto R(\sin x, \cos x)$. Međutim ova smena nije uvek i najbolje rešenje jer često dovodi do integrala racionalne funkcije sa polinomima u brojiocu i imeniocu koji imaju velike stepene. Navešćemo primere integrala ovog tipa kod kojih je podesnije koristiti neke druge smene.

2. Najpre uočimo da ako je $x \mapsto R(x)$ parna racionalna funkcija, da se ona može napisati u obliku $R(x) = R_1(x^2)$ gde je R_1 takođe racionalna funkcija. [1]

[1] Ako je racionalna funkcija $R(x) = \frac{P(x)}{Q(x)}$, gde su P i Q polinomi, parna, onda je $R(-x) = R(x)$ i kako je $P(x) = P_1(x^2) + xP_2(x^2)$ i $Q(x) = Q_1(x^2) + xQ_2(x^2)$ gde su P_1, P_2, Q_1 i Q_2 takođe polinomi, sledi da je

$$\begin{aligned}
\frac{P_1(x^2)+xP_2(x^2)}{Q_1(x^2)+xQ_2(x^2)} &= \frac{P_1((-x)^2)-xP_2((-x)^2)}{Q_1((-x)^2)-xQ_2((-x)^2)}, \\
\text{tj.}\quad \frac{P_1(x^2)+xP_2(x^2)}{Q_1(x^2)+xQ_2(x^2)} &= \frac{P_1(x^2)-xP_2(x^2)}{Q_1(x^2)-xQ_2(x^2)}
\end{aligned}$$

Odavde,

$$(P_1(x^2)+xP_2(x^2))(Q_1(x^2)-xQ_2(x^2)) = (P_1(x^2)-xP_2(x^2))(Q_1(x^2)+xQ_2(x^2)),$$

i prema tome,

$$P_1(x^2)Q_2(x^2) = P_2(x^2)Q_1(x^2).$$

Takođe, ako je racionalna funkcija dveju promenljivih $R = R(x_1, x_2)$ parna po jednoj promenljivoj, recimo x_1, tj. ako važi $R(-x_1, x_2) = R(x_1, x_2)$, onda se ona može napisati u obliku

$$R(x_1, x_2) = R_1(x_1^2, x_2),$$

gde je R_1 takođe racionalna funkcija.

Ako je pak funkcija $R = R(x_1, x_2)$ neparna po x_1, tj. važi $R(-x_1, x_2) = -R(x_1, x_2)$, onda je $(x_1, x_2) \mapsto \frac{R(x_1,x_2)}{x_1}$ parna po x_1, pa je $\frac{R(x_1,x_2)}{x_1} = R^*(x_1^2, x_2)$, gde je R^* racionalna funkcija, i prema tome

$$R(x_1, x_2) = R^*(x_1^2, x_2) \cdot x_1.$$

Stoga, ako je podintegralna funkcija racionalna po $\sin x$ i $\cos x$ i uz to i neparna po $\sin x$, tj. važi $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, onda je

$$\begin{aligned} R(\sin x, \cos x)dx &= R^*(\sin^2 x, \cos x) \sin x dx \\ &= -R^*(1 - \cos^2 x, \cos x) d(\cos x), \end{aligned}$$

pa se smenom

$$\cos x = t$$

integral $\int R(\sin x, \cos x)dx$ svodi na integral racionalne funkcije.

Primer 1.3.

$$\begin{aligned} \int \frac{dx}{\sin x \cos 2x} &= \int \frac{\sin x\ dx}{\sin^2 x(\cos^2 x - \sin^2 x)} = \int \frac{\sin x\ dx}{(1-\cos^2 x)(2\cos^2 x - 1)} \\ &= \left| \begin{array}{c} \cos x = t \Longrightarrow -\sin x dx = dt \\ \sin x dx = -dt \end{array} \right| = -\int \frac{dt}{(1-t^2)(2t^2-1)}. \bullet \end{aligned}$$

Ako je podintegralna funkcija racionalna po $\sin x$ i $\cos x$ i osim toga i neparna po $\cos x$, tj. važi $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, onda je

$$\begin{aligned} R(\sin x, \cos x)dx &= \widetilde{R}(\sin x, \cos^2 x) \cos x dx \\ &= \widetilde{R}(\sin x, 1 - \sin^2 x) d(\sin x), \end{aligned}$$

gde je $\widetilde{R}$ takođe racionalna funkcija, pa se smenom

$$\sin x = t$$

Iz poslednje jednakosti dobijamo da je

$$\begin{aligned} R(x) &= \frac{P_1(x^2) + xP_2(x^2)}{Q_1(x^2) + xQ_2(x^2)} = \frac{Q_2(x^2)(P_1(x^2) + xP_2(x^2))}{Q_2(x^2)(Q_1(x^2) + xQ_2(x^2))} \\ &= \frac{P_1(x^2)Q_2(x^2) + xP_2(x^2)Q_2(x^2)}{Q_2(x^2)(Q_1(x^2) + xQ_2(x^2))} = \frac{P_2(x^2)Q_1(x^2) + xP_2(x^2)Q_2(x^2)}{Q_2(x^2)(Q_1(x^2) + xQ_2(x^2))} \\ &= \frac{P_2(x^2)(Q_1(x^2) + xQ_2(x^2))}{Q_2(x^2)(Q_1(x^2) + xQ_2(x^2))} = \frac{P_2(x^2)}{Q_2(x^2)} = R_1(x^2). \end{aligned}$$

integral $\int R(\sin x, \cos x)dx$ svodi na integral racionalne funkcije.

Tako se, na primer, integral $\int \sin^m x \cos^n x dx$, gde su m i n celi brojevi, za slučaj da je m neparan broj, rešava smenom $\cos x = t$, dok, za slučaj da je n neparan, smenom $\sin x = t$.

Primer 1.4.

$$\begin{aligned}
\int \frac{\sin^5 x}{\cos^2 x}dx &= \int \frac{\sin^4 x}{\cos^2 x}\sin x\,dx = \left| \begin{array}{c} \cos x = t \Longrightarrow \\ -\sin x dx = dt \Longrightarrow \sin x dx = -dt \\ \sin^2 x = 1 - t^2 \end{array} \right| = \\
&= -\int \frac{(1-t^2)^2}{t^2}dt = \int \frac{1-2t^2+t^4}{t^2} = \int \left(\frac{1}{t^2} - 2 + t^2\right) dt \\
&= -\frac{1}{t} - 2t + \frac{t^3}{3} + C \\
&= -\frac{1}{\cos x} - 2\cos x + \frac{\cos^3 x}{3} + C,
\end{aligned}$$

na svakom intervalu koji ne sadrži tačke $\frac{k\pi}{2}$, $k \in \mathbb{Z}$. •

Primer 1.5.

$$\begin{aligned}
\int \sin^4 x \cos^3 x\,dx &= \int \sin^4 x \cos^2 x \cos x\,dx = \int \sin^4 x(1-\sin^2 x)\cos x\,dx \\
&= \left| \sin x = t \Longrightarrow \cos x\,dx = dt \right| = \int t^4(1-t^2)dt \\
&= \int (t^4 - t^6)dt = \frac{t^5}{5} - \frac{t^7}{7} + C \\
&= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C, \ x \in \mathbb{R}.
\end{aligned}$$ •

Primer 1.6.

$$\begin{aligned}
\int \frac{dx}{\sin^2 x\,\cos x} &= \int \frac{\cos x\,dx}{\sin^2 x \cos^2 x} \\
&= \int \frac{\cos x\,dx}{\sin^2 x\,(1-\sin^2 x)} = \left| \sin x = t \Longrightarrow \cos x\,dx = dt \right| \\
&= \int \frac{dt}{t^2(1-t^2)}
\end{aligned}$$

Ako su i m i n neparni brojevi, $m = 2k+1$, $n = 2l+1$, $k, l \in \mathbb{Z}$, onda je osim smena $\sin x = t$ ili $\cos x = t$, celishodno upotrebiti i smenu $\cos 2x = t$. Naime,

$$\begin{aligned}
\int \sin^{2k+1} x \cos^{2l+1} x\ dx &= \int \sin^{2k} x \cos^{2l} x \sin x \cos x\ dx \\
&= \int \left(\frac{1-\cos 2x}{2}\right)^k \left(\frac{1+\cos 2x}{2}\right)^l \left(\frac{1}{2}\sin 2x\right) dx \\
&= \left| \begin{array}{c} \cos 2x = t \Longrightarrow -2\sin 2x\,dx = dt \\ \Longrightarrow \sin 2x\,dx = -\frac{1}{2}dt \end{array} \right| \\
&= -\frac{1}{2^{k+l+2}} \int (1-t)^k (1+t)^l dt,
\end{aligned}$$

što je integral racionalne funkcije (jer $k, l \in \mathbb{Z}$).

3. Ako je racionalna funkcija dveju promenljivih $R = R(x_1, x_2)$ parna i po jednoj i po drugoj promenljivoj, tj. ako važi $R(-x_1, -x_2) = R(x_1, x_2)$, onda je

$$R(x_1, x_2) = R\left(\frac{x_1}{x_2}x_2, x_2\right) = R_1\left(\frac{x_1}{x_2}, x_2\right),$$

gde je R_1 takođe racionalna funkcija. Kako je

$$R_1\left(\frac{x_1}{x_2}, x_2\right) = R(x_1, x_2) = R(-x_1, -x_2) = R_1\left(\frac{-x_1}{-x_2}, -x_2\right) = R_1\left(\frac{x_1}{x_2}, -x_2\right),$$

sledi da je funkcija $(x_1, x_2) \mapsto R_1(\frac{x_1}{x_2}, x_2)$ parna po x_2, te je

$$R_1\left(\frac{x_1}{x_2}, x_2\right) = R_2\left(\frac{x_1}{x_2}, x_2^2\right), \quad R_2 \text{ je racionalna funkcija.}$$

Prema tome, ako je podintegralna funkcija racionalna po $\sin x$ i $\cos x$ i takva da zamenom $\sin x$ sa $-\sin x$ i $\cos x$ sa $-\cos x$ ostaje nepromenjena, tj. $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, onda je

$$R(\sin x, \cos x) = R_2\left(\frac{\sin x}{\cos x}, \cos^2 x\right) = R_2\left(\operatorname{tg} x, \frac{1}{1 + \operatorname{tg}^2 x}\right) = R_3(\operatorname{tg} x),$$

gde je R_3 racionalna funkcija, pa se smenom $\operatorname{tg} x = t$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, integral $\int R(\sin x, \cos x)dx$ svodi na integral racionalne funkcije:

$$\int R(\sin x, \cos x)dx = \int R_3(t)\frac{dt}{1 + t^2}.$$

Ovde je:

$$\begin{aligned}
\sin^2 x &= \frac{\sin^2 x}{\sin^2 x + \cos^2 x} = \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}} = \frac{\operatorname{tg}^2 x}{1 + \operatorname{tg}^2 x} = \frac{t^2}{1 + t^2},\\
\cos^2 x &= \frac{\cos^2 x}{\sin^2 x + \cos^2 x} = \frac{\frac{\cos^2 x}{\cos^2 x}}{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}} = \frac{1}{1 + \operatorname{tg}^2 x} = \frac{1}{1 + t^2},\\
\sin x \cos x &= \frac{\sin x \cos x}{\sin^2 x + \cos^2 x} = \frac{\frac{\sin x \cos x}{\cos^2 x}}{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}} = \frac{\operatorname{tg} x}{1 + \operatorname{tg}^2 x} = \frac{t}{1 + t^2},\\
&\operatorname{tg} x = t, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \Longrightarrow x = \operatorname{arctg} t \Longrightarrow dx = \frac{dt}{1 + t^2}. \bullet
\end{aligned}$$

Primer 1.7.

$$\begin{aligned}
\int \frac{\cos x}{\sin x + \cos x}dx &= \int \frac{\frac{\cos x}{\cos x}}{\frac{\sin x + \cos x}{\cos x}}dx = \int \frac{1}{\operatorname{tg} x + 1}dx\\
&= \left| \begin{array}{c} \operatorname{tg} x = t,\ -\frac{\pi}{2} < x < \frac{\pi}{2},\ x \neq -\frac{\pi}{4} \Longrightarrow x = \operatorname{arctg} t \\ \Longrightarrow dx = \frac{dt}{1+t^2} \end{array} \right|\\
&= \int \frac{1}{t + 1} \cdot \frac{dt}{1 + t^2}. \bullet
\end{aligned}$$

Između ostalih, u ovaj tip integrala spadaju i integrali tipa

$$\int R(\sin^2 x, \cos^2 x, \sin x\cos x)dx.$$

Primer 1.8.

$$\int \frac{\cos^2 x}{3+\sin^2 x}dx = \left| \begin{array}{c} \operatorname{tg} x = t,\ -\frac{\pi}{2} < x < \frac{\pi}{2} \Longrightarrow x = \operatorname{arctg} t \\ \Longrightarrow dx = \frac{dt}{1+t^2} \\ \cos^2 x = \frac{1}{1+t^2},\ \sin^2 x = \frac{t^2}{1+t^2} \end{array} \right| = \int \frac{\frac{1}{1+t^2}}{3+\frac{t^2}{1+t^2}} \cdot \frac{dt}{1+t^2}$$

$$= \int \frac{dt}{(4t^2+1)(1+t^2)}. \bullet$$

Primer 1.9.

$$\begin{aligned} \int \frac{2+\sin x\cos x}{\cos x(\sin^3 x+\cos^3 x)}dx &= \int \frac{2+\sin x\cos x}{\cos x(\sin x+\cos x)(\sin^2 x-\sin x\cos x+\cos^2 x)}dx \\ &= \int \frac{2+\sin x\cos x}{\cos x(\sin x+\cos x)(1-\sin x\cos x)}dx \\ &= \int \frac{2+\sin x\cos x}{(\sin x\cos x+\cos^2 x)(1-\sin x\cos x)}dx \\ &= \left| \begin{array}{c} \operatorname{tg} x = t,\ -\frac{\pi}{2} < x < \frac{\pi}{2},\ x \neq -\frac{\pi}{4} \Longrightarrow x = \operatorname{arctg} t \\ \Longrightarrow dx = \frac{dt}{1+t^2} \\ \cos^2 x = \frac{1}{1+t^2},\ \sin^2 x = \frac{t^2}{1+t^2},\ \sin x\cos x = \frac{t}{1+t^2} \end{array} \right| \\ &= \int \frac{2+\frac{t}{1+t^2}}{\left(\frac{t}{1+t^2}+\frac{1}{1+t^2}\right)\left(1-\frac{t}{1+t^2}\right)} \cdot \frac{dt}{1+t^2} \\ &= \int \frac{2t^2+t+2}{(t+1)(t^2-t+1)(1+t^2)}\,dt. \bullet \end{aligned}$$

Primer 1.10.

$$\begin{aligned} \int \frac{\sin^2 x\,dx}{\cos^3 x\,(\sin x+\cos x)} &= \int \frac{\sin^2 x\,dx}{\cos^2 x\,(\sin x\cos x+\cos^2 x)} \\ &= \left| \begin{array}{c} \operatorname{tg} x = t,\ -\frac{\pi}{2} < x < \frac{\pi}{2},\ x \neq -\frac{\pi}{4} \Longrightarrow x = \operatorname{arctg} t \\ dx = \frac{dt}{1+t^2} \end{array} \right| \\ &= \int \frac{\frac{t^2}{1+t^2}}{\frac{1}{1+t^2}\left(\frac{t}{1+t^2}+\frac{1}{1+t^2}\right)} \cdot \frac{dt}{1+t^2} \\ &= \int \frac{t^2 dt}{t+1}. \end{aligned}$$

Mogli smo podintegralnu funkciju transformisati i na sledeći način:

$$\begin{aligned} \frac{\sin^2 x}{\cos^3 x(\sin x+\cos x)} &= \frac{\frac{\sin^2 x}{\cos^4 x}}{\frac{\cos^3 x(\sin x+\cos x)}{\cos^4 x}} = \frac{\frac{1}{\cos^2 x}\cdot \operatorname{tg}^2 x}{\operatorname{tg} x+1} \\ &= \frac{(1+\operatorname{tg}^2 x)\operatorname{tg}^2 x}{\operatorname{tg} x+1}. \bullet \end{aligned}$$

Intgral $\int \sin^m x \cos^n x dx$, gde su m i n parni celi brojevi, takođe spada u ovaj tip integrala.

Primer 1.11.

$$\begin{aligned}
\int \frac{dx}{\sin^4 x} &= \left| \begin{array}{c} \operatorname{tg} x = t,\ x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\} \Rightarrow x = \operatorname{arctg} t \Rightarrow dx = \frac{dt}{1+t^2} \\ \sin^2 x = \frac{t^2}{1+t^2} \end{array} \right| \\
&= \int \frac{1}{\frac{t^4}{(1+t^2)^2}} \frac{dt}{1+t^2} = \int \frac{(1+t^2)dt}{t^4} \\
&= \int \left(\frac{1}{t^4} + \frac{1}{t^2} \right) dt = -\frac{1}{3}\frac{1}{t^3} - \frac{1}{t} + C \\
&= -\frac{1}{3}\frac{1}{tg^3 x} - \frac{1}{tgx} + C, \text{ na intervalima } (-\frac{\pi}{2}, 0) \text{ i } (0, \frac{\pi}{2}).
\end{aligned}$$

Kako je $x \mapsto -\frac{1}{3}\frac{1}{tg^3 x} - \frac{1}{tgx}$ periodična funkcija sa periodom π, to je

$$\int \frac{dx}{\sin^4 x} = -\frac{1}{3}\frac{1}{tg^3 x} - \frac{1}{tgx} + C, \text{ na intervalima } (-\frac{\pi}{2} + k\pi, k\pi) \text{ i } (k\pi, \frac{\pi}{2} + k\pi),\ k \in \mathbb{Z}. \bullet$$

Međutim, ako su u integralu $\int \sin^m x \cos^n x dx$, brojevi m i n parni i nenegativni celi brojevi, onda je podesnije primeniti takozvane formule za snižavanje stepena:

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

Primer 1.12.

$$\int \cos^2 x\, dx = \int \frac{1 + \cos 2x}{2} dx = \frac{1}{2} \int (1 + \cos 2x) dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C,\ x \in \mathbb{R}. \bullet$$

Primer 1.13.

$$\begin{aligned}
\int \sin^4 x\, dx &= \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx = \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) dx \\
&= \frac{1}{4} \int \left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2} \right) dx \\
&= \frac{1}{4} \int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x \right) dx \\
&= \frac{1}{4} \left(\frac{3}{2}x - \sin 2x - \frac{1}{8}\sin 4x \right) + C \\
&= \frac{3}{8}x - \frac{1}{4}\sin 2x - \frac{1}{32}\sin 4x + C,\ x \in \mathbb{R}. \bullet
\end{aligned}$$

Primer 1.14.

$$
\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)dx \\
&= \frac{1}{8}\int (1+\cos 2x-\cos^2 2x-\cos^3 2x)dx \\
&= \frac{1}{8}\left(x+\frac{1}{2}\sin 2x-\frac{1}{2}\int (1+\cos 4x)dx-\frac{1}{2}\int (1-\sin^2 x)d(\sin 2x)\right) \\
&= \frac{x}{16}-\frac{\sin 4x}{64}+\frac{\sin^3 2x}{48}+C.
\end{aligned}
$$

4. Integrali oblika $\int R(\operatorname{tg} x)dx$ se rešavaju smenom $\operatorname{tg} x = t$. Primetimo da su svi integrali iz tačke **3.** upravo ovog tipa, jer se podintegralna funkcija $R(\sin x, \cos x)$ može transformisati u oblik $R_1(\operatorname{tg} x)$, gde je R_1 takođe racionalna funkcija.

Integrali oblika $\int R(\operatorname{ctg} x)dx$ se rešavaju smenom $\operatorname{ctg} x = t$, $0 < x < \pi$.

Primer 1.15.

$$
\int \frac{1-\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x} dx = \left| \begin{array}{c} \operatorname{tg} x = t, \ -\frac{\pi}{2} < x < \frac{\pi}{2} \Longrightarrow x = \operatorname{arctg} t \\ \Longrightarrow dx = \frac{dt}{1+t^2} \end{array} \right| = \int \frac{1-t^2}{1+t^2} \cdot \frac{dt}{1+t^2}. \bullet
$$

5. Kod integrala oblika $\int \sin ax \cos bx \, dx$, $\int \cos ax \cos bx \, dx$, $\int \sin ax \sin bx \, dx$, $a, b \in \mathbb{R} \setminus \{0\}$, podintegralne funkcije transformišemo redom na sledeći način:

$$
\begin{aligned}
\sin ax \cos bx &= \frac{1}{2}[\sin(a+b)x + \sin(a-b)x], \\
\cos ax \cos bx &= \frac{1}{2}[\cos(a+b)x + \cos(a-b)x], \\
\sin ax \sin bx &= \frac{1}{2}[\cos(a-b)x - \cos(a+b)x].
\end{aligned}
$$

Primer 1.16.

$$
\int \sin 3x \sin 5x \, dx = \frac{1}{2}\int (\cos 2x - \cos 8x)dx = \frac{1}{2}(\frac{1}{2}\sin 2x - \frac{1}{8}\sin 8x) + C, \ x \in \mathbb{R}. \bullet
$$

Rekurzivne formule

U opštem slučaju termin rekuzivna formula označava formulu u kojoj se izraz koji zavisi od nekog parametra izražava preko izraza istog oblika kome odgovara ista ili neka druga vrednost tog parametra. U slučaju integrala, metod rekurzivnih formula se koristi za određivanje integrala funkcija koje zavise od celobrojnog parametra n, na taj način što se polazni integral koji zavisi od n izražava preko integrala istog tipa koji zavisi od parametra m manjeg od n.

Preciznije, neka su date funkcije f_n, $n = 0, 1, \dots$, na intervalu I, $f_n : I \to \mathbb{R}$, i neka postoje integrali $I_n = \int f_n(x)dx$. Formula oblika

$$I_n = \Phi(I_{n-1}, I_{n-2}, \dots, I_{n-k}),\ k \le n$$

zove se *rekrzivna formula* za niz integrala (I_n), $n \in \mathbb{N}_0$. [2]

Primer 1.17. Neka je $I_n = \int \operatorname{tg}^n x\, dx,\ n \in \mathbb{N}_0$. Nađimo I_0 i I_1:

$$\begin{aligned} I_0 &= \int dx = x + C, \\ I_1 &= \int \operatorname{tg} x\, dx = \int \frac{\sin x\, dx}{\cos x} = \left| \begin{array}{c} \cos x = t \Longrightarrow -\sin x\, dx = dt \\ \Longrightarrow \sin x\, dx = -dt \end{array} \right| \\ &= -\int \frac{dt}{t} = -\ln|t| + C = -\ln|\cos x| + C \end{aligned}$$

Za $n \ge 2$ imamo:

$$\begin{aligned} I_n &= \int \operatorname{tg}^n x\, dx = \int \operatorname{tg}^{n-2} x \operatorname{tg}^2 x\, dx = \int \operatorname{tg}^{n-2} x \frac{\sin^2 x}{\cos^2 x}\, dx \\ &= \int \operatorname{tg}^{n-2} x \frac{1-\cos^2 x}{\cos^2 x}\, dx = \int \operatorname{tg}^{n-2} x \frac{1}{\cos^2 x}\, dx - \int \operatorname{tg}^{n-2} x\, dx \\ &= \int \operatorname{tg}^{n-2} x \frac{1}{\cos^2 x}\, dx - I_{n-2}. \end{aligned}$$

Kako je

$$\begin{aligned} \int \operatorname{tg}^{n-2} x \frac{1}{\cos^2 x}\, dx &= \left|\ \operatorname{tg} x = t \Rightarrow \tfrac{1}{\cos^2 x} dx = dt\ \right| = \int t^{n-2} dt = \frac{t^{n-1}}{n-1} + C \\ &= \frac{\operatorname{tg}^{n-1} x}{n-1} + C, \end{aligned}$$

sledi

$$I_n = \frac{\operatorname{tg}^{n-1} x}{n-1} - I_{n-2},\ n \ge 2. \tag{1.4}$$

Tako, za $n = 2$ iz formule (1.4) dobijamo:

$$I_2 = \operatorname{tg} x - I_0 = \operatorname{tg} x - x + C,$$

dok za $n = 3$ imamo:

$$I_3 = \frac{1}{2}\operatorname{tg}^2 x - I_1 = \frac{1}{2}\operatorname{tg}^2 x + \ln|\cos x| + C. \bullet$$

[2] Sa $\mathbb{N}_0$ označavamo skup $\mathbb{N} \cup \{0\}$.

Primer 1.18. Neka je $I_n = \int \sin^n x\,dx$ i $J_n = \int \cos^n x\,dx$, $n \in \mathbb{N}_0$. Za $n \geq 2$ primenjujemo metod parcijalne integracije:

$$\begin{aligned}
I_n &= \int \sin^n x\,dx = \left| \begin{array}{c} dv = \sin x\,dx \Longrightarrow v = -\cos x \\ u = \sin^{n-1} x \Longrightarrow du = (n-1)\sin^{n-2} x\,\cos x dx \end{array} \right| \\
&= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x dx \\
&= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)dx \\
&= -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n.
\end{aligned}$$

Odavde sledi

$$nI_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2},$$

tj.

$$I_n = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}I_{n-2}.$$

Kako je

$$I_0 = \int dx = x + C \quad \text{i} \quad I_1 = \int \sin x dx = -\cos x + C,$$

to je

$$I_2 = -\frac{1}{2}\sin x \cos x + \frac{1}{2}I_0 = -\frac{1}{2}\sin x \cos x + \frac{1}{2}x + C.$$

Takođe,

$$\begin{aligned}
I_3 &= \int \sin^3 x\,dx = -\frac{1}{3}\sin^2 x \cos x + \frac{2}{3}I_1 \\
&= -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C
\end{aligned}$$

i

$$\begin{aligned}
I_4 &= \int \sin^4 x\,dx = -\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}I_2 \\
&= -\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\left(-\frac{1}{2}\sin x \cos x + \frac{1}{2}x\right) + C \\
&= -\frac{1}{4}\sin^3 x \cos x - \frac{3}{8}\sin x \cos x + \frac{3}{8}x + C.
\end{aligned}$$

Slično se dokazuje da je

$$J_n = \int \cos^n\,dx = \tfrac{1}{n}\cos^{n-1} x \sin x + \tfrac{n-1}{n}J_{n-2},$$
$$J_0 = \int dx = x + C, \quad J_1 = \int \cos x dx = \sin x + C.$$

1.2 Integracija iracionalnih funkcija

Primer 1.19. Nađimo $I = \int \frac{dx}{(x-1)\sqrt{x^2-2}}$. Domen podintegralne funkcije $f(x) = \frac{1}{(x-1)\sqrt{x^2-2}}$ je $D_f = (-\infty, -\sqrt{2}) \cup (\sqrt{2}, +\infty)$. Nađimo integral ove funkcije najpre na intervalu $(\sqrt{2}, +\infty)$.

$$\begin{aligned}
I &= \left| \begin{array}{c} x > \sqrt{2},\ x-1 = \frac{1}{t} \Rightarrow t > 0 \\ x = \frac{1}{t} + 1 \\ dx = -\frac{1}{t^2}dt \end{array} \right| = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}+1\right)^2 - 2}} \\
&= -\int \frac{dt}{t\sqrt{\frac{1}{t^2} + \frac{2}{t} + 1 - 2}} = -\int \frac{dt}{t\sqrt{\frac{1+2t-t^2}{t^2}}} \\
&= -\int \frac{dt}{t\frac{\sqrt{1+2t-t^2}}{|t|}} = -\int \frac{dt}{t\frac{\sqrt{1+2t-t^2}}{t}} = -\int \frac{dt}{\sqrt{1+2t-t^2}} \\
&= -\int \frac{dt}{\sqrt{-(t^2-2t-1)}} = -\int \frac{dt}{\sqrt{-(t^2-2t+1-2)}} \\
&= -\int \frac{dt}{\sqrt{-((t-1)^2-2)}} = -\int \frac{dt}{\sqrt{2-(t-1)^2}} \\
&= -\arcsin \frac{t-1}{\sqrt{2}} + C = -\arcsin \frac{\frac{1}{x-1} - 1}{\sqrt{2}} + C \\
&= -\arcsin \frac{2-x}{\sqrt{2}(x-1)} + C = \arcsin \frac{x-2}{\sqrt{2}(x-1)} + C.
\end{aligned}$$

Nađimo sada integral ove funkcije na intervalu $(-\infty, -\sqrt{2})$.

$$\begin{aligned}
I &= \left| \begin{array}{c} x < -\sqrt{2},\ x-1 = \frac{1}{t} \Rightarrow t < 0 \\ x = \frac{1}{t} + 1 \\ dx = -\frac{1}{t^2}dt \end{array} \right| = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}+1\right)^2 - 2}} \\
&= -\int \frac{dt}{t\sqrt{\frac{1}{t^2} + \frac{2}{t} - 1}} = -\int \frac{dt}{t\sqrt{\frac{1+2t-t^2}{t^2}}} \\
&= -\int \frac{dt}{t\frac{\sqrt{1+2t-t^2}}{|t|}} = -\int \frac{dt}{t\frac{\sqrt{1+2t-t^2}}{-t}} \\
&= \int \frac{dt}{\sqrt{1+2t-t^2}} = \int \frac{dt}{\sqrt{2-(t-1)^2}} \\
&= \arcsin \frac{t-1}{\sqrt{2}} + C = \arcsin \frac{\frac{1}{x-1} - 1}{\sqrt{2}} + C \\
&= \arcsin \frac{2-x}{\sqrt{2}(x-1)} + C = -\arcsin \frac{x-2}{\sqrt{2}(x-1)} + C.
\end{aligned}$$

Prema tome,

$$\begin{aligned} I &= \int \frac{dx}{(x-1)\sqrt{x^2-2}} \\ &= (\operatorname{sgn} x)\arcsin\frac{x-2}{\sqrt{2}(x-1)} + C, \text{ na intervalima } (-\infty,-\sqrt{2}) \text{ i } (\sqrt{2},+\infty). \bullet \end{aligned}$$

Primer 1.20. Nađimo $\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}}$. Domen podintegralne funkcije $f(x) = \frac{1}{(x+1)\sqrt{x^2+2x+2}}$ je $D_f = (-\infty,-1)\cup(-1,+\infty)$. Posmatrajmo najpre interval $(-1,+\infty)$.

$$\begin{aligned} \int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}} &= \left| \begin{array}{c} x > -1,\ x+1 = \frac{1}{t}, \\ \Longrightarrow t > 0,\ x = \frac{1}{t} - 1, \\ dx = -\frac{1}{t^2}dt \end{array} \right| \\ &= \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t^2}\sqrt{\left(\frac{1}{t}-1\right)^2 + 2\left(\frac{1}{t}-1\right) + 2}} \\ &= -\int \frac{dt}{\sqrt{\frac{1}{t^2}+1}} = -\int \frac{dt}{\frac{\sqrt{1+t^2}}{|t|}} \\ &= -\int \frac{tdt}{\sqrt{1+t^2}} = \left| \begin{array}{c} 1+t^2 = z \\ 2tdt = dz \Rightarrow tdt = \frac{1}{2}dz \end{array} \right| \\ &= -\int \frac{dz}{2\sqrt{z}} = \sqrt{z} + C = -\sqrt{1+t^2} + C \\ &= -\sqrt{1+\frac{1}{(x+1)^2}} + C = -\sqrt{\frac{x^2+2x+2}{(x+1)^2}} + C \\ &= -\frac{\sqrt{x^2+2x+2}}{|x+1|} + C = -\frac{\sqrt{x^2+2x+2}}{x+1} + C. \end{aligned}$$

Ako je $x < -1$ i opet $x+1 = \frac{1}{t}$, onda je $x+1 < 0$ i stoga $t < 0$, pa je $|t| = -t$ i

$$\begin{aligned} \int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}} &= -\int \frac{dt}{\frac{\sqrt{1+t^2}}{|t|}} \\ &= \int \frac{tdt}{\sqrt{1+t^2}} = \sqrt{1+t^2} + C \\ &= \sqrt{1+\frac{1}{(x+1)^2}} + C = \sqrt{\frac{x^2+2x+2}{(x+1)^2}} + C \\ &= \frac{\sqrt{x^2+2x+2}}{|x+1|} + C = -\frac{\sqrt{x^2+2x+2}}{x+1} + C. \end{aligned}$$

Prema tome,

$$\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}} = -\frac{\sqrt{x^2+2x+2}}{x+1} + C, \text{ na intervalima } (-\infty,-1) \text{ i } (-1,+\infty). \bullet$$

Ojlerove smene

Treća Ojlerova smena. Neka su nule x_1 i x_2 kvadratnog ax^2+bx+c realne i različite i neka je $x_1 < x_2$. Tada možemo koristiti smenu

$$\sqrt{ax^2+bx+c} = t(x-x_1) \text{ ili } \sqrt{ax^2+bx+c} = t(x-x_2).$$

1. Neka je $\sqrt{ax^2+bx+c} = t(x-x_1)$. Kako je $ax^2+bx+c = a(x-x_1)(x-x_2)$, to je

$$\begin{aligned} \sqrt{a(x-x_1)(x-x_2)} &= t(x-x_1) \qquad (1.5) \\ a(x-x_1)(x-x_2) &= t^2(x-x_1)^2 \\ t^2 &= \frac{a(x-x_2)}{x-x_1}. \qquad (1.6) \end{aligned}$$

Iz (1.6) sledi

$$\begin{aligned} t^2(x-x_1) &= ax-ax_2 \\ (t^2-a)x &= t^2x_1-ax_2 \\ x &= \frac{x_1t^2-ax_2}{t^2-a}. \qquad (1.7) \end{aligned}$$

Označimo desnu stranu jednakosti u (1.7) sa $\varphi(t)$. Sada je

$$dx = \varphi'(t)dt$$

i

$$\int R(x, \sqrt{ax^2+bx+c})dx = \int R(\varphi(t), t(\varphi(t)-x_1))\varphi'(t)dt. \qquad (1.8)$$

Integral na desnoj strani u (1.8) je integral racionalne funkcje.

1.1 Ako je $a > 0$, domen podintegralne funkcije je podskup skupa $(-\infty, x_1] \cup [x_2, +\infty)$ (nekad je domen i jednak ovom skupu).

Za $x \in (-\infty, x_1)$ je $x - x_1 < 0$, pa iz (1.5) sledi da je $t < 0$ i stoga onda iz (1.6) dobijamo

$$t = -\sqrt{\frac{a(x-x_2)}{x-x_1}}.$$

Za $x \in (x_2, +\infty)$ je $x - x_1 > 0$, pa iz (1.5) sledi da je $t > 0$ i zato

$$t = \sqrt{\frac{a(x-x_2)}{x-x_1}}.$$

1.2. Ako je $a < 0$, domen podintegralne funkcije je podskup intervala $[x_1, x_2]$ (nekad je domen i jednak ovom intervalu). Za $x \in (x_1, x_2)$ sledi $x - x_1 > 0$, pa iz (1.5) sledi da je $t > 0$ i stoga

$$t = \sqrt{\frac{a(x-x_2)}{x-x_1}}.$$

2. Neka je $\sqrt{ax^2+bx+c} = t(x-x_2)$. Sada je

$$\begin{aligned} \sqrt{a(x-x_1)(x-x_2)} &= t(x-x_2) \qquad (1.9) \\ a(x-x_1)(x-x_2) &= t^2(x-x_2)^2 \\ t^2 &= \frac{a(x-x_1)}{x-x_2}. \qquad (1.10) \end{aligned}$$

2.1. Neka je $a>0$. Za $x \in (-\infty, x_1)$ je $x-x_2<0$, pa iz (1.9) sledi da je $t<0$ i stoga onda iz (1.10) dobijamo

$$t = -\sqrt{\frac{a(x-x_1)}{x-x_2}}.$$

Za $x \in (x_2, +\infty)$ je $x-x_2>0$, pa iz (1.9) sledi da je $t>0$ i zato

$$t = \sqrt{\frac{a(x-x_1)}{x-x_2}}.$$

2.2. Neka je $a<0$. Za $x \in (x_1, x_2)$ sledi $x-x_2<0$, pa iz (1.9) sledi da je $t<0$ i stoga

$$t = -\sqrt{\frac{a(x-x_1)}{x-x_2}}.$$

(Zato je kod treće Ojlerove smene, za slučaj da je $a<0$ i $x_1<x_2$, (osim smene $\sqrt{ax^2+bx+c} = t(x-x_1)$) podesnije izabrati smenu

$$\sqrt{ax^2+bx+c} = t(x_2-x) \qquad (1.11)$$

umesto smene $\sqrt{ax^2+bx+c} = t(x-x_2)$, jer za $x \in (x_1,x_2)$ je $x_2-x>0$ pa je $t>0$ i iz (1.11) sledi

$$\begin{aligned} a(x-x_1)(x-x_2) &= t^2(x-x_2)^2 \\ t^2 &= \frac{a(x-x_1)}{x-x_2} \\ t &= \sqrt{\frac{a(x-x_1)}{x-x_2}}. \end{aligned}$$

Primer 1.21. Nađimo $I = \int (x+\sqrt{(x-1)(x-2)})dx$.

Koristimo smenu

$$\sqrt{(x-1)(x-2)} = t(x-1).$$

Odavde sledi

$$\begin{aligned}
(x-1)(x-2) &= t^2(x-1)^2 \\
t^2 &= \frac{x-2}{x-1} \\
xt^2 - t^2 &= x-2 \\
x(t^2-1) &= t^2-2 \\
x &= \frac{t^2-2}{t^2-1} \\
dx &= \frac{2t}{(t^2-1)^2}dt.
\end{aligned}$$

Sada je

$$\begin{aligned}
I &= \int \left(\frac{t^2-2}{t^2-1} + t\left(\frac{t^2-2}{t^2-1} - 1 \right) \right) \frac{2t}{(t^2-1)^2} dt = \ldots \\
&= \frac{1}{8} \ln \left| \frac{t-1}{t+1} \right| - \frac{1}{2}\frac{1}{t-1} + \frac{3}{4}\frac{1}{t+1} + \frac{1}{4}\frac{1}{t-1} + C.
\end{aligned}$$

Za $x \in (-\infty, 1)$ je $t < 0$ i $t = -\sqrt{\frac{x-2}{x-1}}$.

Za $x \in (2, +\infty)$ je $t > 0$ i $t = \sqrt{\frac{x-2}{x-1}}$. •

Primer 1.22. Odrediti

$$I = \int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$$

smenom

(i) $\sqrt{1-x^2} = t(x+1)$; (ii) $\sqrt{1-x^2} = t(1-x)$. •

Trigonometrijske i hiperboličke smene kod integrala $\int R(x, \sqrt{ax^2+bx+c})dx$

Primer 1.23. Odrediti $I = \int \frac{\sqrt{x^2-9}}{x} dx$. Tražićemo neodređeni integral funkcije $f(x) = \frac{\sqrt{x^2-9}}{x}$ najpre na intervalu $(3, +\infty)$, a potom intervalu $(-\infty, -3)$.

Neka je $x > 3$.

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \left| \begin{array}{c} x = \frac{3}{\cos t},\ t \in (0, \frac{\pi}{2}) \\ \cos t = \frac{3}{x} \Rightarrow t = \arccos\frac{3}{x} \\ dx = -\frac{3}{\cos^2 t}\cdot(-\sin t)dt = \frac{3\sin t}{\cos^2 t}dt \end{array} \right|$$

$$= \int \frac{\sqrt{\frac{9}{\cos^2 t} - 9}}{\frac{3}{\cos t}} \cdot \frac{3\sin t}{\cos^2 t}dt = 3\int \sqrt{\frac{1-\cos^2 t}{\cos^2 t}} \cdot \frac{\sin t}{\cos t}\,dt$$

$$= 3\int \sqrt{\frac{\sin^2 t}{\cos^2 t}} \cdot \frac{\sin t}{\cos t}\,dt = 3\int \left|\frac{\sin t}{\cos t}\right| \cdot \frac{\sin t}{\cos t}\,dt$$

$$= 3\int \frac{\sin^2 t}{\cos^2 t}\,dt = 3\int \frac{1-\cos^2 t}{\cos^2 t}\,dt$$

$$= 3\int \frac{1}{\cos^2 t}\,dt - 3\int dt = 3\,\mathrm{tg}\,t - 3t + C$$

$$= 3\frac{\sqrt{1-\cos^2 t}}{\cos t} - 3t + C = 3\frac{\sqrt{1-\frac{9}{x^2}}}{\frac{3}{x}} - 3\arccos\frac{3}{x} + C$$

$$= \sqrt{x^2-9} - 3\arccos\frac{3}{x} + C.$$

Neka je $x < -3$.

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \left| \begin{array}{c} x = \frac{3}{\cos t},\ t \in (\frac{\pi}{2}, \pi) \\ \cos t = \frac{3}{x} \Rightarrow t = \arccos\frac{3}{x} \\ dx = -\frac{3}{\cos^2 t}\cdot(-\sin t)dt = \frac{3\sin t}{\cos^2 t}dt \end{array} \right|$$

$$= \int \frac{\sqrt{\frac{9}{\cos^2 t} - 9}}{\frac{3}{\cos t}} \cdot \frac{3\sin t}{\cos^2 t}dt = 3\int \sqrt{\frac{1-\cos^2 t}{\cos^2 t}} \cdot \frac{\sin t}{\cos t}\,dt$$

$$= 3\int \sqrt{\frac{\sin^2 t}{\cos^2 t}} \cdot \frac{\sin t}{\cos t}\,dt = 3\int |\mathrm{tg}\,t| \cdot \mathrm{tg}\,t\,dt$$

$$= -3\int \mathrm{tg}^2\,dt = -3\int \frac{1-\cos^2 t}{\cos^2 t}\,dt$$

$$= -3\int \frac{1}{\cos^2 t}\,dt + 3\int dt = -3\,\mathrm{tg}\,t + 3t + C$$

$$= -3\frac{\sqrt{1-\cos^2 t}}{\cos t} + 3t + C = -3\frac{\sqrt{1-\frac{9}{x^2}}}{\frac{3}{x}} + 3\arccos\frac{3}{x} + C$$

$$= -x\sqrt{\frac{x^2-9}{x^2}} + 3\arccos\frac{3}{x} + C = -x\frac{\sqrt{x^2-9}}{|x|} + 3\arccos\frac{3}{x} + C$$

$$= = -x\frac{\sqrt{x^2-9}}{-x} + 3\arccos\frac{3}{x} + C = \sqrt{x^2-9} + 3\arccos\frac{3}{x} + C.$$

II način: Neka je $x < -3$.

$$\begin{aligned}
\int \frac{\sqrt{x^2-9}}{x}\,dx &= \left| \begin{array}{c} x = -\frac{3}{\cos t},\ t \in (0, \frac{\pi}{2}) \\ \cos t = -\frac{3}{x} \Rightarrow t = \arccos\left(-\frac{3}{x}\right) \\ dx = -(-\frac{3}{\cos^2 t}) \cdot (-\sin t)dt = -\frac{3\sin t}{\cos^2 t}dt \end{array} \right| \\
&= \int \frac{\sqrt{\frac{9}{\cos^2 t} - 9}}{-\frac{3}{\cos t}} \cdot \left(-\frac{3\sin t}{\cos^2 t}\right) dt = 3\int \sqrt{\frac{1-\cos^2 t}{\cos^2 t}} \cdot \frac{\sin t}{\cos t}\,dt \\
&= 3\int \sqrt{\frac{\sin^2 t}{\cos^2 t}} \cdot \frac{\sin t}{\cos t}\,dt = 3\int |\mathrm{tg}\, t| \cdot \mathrm{tg}\, t\,dt \\
&= 3\int \mathrm{tg}^2\ dt = 3\int \frac{1-\cos^2 t}{\cos^2 t}\,dt \\
&= 3\int \frac{1}{\cos^2 t}\,dt - 3\int dt = 3\,\mathrm{tg}\, t - 3t + C' \\
&= 3\frac{\sqrt{1-\cos^2 t}}{\cos t} - 3t + C' = 3\frac{\sqrt{1-\frac{9}{x^2}}}{-\frac{3}{x}} - 3\arccos\left(-\frac{3}{x}\right) + C' \\
&= -x\sqrt{\frac{x^2-9}{x^2}} - 3\left(\pi - \arccos\frac{3}{x}\right) + C' = -x\frac{\sqrt{x^2-9}}{|x|} + 3\arccos\frac{3}{x} + C \\
&= -x\frac{\sqrt{x^2-9}}{-x} + 3\arccos\frac{3}{x} + C = \sqrt{x^2-9} + 3\arccos\frac{3}{x} + C.
\end{aligned}$$

Prema tome,

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \sqrt{x^2-9} - 3(\mathrm{sgn}\, x)\arccos\frac{3}{x} + C,\ \text{na intervalima } (-\infty,-3) \text{ i } (3,+\infty).$$

Primer 1.24. Integrali $\int \frac{1}{\sqrt{x^2+1}}\,dx$ i $\int \frac{1}{\sqrt{x^2-1}}\,dx$ su navedeni u tablici integrala i za razliku od ostalih tabličnih integrala ne proizilaze direktno iz tablice izvoda. Jedan od načina da se oni odrede je korišćenje trigonometrijskih ili hiperboličkih smena.

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+1}}\,dx &= \left| \begin{array}{c} x = \mathrm{sh}\, t,\ t \in (-\infty,+\infty) \Longrightarrow t = arshx \\ dx = \mathrm{ch}\, t\,dt \end{array} \right| \\
&= \int \frac{1}{\sqrt{\mathrm{sh}^2 t + 1}} \cdot \mathrm{ch}\, t\,dt = \int \frac{\mathrm{ch}\, t\,dt}{\sqrt{\mathrm{ch}^2 t}} = \int \frac{\mathrm{ch}\, t\,dt}{\mathrm{ch}\, t} \\
&= \int dt = t + C = arshx + C \\
&= \ln(x + \sqrt{x^2+1}) + C,\quad x \in \mathbb{R}.
\end{aligned}$$

Nađimo sada neodređeni integral funkcije $f(x) = \frac{1}{\sqrt{x^2-1}}$ i to najpre na intervalu $(1,+\infty)$, a potom i na intervalu $(-\infty,-1)$.

Neka je $x > 1$.

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2-1}}\,dx &= \left| \begin{array}{c} x = \frac{1}{\cos t},\ t \in (0, \frac{\pi}{2}) \\ \cos t = \frac{1}{x} \Rightarrow t = \arccos \frac{1}{x} \\ dx = -\frac{1}{\cos^2 t} \cdot (-\sin t) dt = \frac{\sin t}{\cos^2 t} dt \end{array} \right| \\
&= \int \frac{\frac{\sin t}{\cos^2 t}}{\sqrt{\frac{1}{\cos^2 t} - 1}}\,dt = \int \frac{\frac{\sin t}{\cos^2 t}}{\sqrt{\frac{\sin^2 t}{\cos^2 t}}}\,dt \\
&= \int \frac{\frac{\sin t}{\cos^2 t}}{\frac{\sin t}{\cos t}}\,dt = \int \frac{dt}{\cos t} = \ln\left|\frac{1+\operatorname{tg}\frac{t}{2}}{1-\operatorname{tg}\frac{t}{2}}\right| + C'' \\
&= \ln\left|\frac{1+\sqrt{\frac{1-\cos t}{1+\cos t}}}{1-\sqrt{\frac{1-\cos t}{1+\cos t}}}\right| + C'' = \ln\left|\frac{1+\sqrt{\frac{1-\frac{1}{x}}{1+\frac{1}{x}}}}{1-\sqrt{\frac{1-\frac{1}{x}}{1+\frac{1}{x}}}}\right| + C'' \\
&= \ln\left|\frac{1+\sqrt{\frac{x-1}{x+1}}}{1-\sqrt{\frac{x-1}{x+1}}}\right| + C'' = \ln\left|\frac{1+\frac{\sqrt{x-1}}{\sqrt{x+1}}}{1-\frac{\sqrt{x-1}}{\sqrt{x+1}}}\right| + C'' \\
&= \ln\left|\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}\right| + C'' = \ln\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} + C'' \\
&= \ln\left(\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} \cdot \frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\right) + C'' \\
&= \ln\frac{(\sqrt{x+1}+\sqrt{x-1})^2}{2} + C'' = \ln(\sqrt{x+1}+\sqrt{x-1})^2 - \ln 2 + C'' \\
&= \ln(\sqrt{x+1}+\sqrt{x-1})^2 + C' = \ln(x+1+2\sqrt{x^2-1}+x-1) + C' \\
&= \ln 2(x+\sqrt{x^2-1}) + C' = \ln(x+\sqrt{x^2-1}) + \ln 2 + C' \\
&= \ln(x+\sqrt{x^2-1}) + C.
\end{aligned}
$$

Neka je $x < -1$.

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-1}}\,dx &= \left| \begin{array}{c} x = -\frac{1}{\cos t},\ t \in (0, \frac{\pi}{2}) \\ \cos t = -\frac{1}{x} \Rightarrow t = \arccos\left(-\frac{1}{x}\right) \\ dx = -(-\frac{1}{\cos^2 t}) \cdot (-\sin t)dt = -\frac{\sin t}{\cos^2 t}dt \end{array} \right| \\
&= \int \frac{-\frac{\sin t}{\cos^2 t}}{\sqrt{\frac{1}{\cos^2 t} - 1}}\,dt = \int \frac{-\frac{\sin t}{\cos^2 t}}{\sqrt{\frac{\sin^2 t}{\cos^2 t}}}\,dt \\
&= \int \frac{-\frac{\sin t}{\cos^2 t}}{\frac{\sin t}{\cos t}}\,dt = -\int \frac{dt}{\cos t} = -\ln\left|\frac{1+\operatorname{tg}\frac{t}{2}}{1-\operatorname{tg}\frac{t}{2}}\right| + C'' \\
&= \ln\left|\frac{1-\operatorname{tg}\frac{t}{2}}{1+\operatorname{tg}\frac{t}{2}}\right| + C'' = \ln\left|\frac{1-\sqrt{\frac{1-\cos t}{1+\cos t}}}{1+\sqrt{\frac{1-\cos t}{1+\cos t}}}\right| + C'' \\
&= \ln\left|\frac{1-\sqrt{\frac{1-\frac{1}{x}}{1+\frac{1}{x}}}}{1+\sqrt{\frac{1-\frac{1}{x}}{1+\frac{1}{x}}}}\right| + C'' = \ln\left|\frac{1-\sqrt{\frac{x-1}{x+1}}}{1+\sqrt{\frac{x-1}{x+1}}}\right| + C'' \\
&= \ln\left|\frac{1-\frac{\sqrt{1-x}}{\sqrt{-x-1}}}{1+\frac{\sqrt{1-x}}{\sqrt{-x-1}}}\right| + C'' = \ln\left|\frac{\sqrt{-x-1}-\sqrt{1-x}}{\sqrt{-x-1}+\sqrt{1-x}}\right| + C'' \\
&= \ln\frac{\sqrt{1-x}-\sqrt{-x-1}}{\sqrt{1-x}+\sqrt{-x-1}} + C'' \\
&= \ln\left(\frac{\sqrt{1-x}-\sqrt{-x-1}}{\sqrt{1-x}+\sqrt{-x-1}} \cdot \frac{\sqrt{1-x}-\sqrt{-x-1}}{\sqrt{1-x}-\sqrt{-x-1}}\right) + C'' \\
&= \ln\frac{(\sqrt{1-x}-\sqrt{-x-1})^2}{2} + C'' \\
&= \ln(\sqrt{1-x}-\sqrt{-x-1})^2 - \ln 2 + C'' \\
&= \ln(1-x-2\sqrt{x^2-1}-x-1) + C' \\
&= \ln(2(-x-\sqrt{x^2-1})) + C' = \ln(-x-\sqrt{x^2-1}) + \ln 2 + C' \\
&= \ln|x+\sqrt{x^2-1}| + C.
\end{aligned}$$

Prema tome,

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}| + C,\ \text{ na intervalima } (-\infty,-1) \text{ i } (1,+\infty).$$

Glava 2

Određeni integral

2.1 Integrabilnost deo po deo neprekidnih funkcija

Lema 2.1. *Neka je su funkcije f i g definisane na $[a,b]$ i neka je $f(x) = g(x)$ za $x \in (a,b)$. Ako je funkcija f integrabilna na $[a,b]$, onda je i funkcija g integrabilna na $[a,b]$ i važi*

$$\int_a^b g(x)dx = \int_a^b f(x)dx.$$

Dokaz. Neka je funkcija f integrabilna na $[a,b]$ i neka je $I = \int_a^b f(x)dx$. Iz integrabilnosti funkcije f sledi da je ona ograničena na segmentu $[a,b]$, tj. postoji broj $M > 0$ tako da je $|f(x)| \leq M$ za sve $x \in [a,b]$. Neka je $K = \max\{M, |g(a)|, |g(b)|\}$. Neka je (P_n) niz podela segmenta $[a,b]$, $P_n = \{x_0^{(n)}, x_1^{(n)}, \ldots, x_{k_n}^{(n)}\}$, $n \in \mathbb{N}$, takav da je $\lim\limits_{n\to\infty} d(P_n) = 0$, i neka je $\xi^{(n)} = (\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{k_n}^{(n)})$ proizvoljan izbor tačaka $\xi_i^{(n)} \in [x_{i-1}^{(n)}, x_i^{(n)}]$, $i = 1, \ldots, k_n$, $n \in \mathbb{N}$. Tada je

$$\lim_{n\to\infty} \sigma(f, P_n, \xi^{(n)}) = \lim_{n\to\infty} \sum_{i=1}^{k_n} f(\xi_i^{(n)})(x_i^{(n)} - x_{i-1}^{(n)}) = \lim_{n\to\infty} \sum_{i=1}^{k_n} f(\xi_i^{(n)})\Delta x_i^{(n)} = I, \quad (2.1)$$

gde je $\Delta x_i^{(n)} = x_i^{(n)} - x_{i-1}^{(n)}$, $i = 1, 2, \ldots k_n$, $n \in \mathbb{N}$.

Kako je

$$\begin{aligned}
0 &\leq |\sigma(g, P_n, \xi^{(n)}) - I| \leq |\sigma(g, P_n, \xi^{(n)}) - \sigma(f, P_n, \xi^{(n)})| + |\sigma(f, P_n, \xi^{(n)}) - I| \\
&= |\sum_{i=1}^{k_n} g(\xi_i^{(n)})\Delta x_i^{(n)} - \sum_{i=1}^{k_n} f(\xi_i^{(n)})\Delta x_i^{(n)}| + |\sigma(f, P_n, \xi^{(n)}) - I| \\
&= |g(\xi_1^{(n)})\Delta x_1^{(n)} + g(\xi_{k_n}^{(n)})\Delta x_{k_n}^{(n)} - f(\xi_1^{(n)})\Delta x_1^{(n)} - f(\xi_{k_n}^{(n)})\Delta x_{k_n}^{(n)}| + |\sigma(f, P_n, \xi^{(n)}) - I| \\
&\leq |g(\xi_1^{(n)})\Delta x_1^{(n)}| + |g(\xi_{k_n}^{(n)})\Delta x_{k_n}^{(n)}| + |f(\xi_1^{(n)})\Delta x_1^{(n)}| + |f(\xi_{k_n}^{(n)})\Delta x_{k_n}^{(n)}| + |\sigma(f, P_n, \xi^{(n)}) - I|
\end{aligned}$$

i kako je

$$|g(\xi_1^{(n)})\Delta x_1^{(n)}| \le Kd(P_n), \quad |g(\xi_{k_n}^{(n)})\Delta x_{k_n}^{(n)}| \le Kd(P_n),$$
$$|f(\xi_1^{(n)})\Delta x_1^{(n)}| \le Kd(P_n), \quad |f(\xi_{k_n}^{(n)})\Delta x_{k_n}^{(n)}| \le Kd(P_n),$$

to je

$$0 \le |\sigma(g, P_n, \xi^{(n)}) - I| \le 4Kd(P_n) + |\sigma(f, P_n, \xi^{(n)}) - I|. \tag{2.2}$$

S obzirom da je $\lim_{n\to\infty} d(P_n) = 0$, iz (2.1) i (2.2) sledi

$$\lim_{n\to\infty} \sigma(f, P_n, \xi^{(n)}) = I,$$

što znači da je funkcija g integrabilna na segmentu $[a, b]$ i da je

$$\int_a^b g(x)dx = I = \int_a^b f(x)dx.\Box$$

Dokaz. (II način) Neka je funkcija f integrabilna na $[a, b]$ i neka je $I = \int_a^b f(x)dx$. Iz integrabilnosti funkcije f sledi da je ona ograničena na segmentu $[a, b]$, tj. postoji broj $M > 0$ tako da je $|f(x)| \le M$ za sve $x \in [a, b]$. Neka je $K = \max\{M, |g(a)|, |g(b)|\}$. Neka je $\epsilon > 0$ proizvoljno. Tada postoji $\delta^* > 0$ tako da za svaku podelu $P = \{x_0, x_1, \ldots, x_n\}$ segmenta $[a, b]$ takvu da je $d(P) < \delta^*$ i za svaki izbor $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ tačaka $\xi_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$ važi nejednakost

$$|\sigma(f, P, \xi) - I| < \frac{\epsilon}{2}. \tag{2.3}$$

Neka je $\delta = \min\left\{\delta^*, \dfrac{\epsilon}{8K}\right\}$ i neka je $P = \{x_0, x_1, \ldots, x_n\}$ podela segmenta $[a, b]$ takva da je $d(P) < \delta$ i neka je $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ proizvoljan izbor tačaka $\xi_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$. Tada je

$$\begin{aligned} |\sigma(g, P, \xi) - I| &\le |\sigma(g, P, \xi) - \sigma(f, P, \xi)| + |\sigma(f, P, \xi) - I| \\ &\le |\sum_{i=1}^n g(\xi_i)\Delta x_i - \sum_{i=1}^n f(\xi_i)\Delta x_i| + |\sigma(f, P, \xi) - I| \\ &\le |g(\xi_1)\Delta x_1 + g(\xi_n)\Delta x_n - f(\xi_1)\Delta x_1 - f(\xi_n)\Delta x_n| + |\sigma(f, P, \xi) - I| \end{aligned}$$

i kako je

$$|g(\xi_1)\Delta x_1| \le Kd(P) < K\delta, \quad |g(\xi_n)\Delta x_n| \le Kd(P) < K\delta,$$
$$|f(\xi_1)\Delta x_1| \le Kd(P) < K\delta, \quad |f(\xi_n)\Delta x_n| \le Kd(P) < K\delta,$$

to je, s obzirom na (2.3),

$$|\sigma(g, P, \xi) - I| < 4K\delta + |\sigma(f, P, \xi) - I| < 4K\frac{\epsilon}{8K} + \frac{\epsilon}{2} = \epsilon.$$

Ovo znači da je funkcija g integrabilna na segmentu $[a, b]$ i da je $\int_a^b g(x)dx = I = \int_a^b f(x)dx$. ■