Visualization of the spaces $W(u, v; \ell_p)$ and their duals

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Abstract. We consider the weighted means spaces $W(u, v; \ell_p)$ and their $\alpha$-, $\beta$- and $\gamma$-duals. The duality of the spaces is visualized in three dimensional real space by representing the norm as a potential surface and the dual norm as the corresponding Wulff’s crystal.

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INTRODUCTION

The weighted means spaces $W(u, v; \ell_p)$ and their $\alpha$-, $\beta$- and $\gamma$-duals were studied in [1]. It is known that those spaces are $BK$ spaces, that is, Banach sequence spaces with continuous coordinates (see [2] for $BF$ spaces). We use our software $MV$ Graphics to represent the norm as a potential surface and the $\beta$-dual norm as the corresponding Wulff’s crystal.

According to Wulff’s principle [3], the shape of a crystal is uniquely determined by its surface energy function. A surface energy function is a real valued function depending on a direction in space. These crystals are referred to as Wulff’s crystals.

Notations and Definitions

We denote by $\omega$, and $cs$ and $bs$ the sets of all complex sequences $x = (x_k)_{k=0}^\infty$, and of all convergent and bounded series, and write $\ell_p = \{ x \in \omega : \sum_{k=0}^\infty |x_k|^p < \infty \}$ for the set of all absolutely $p$–summable series. Also $e = (e_k)_{k=0}^\infty$ is the sequence with $e_k = 1$ for all $k$. We write $\Sigma = (\Sigma_{nk})_{n,k=0}^\infty$ for the infinite matrix with $\Sigma_{nk} = 1$ $(0 \leq k \leq n)$ and $\Sigma_{nk} = 0$ $(k > n)$ for $n = 0, 1, \ldots$

If $x$ and $y$ are sequences and $X$ and $Y$ are subsets of $\omega$, then we write $x \cdot y = (x_ky_k)_{k=0}^\infty$, $x^{-1} * Y = \{ a \in \omega : a \cdot x \in Y \}$ and $M(X, Y) = \bigcup_{x \in X} x^{-1} * Y = \{ a \in \omega : a \cdot x \in Y \text{ for all } x \in X \}$ for the multiplier space of $X$ and $Y$; in particular, we use the notations $x^\alpha = x^{-1} * cs$ and $X^\alpha = M(X, \ell_1)$, $X^\beta = M(X, cs)$ and $X^\gamma = M(X, bs)$ for the $\alpha$–, $\beta$– and $\gamma$-duals of $X$.

Given any infinite matrix $A = (a_{nk})_{n,k=0}^\infty$ of complex numbers and any sequence $x$, we write $A_n = (a_{nk})_{k=0}^\infty$ for the sequence in the $n$th row of $A$, $A_n x = \sum_{k=0}^\infty a_{nk} x_k$ $(n = 0, 1, \ldots)$ and $A x = (A_n x)_{n=0}^\infty$ for the $A$ transform of $x$, provided $A_n \in \ell^B$ for all $n$. If $X$ and $Y$ are subsets of $\omega$, then $X_A = \{ x \in \omega : A x \in X \}$ denotes the matrix domain of $A$ in $X$.

We write $\mathcal{W}$ for the set of all sequences $u$ with $u_k \neq 0$ for all $k$; if $u \in \mathcal{W}$ then $1/u = (1/u_k)_{k=0}^\infty$. Let $n+1 = (n+1)_{n=0}^\infty$. The matrices $\Delta = (\Delta_{nk})_{n,k=0}^\infty$ and $\Delta^+ = (\Delta^+_{nk})_{n,k=0}^\infty$ are defined by $\Delta_{mk} = \Delta^+_{mk} = 1$, $\Delta_{n+1,n} = \Delta_{n+1,n}^+ = 1$, $\Delta_{nk} = \Delta_{nk}^+ = 0$ (otherwise) for all $n, k = 0, 1, \ldots$
THE SPACES $W(u,v;\ell_p)$ AND THEIR DUALS

We consider the sequence spaces $W(u,v;\ell_p) = v^{-1} \ast (u^{-1} \ast \ell_p)_\Sigma$ for sequences $u,v \in U$ and $(1 \leq p < \infty)$ and their duals. If we write

$$ W(u,v) = \begin{pmatrix} u_0 v_0 & 0 & \cdots & 0 & 0 & \cdots \\ u_1 v_0 & u_1 v_1 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_n v_0 & \cdots & \cdots & u_n v_n & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} $$

then $W(u,v;\ell_p) = (l_p)_W(u,v)$.

The $\alpha$-, $\beta$- and $\gamma$-duals of the spaces $W(u,v;\ell_p)$ are given in the results which we state here for the reader’s convenience.

Theorem 1. [1, Theorem 3.1] Let $u,v \in U$. We write $b = (1/u) \cdot \Delta^+(a/v)$ for $a \in \omega, q$ for the conjugate number of $p$, that is, $q = \infty$ for $p = 1$ and $q = p/(p-1)$ for $1 < p < \infty$, and $S_q(u,v) = \{a \in \omega : b \in \ell_q\}$. Then we have

(a) $(W(u,v;\ell_p))^\alpha = S_q(u,v) \cap ((1/(u \cdot v))^{-1} \ast \ell_q)$;
(b) $(W(u,v;\ell_p))^\beta = (W(u,v;\ell_p))^\gamma = S_q(u,v) \cap ((1/(u \cdot v))^{-1} \ast \ell_q)$.

VISUALIZATION OF NORMS AS POTENTIAL SURFACES AND DUAL NORMS AS WULFF’S CRYSTAL

Let $\partial B^2$ denote the unit sphere in Euclidean $\mathbb{R}^3$ space, that is,

$$ \partial B^2 = \{ \bar{x} = (x_1,x_2,x_3) \in \mathbb{R}^3 : \|\bar{x}\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2} = 1 \}, $$

and let $F : \partial B^2 \to \mathbb{R}$ be a function. Then, we may consider the set

$$ PM = \{ \bar{x} = F(\bar{e}) \bar{e} \in \mathbb{R}^3 : \bar{e} \in \partial B^2 \} $$

as a natural representation of $F$.

In particular, for

$$ \bar{e} = \bar{e}(u^1,u^2) = (\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1), $$

where $(u^1,u^2) \in R = (-\pi/2,\pi/2) \times (0,2\pi)$, we obtain a potential surface with a parametric representation

$$ PS = \{ \bar{x} = f(u^1,u^2)(\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1) : (u^1,u^2) \in R \}, $$

where $f(u^1,u^2) = F(\bar{e}(u^1,u^2))$.

We can visualize a norm $\| \cdot \|$ in three-dimensional space as a potential surface by putting $F(\bar{x}) = F(x_1,x_2,x_3) = \|\bar{x}\|$.

We can visualize the $\beta$–dual norm by the shape of Wulff’s crystal corresponding to the norm $F$ by using the following result the general case of which can be found in [4, Corollary 5.5].

Theorem 2. Let $\| \cdot \|$ be a norm on $\mathbb{R}^3$ and $\phi_{\| \cdot \|} : \mathbb{R}^3 \to \mathbb{R}$ be defined by $\phi_{\| \cdot \|}(x) = \bar{w} \cdot \bar{x} = w_1 x_1 + w_2 x_2 + w_3 x_3$, for each $\bar{w} \in \partial B^2$ and $\bar{x} \in \mathbb{R}^3$. Then, the boundary $\partial C_{\| \cdot \|}$ of Wulff’s crystal corresponding to $\| \cdot \|$ is given by

$$ \partial C_{\| \cdot \|} = \left\{ \bar{x} = \frac{1}{\|\phi_{\| \cdot \|}\|} \cdot \bar{e} \in \mathbb{R}^3 : \bar{e} \in \partial B^2 \right\}, \quad (1) $$

where $\|\phi_{\| \cdot \|}\|^* \cdot \bar{e}$ is the norm of the functional $\phi_{\| \cdot \|}$, that is, the dual norm of $\| \cdot \|$. 

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Dual spaces can be applied in crystalography [4–6].

Now we compute the $W(u, v; \ell_p)$ norms and their dual norms of vectors in $\mathbb{R}^3$. We introduce the norm of the $W(u, v; \ell_p)$ spaces on $\mathbb{R}^3$ in a natural way as

$$\| (x, y, z) \|_{W(u, v; \ell_p)} = \left( |u_0 v_0 x|^p + |u_1 (v_0 x + v_1 y)|^p + |u_2 (v_0 x + v_1 y + v_2 z)|^p \right)^{1/p}$$

for all $(x, y, z) \in \mathbb{R}^3$ and their $\beta$–duals for $p = 1$

$$\| (a, b, c) \|_{W(u, v; \ell_1)}^\beta = \max \left\{ \left| \frac{1}{u_0} \left( \frac{a}{v_0} - \frac{b}{v_1} \right) \right|, \left| \frac{1}{u_1} \left( \frac{b}{v_1} - \frac{c}{v_2} \right) \right|, \left| \frac{1}{u_2} \right| \right\}$$

and for $p > 1$ and $q = p/(p - 1)$

$$\| (a, b, c) \|_{W(u, v; \ell_p)}^\beta = \left( \left| \frac{1}{u_0} \left( \frac{a}{v_0} - \frac{b}{v_1} \right) \right|^q + \left| \frac{1}{u_1} \left( \frac{b}{v_1} - \frac{c}{v_2} \right) \right|^q + \left| \frac{1}{u_2} \right|^q \right)^{1/q}$$

for all $(a, b, c) \in \mathbb{R}^3$.

For example see Figures 1 to 4.
FIGURE 3. Potential surface given by the norm of $W(u, v; \ell_p)$ for $p = 1.8$, $u = (3, 5, 3)$ and $v = (6, 2, 4)$ and corresponding Wulff’s crystal.

FIGURE 4. Potential surface given by the norm of $W(u, v; \ell_p)$ for $p = 5$, $u = (-2, 1, 2)$ and $v = (3, -2, -3)$ and corresponding Wulff’s crystal.

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REFERENCES