POLYNOMIALLY MEROMORPHIC OPERATORS

B.P. Duggal *
8 Redwood Grove, Northfield Avenue, Ealing, London W5 4SZ, United Kingdom

Dragan S. Djordjević
University of Niš, Faculty of Sciences and Mathematics, P.O.Box 224, 1800 Nis,
Serbia

Robin E. Harte
Trinity College, Dublin, Ireland

and

Snežana Č. Živković-Zlatanović
University of Niš, Faculty of Sciences and Mathematics, P.O.Box 224, 1800 Nis,
Serbia

[Accepted 13 July 2016. Published XX August 2016.]

Abstract

Given a Banach space operator $T \in B(\mathcal{X})$, the following statements are equivalent: (i) T is polynomially meromorphic; (ii) T is generalised meromorphic; (iii) $f(T)$ is meromorphic for some function f analytic on, and non-constant on the connected components of, an open neighbourhood of $\sigma(T)$; (iv) there exists a finite sequence of scalars $\{\mu_i\}_{i=1}^n$ and a decomposition $\mathcal{X} = \bigoplus_{i=1}^n \mathcal{X}_i$ such that $T = \bigoplus_{i=1}^n T|_{\mathcal{X}_i} = \bigoplus_{i=1}^n T_i$, where $\mu_i I_i - T_i$ is meromorphic for all $1 \le i \le n$. An operator $T \in B(\mathcal{X})$ with countable spectrum (and at best a single point of accumulation) such that every part, and the inverse of every invertible part, of T is normaloid is a translate of a meromorphic operator.

1. Introduction

A Banach space operator (that is, a bounded linear transformation) $T \in B(\mathcal{X})$ is *a meromorphic operator* if its non-zero spectral points are poles of the resolvent (of T). Compact operators, more generally Riesz operators, are meromorphic; algebraic operators, although not Riesz, are meromorphic operators. The classes consisting of Riesz operators and meromorphic operators, along with sharing some common properties, have some elemental differences. Thus, an operator $T \in B(\mathcal{X})$ in either of

*Corresponding author, e-mail: bpduggal@yahoo.co.uk
AMS Classification 2010: 47A53, 47A55
doi: http://dx.doi.org/10.3318/PRIA.2016.116.07
Cite as follows: B.P. Duggal, Dragan S. Djordjević, Robin E. Harte and Snežana Č. Živković-Zlatanović, Polynomially meromorphic operators, *Mathematical Proceedings of the Royal Irish Academy* **116**A (2016), 71–86; http://dx.doi.org/10.3318/PRIA.2016.116.07

these classes has a countable spectrum, and both T and T^* have the single-valued extension property. Whereas the (Fredholm) essential spectrum (or the Browder spectrum or the Weyl spectrum) of a Riesz operator $T \in B(\mathcal{X})$ consists at best of the singleton set $\{0\}$, however, it is possibly the whole of the spectrum of T in the case in which T is meromorphic.

$T \in B(\mathcal{X})$ is *polynomially meromorphic* (resp., compact, Riesz) if there exists a non-trivial polynomial $p(.)$ such that $p(T)$ is meromorphic (resp., compact, Riesz). The structure of polynomially compact and polynomially Riesz operators has been considered by a number of authors, amongst them Gilfeather [13], Kaashoek and Smyth [18], Han *et al.* [15], Jeribi and Moalla [17] and Živkovic-Zlatanović *et al.* [22]. In this article, we carry out a study (similar in spirit to the one carried out in [18]) on the structure of holomorphically meromorphic operators to prove amongst other results that $f(T)$, $T \in B(\mathcal{X})$, is meromorphic for some $f \in \mathrm{Holo}_c(\sigma(T))$, if and only if there exists a finite subset $\{\mu_1, ..., \mu_n\}$ of complex numbers such that $f(\mu_i) = 0$ for all $1 \leq i \leq n$, a decomposition $\mathcal{X} = \bigoplus_{i=1}^n \mathcal{X}_i$ of $\mathcal{X}$ into a direct sum of closed A invariant subspaces $\mathcal{X}_i$ and a decomposition $T = \bigoplus_{i=1}^n T|_{\mathcal{X}_i} = \bigoplus_{i=1}^n T_i$ such that $\mu_i I_i - T_i$ is meromorphic for all $1 \leq i \leq n$. Here I_i denote the identity of $B(\mathcal{X}_i)$ and $\mathrm{Holo}_c(\sigma(A))$ denotes the set of functions f that are analytic on, and non-constant on each of the connected components of, an open neighbourhood of the spectrum $\sigma(T)$ of T. It is seen that $T \in B(\mathcal{X})$ is polynomially meromorphic if and only if $f(T)$ is meromorphic for some $f \in \mathrm{Holo}_c(\sigma(A))$.

2. Some terminology and notation

A Banach space operator $T \in B(\mathcal{X})$ is *polaroid* if the isolated points of the spectrum $\sigma(T)$ of T, points $\lambda \in \mathrm{iso}\sigma(T)$, are poles of the resolvent of T. Let $\Pi(T)$ denote the set of poles of the resolvent of T. A necessary and sufficient condition for $\lambda \in \Pi(T)$ is that $\mathrm{asc}(\lambda I - T) = \mathrm{dsc}(\lambda I - T) < \infty$, where the *ascent of* T, $\mathrm{asc}(T)$ (resp. *descent of* T, $\mathrm{dsc}(T)$), is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ (resp., $T^n\mathcal{X} = T^{n+1}\mathcal{X}$). (If no such integer exists, then $\mathrm{asc}(T)$, resp. $\mathrm{dsc}(T)$, $= \infty$.)

An operator $T \in B(\mathcal{X})$ is *upper semi-Fredholm* (resp., *lower semi-Fredholm*) if $T\mathcal{X}$ is closed, and the deficiency index $\alpha(T) = \dim T^{-1}(0) < \infty$ (resp., $\beta(T) = \dim \mathcal{X}/T\mathcal{X} < \infty$); T is semi-Fredholm if it is either upper or lower semi-Fredholm, and the (semi-Fredholm) index of T is then the integer $\mathrm{ind}(T) = \alpha(T) - \beta(T)$. In the following, we shall denote the set of semi-Fredholm points of an operator T by $\Phi_{SF}(T)$, the set of *upper semi-Fredholm* (resp., *lower semi-Fredhom*) operators by $\Phi_{SF_+}(\mathcal{X})$ (resp., $\Phi_{SF_-}(\mathcal{X})$), and the semi-group of semi-Fredholm operators by $\Phi_{SF}(\mathcal{X})$. Let $\Phi_{SF_+^-}(\mathcal{X}) = \{T \in \Phi_{SF_+}(\mathcal{X}) : \mathrm{ind}(T) \leq 0\}$ (resp., $\Phi_{SF_-^+}(\mathcal{X}) = \{T \in \Phi_{SF_-}(\mathcal{X}) : \mathrm{ind}(T) \geq 0\}$). T is *Fredholm*, $T \in \Phi_F(\mathcal{X})$, if $T \in \Phi_+(\mathcal{X}) \cap \Phi_-(\mathcal{X})$, and T is *Weyl* (resp., *Browder*) if it is Fredholm of index 0 (resp., Fredholm of finite ascent and descent). The *Weyl spectrum* $\sigma_W(T)$ (resp., *Browder spectrum* $\sigma_B(T)$) of T is the set of complex numbers λ, $\lambda \in \mathbf{C}$, such that $\lambda I - T$ is not Weyl (resp., $\lambda \in \mathbf{C}$ such that $\lambda I - T$ is not Browder).

Berkani [3] has called an operator $T \in B(\mathcal{X})$ *a B-Fredholm operator*, $T \in \Phi_{BF}(\mathcal{X})$, if there exists a natural number n, $n \in \mathbf{N}$, for which $T^n(\mathcal{X})$ is closed and the induced operator $T_n : T^n(\mathcal{X}) \longrightarrow T^n(\mathcal{X})$ is Fredholm in the usual sense, and *a B-Weyl operator*, $T \in \Phi_{BW}(\mathcal{X})$, if, in addition, T_n has index 0. An operator T is *upper semi B-Fredholm* (resp., *lower semi B-Fredholm*), $T \in \Phi_{UBF}(\mathcal{X})$ (resp., $T \in \Phi_{LBF}(\mathcal{X})$), if $T^n(\mathcal{X})$ is closed for some $n \in \mathbf{N}$ and the induced operator T_n is *upper semi-Fredholm* (resp., *lower semi-Fredholm*) in the usual sense [4]; T is *semi B-Fredholm*, $T \in \Phi_{SBF}(\mathcal{X})$, if $T \in \Phi_{UBF}(\mathcal{X})$ or $T \in \Phi_{LBF}(\mathcal{X})$, and T is *B-Fredholm*, $T \in \Phi_{BF}(\mathcal{X})$, if $T \in \Phi_{UBF}(\mathcal{X}) \cap \Phi_{LBF}(\mathcal{X})$. Let $\sigma_{UBF}(T) = \{\lambda : \lambda I - T \notin \Phi_{UBF}(\mathcal{X})\}$ and $\sigma_{LBF}(T) = \{\lambda : \lambda I - T \notin \Phi_{LBF}(\mathcal{X})\}$; then the *B-Fredholm spectrum of* T is the set $\sigma_{BF}(T) = \sigma_{UBF}(T) \cup \sigma_{LBF}(T)$. For a $T \in \Phi_{SBF}(\mathcal{X})$, the index of T is defined by $\mathrm{ind}(T) = \mathrm{ind}(T_d)$, where $d \in \mathbf{N}$ is *the degree of stable iteration* of T (see [4, definition 2.2]). Let

$$\Phi_{SBF_+^-}(T) = \{T \in \Phi_{SBF}(\mathcal{X}) : T \text{ is upper B-Fredholm with } \mathrm{ind}(T) \leq 0\},$$

and let

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbf{C} : \lambda I - T \notin \Phi_{SBF_+^-}(\mathcal{X})\}.$$

We say that a point $\lambda \in \sigma_a(T)$ is a *left pole* (resp., *left pole of finite rank*) of T, denoted $\lambda \in \Pi^a(T)$ (resp., $\lambda \in \Pi_0^a(T)$), if $\lambda I - T \in LD(\mathcal{X})$ (resp., $\lambda I - T \in LD(X)$ and $\alpha(\lambda I - T)) < \infty$), where $LD(\mathcal{X})$ is the *regularity*

$$LD(\mathcal{X}) = \{T \in B(\mathcal{X}) : d = \mathrm{asc}(T) < \infty \text{ and } T^{d+1}(\mathcal{X}) \text{ is closed}\}.$$

The (left Drazin) spectrum induced by the regularity LD will be denoted by $\sigma_{LD}(.)$. Evidently, $\Pi^a(T) = \{\lambda \in \sigma_a(T) : \mathrm{asc}(\lambda I - T) = d < \infty, T^{d+1}\mathcal{X} \text{ is closed}\}$.

An operator $T \in B(\mathcal{X})$ has SVEP (= *the single-valued extension property*) at a point $\lambda_0 \in \mathbf{C}$ if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \longrightarrow \mathcal{X}$ satisfying $(\lambda I - T)f(\lambda) = 0$ is the function $f \equiv 0$. Evidently, every T has SVEP at points in the resolvent $\rho(T) = \mathbf{C} \setminus \sigma(T)$ and the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. *We say that T has SVEP if it has SVEP at every $\lambda \in \mathbf{C}$.* The *quasinilpotent part* $H_0(\lambda I - T)$ and the *analytic core* $K(\lambda I - T)$ of $(\lambda I - T)$ are defined by

$$H_0(\lambda I - T) = \{x \in \mathcal{X} : \lim_{n \longrightarrow \infty} ||(\lambda I - T)^n x||^{\frac{1}{n}} = 0\}$$

and

$$\begin{aligned} K(\lambda I - T) = \{x \in \mathcal{X} : &\text{ there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \\ &\text{for which } x = x_0, (\lambda I - T)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, ...\}. \end{aligned}$$

We note that $H_0(\lambda I - T)$ and $K(\lambda I - T)$ are (generally) non-closed hyperinvariant subspaces of $(\lambda I - T)$ such that $(\lambda I - T)^{-p}(0) \subseteq H_0(\lambda I - T)$ for all $p = 0, 1, 2, ...$ and $(\lambda I - T)K(\lambda I - T) = K(\lambda I - T)$ [1].

Lemma 2.1. *If $\lambda \in iso\sigma(T)$ and $H_0(\lambda I - T) = (\lambda I - T)^{-p}(0)$ for some integer $p \geq 1$, then λ is a pole (of the resolvent) of T.*

Proof. If $\lambda \in \mathrm{iso}\sigma(T)$ and $H_0(\lambda I - T) = (\lambda I - T)^{-p}(0)$, then

$$\begin{aligned} \mathcal{X} &= H_0(\lambda I - T) \oplus K(\lambda I - T) = (\lambda I - T)^{-p}(0) \oplus K(\lambda I - T) \\ \Longrightarrow (\lambda I - T)^p(\mathcal{X}) &= 0 \oplus K(\lambda I - T) \Longrightarrow \mathcal{X} = (\lambda I - T)^{-p}(0) \oplus (\lambda I - T)^p(\mathcal{X}), \end{aligned}$$

that is, λ is a pole (of order p) of T. ■

We say in the following that T *is polar at* λ if $\lambda \in \mathrm{iso}\sigma(T)$ is a pole of T. The operator T *is polaroid* on a subset F of $\mathbf{C}$ if T is polar at every point of F, and T is polaroid if it is polar at every $\lambda \in \mathrm{iso}\sigma(T)$. Observe that:

Lemma 2.2. *$T \in B(\mathcal{X})$ is polaroid if and only if the dual operator T^* is polaroid.*

Proof. A point $\lambda \in \mathrm{iso}\sigma(T)$ is a pole of T if and only if there exists an integer $p > 0$ such that

$$\mathcal{X} = (\lambda I - T)^{-p}(0) \oplus (\lambda I - T)^p(\mathcal{X}) \Longleftrightarrow \mathcal{X}^* = (\lambda I^* - T^*)^p(\mathcal{X}^*) \oplus (\lambda I^* - T^*)^{-p}(0),$$

since $(\lambda I - T)^{-p}(0)$ and $(\lambda I - T)^p(\mathcal{X})$ are closed. ■

Let $RD(\mathcal{X})$ denote the regularity $\{T \in B(\mathcal{X}) : \mathrm{dsc}(T) = d < \infty, T^d\mathcal{X}$ is closed $\}$, $\Pi^s(T) = \{\lambda \in \sigma_s(T) : \lambda I - T \in RD(\mathcal{X})\}$ the set of *right poles* of T, $\Phi_{SBF_-^+}(\mathcal{X})$ the set of $T \in \Phi_{SBF}(\mathcal{X})$ that are lower semi B-Fredholm with $\mathrm{ind}(T) \geq 0$, and let $\sigma_{SBF_-^+}(T) = \{\lambda : \lambda I - T \notin \Phi_{SBF_-^+}(\mathcal{X})\}$. Then, this follows from a straightforward argument, $\Pi(T) = \Pi^a(T) \cap \Pi^s(T)$ and $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T) \cup \sigma_{SBF_-^+}(T) = \sigma_{SBF_-^+}(T^*) \cup \sigma_{SBF_+^-}(T^*) = \sigma_{BW}(T^*)$. The regularities $LD(\mathcal{X})$ and $RD(\mathcal{X})$ give rise to the *left Drazin spectrum* $\sigma_{LD}(T) = \{\lambda : \lambda I - T \notin LD(\mathcal{X})\}$ and the *right Drazin spectrum* $\sigma_{RD}(T) = \{\lambda : \lambda I - T \notin RD(\mathcal{X})\}$. The *Drazin spectrum* $\sigma_D(T) = \{\lambda : \lambda I - T$ is not Drazin invertible$\}$ is then $\sigma_D(T) = \sigma_{LD}(T) \cup \sigma_{RD}(T)$. It is easily verified that $\sigma_{BF}(T) \subseteq \sigma_{BW}(T) \subseteq \sigma_{BB}(T) = \sigma_D(T)$, where $\sigma_{BB}(T) = \{\lambda : \lambda \in \sigma_{BF}(T)$ or $\mathrm{asc}(\lambda I - T) \neq \mathrm{dsc}(\lambda I - T)\}$ is the B-Browder spectrum of T. It is clear that an operator T is Drazin invertible if and only if both $\mathrm{asc}(T)$ and $\mathrm{dsc}(T)$ are finite; also, if $\lambda \in \Pi(T)$, then $\lambda I - T$ is Drazin invertible, and hence B-Fredholm.

3. Some complementary results

$T \in B(\mathcal{X})$ is said to have *uniform descent for* $n \geq d \in \boldsymbol{N}$, if $R(T) + T^{-n}(0) = R(T) + T^{-d}(0)$ for all $n \geq d$. If, in addition, $R(T) + T^{-d}(0)$ is closed, then T is said to have *topological uniform descent* for $n \geq d$. Evidently, if either of the deficiency indices $\alpha(T)$ and $\beta(T)$ or the chain lengths $\mathrm{asc}(T)$ and $\mathrm{dsc}(T)$ is finite, then T has uniform descent [14].

Some of the following lemmas are well known.

Lemma 3.1. *[10, lemma 3.1] If $\lambda \in \Pi^a(T)$, then $\lambda I - T$ is of topological uniform descent, $\lambda \in iso\sigma_a(T)$ and $\lambda \notin \sigma_{SBF_+^-}(T)$.*

Using Banach space duality, Lemma 3.1 implies the following.

Corollary 3.1. *If $\lambda \in \Pi^s(T)$, then $\lambda \in iso\sigma_s(T)$ and $\lambda \notin \sigma_{SBF_-^+}(T)$.*

Let $\mathrm{Holo}(\sigma(T))$ denote the set of functions f that are analytic on an open neighbourhood of $\sigma(T)$, and $\mathrm{Holo}_c(\sigma(T)) = \{f \in \mathrm{Holo}(\sigma(T)) : f$ is non-constant on each of the connected components of the set on which it is defined$\}$. Recall from [3, theorem 3.4] that $f(\sigma_{BF}(T)) = \sigma_{BF}(f(T))$ for all $f \in \mathrm{Holo}(\sigma(T))$.

Lemma 3.2. *If T has SVEP at points $\lambda \notin \sigma_{BF}(T)$, then $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ for all $f \in Holo(\sigma(T))$.*

Proof. Evidently, $\sigma_D(T) \supseteq \sigma_{BW}(T)$. Let $\lambda \notin \sigma_{BW}(T)$. Then there exists a natural number n such that $(\lambda I - T)_n$ is B-Fredholm and $\mathrm{ind}(\lambda I - T)_n = 0$. Since $\sigma_{BF}(T) \subseteq \sigma_{BW}(T)$, T has SVEP at λ implies $\mathrm{asc}((\lambda I - T)_n) = \mathrm{dsc}((\lambda I - T)_n) < \infty$ [1, theorem 3.4]. Hence, $\lambda \notin \sigma_D(T)$, and we conclude that $(\sigma_{BW}(T) = \sigma_D(T)$ and by the spectral mapping theorem for $\sigma_D(T)$ that) $f(\sigma_{BW}(T)) = f(\sigma_D(T)) = \sigma_D(f(T)) \supseteq \sigma_{BW}(f(T))$ for every $f \in \mathrm{Holo}(\sigma(T))$. Suppose now that $\mu \notin \sigma_{BW}(f(T))$. Then $\mu I - f(T)$ is B-Fredholm and $\mathrm{ind}(\mu I - f(T)) = 0$. Since $f(\sigma_{BF}(T)) = \sigma_{BF}(f(T))$ for all $f \in \mathrm{Holo}(\sigma(T))$, there exists a $\nu \notin \sigma_{BW}(T)$ such that $\mu = f(\nu)$. Since T has SVEP at ν, $f(T)$ has SVEP at μ, and hence $\mu \notin \sigma_D(f(T)) = f(\sigma_{BW}(T))$. Thus, $f(\sigma_{BW}(T)) \subseteq \sigma_{BW}(f(T))$, and the proof is complete. ■

It is known that the left Drazin spectrum $\sigma_{LD}(.)$ and the right Drazin spectrum $\sigma_{RD}(.)$ induced by the regularities $LD(\mathcal{X})$ and $RD(\mathcal{X})$ satisfy the spectral mapping theorem for all analytic functions that are locally non-constant [21]. The following lemma is proved in [5, theorem 2.5].

Lemma 3.3. *If T is an operator of topological uniform descent, then T has SVEP at $0 \Longleftrightarrow asc(T) < \infty$.*

Dually, Lemma 3.3 implies that if T is an operator of topological uniform descent, then T^* has SVEP at $0 \Longleftrightarrow \mathrm{dsc}(T) < \infty$. Thus, if T has SVEP at points $\lambda \notin \sigma_{SBF_+}(T)$ (resp., T^* has SVEP at points $\lambda \notin \sigma_{SBF_-}(T)$), then $\sigma_{SBF_+}(T) = \sigma_{LD}(T)$ (resp., $\sigma_{SBF_-}(T) = \sigma_{RD}(T)$). Observe that if T^* has SVEP (resp., T has SVEP), then $\sigma_{LD}(T) = \sigma_D(T)$ (resp., $\sigma_{RD}(T) = \sigma_D(T)$). This is encapsulated in the following lemma.

Lemma 3.4. *[10, lemma 3.5] If T has SVEP at points $\lambda \notin \sigma_{SBF_-^+}(T)$ (resp., T^**

has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T)$), then $\sigma_{SBF_-^+}(T) = \sigma_{BW}(T) = \sigma_D(T)$ (resp., $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T) = \sigma_D(T)$).

Lemma 3.5. *[10, lemma 3.6] If T has SVEP at points $\lambda \notin \sigma_{SBF_-^+}(T)$ (resp., T^* has SVEP at points $\lambda \notin \sigma_{SBF_+^-}(T)$), then $f(\sigma_{SBF_-^+}(T)) = \sigma_{SBF_-^+}(f(T))$ (resp., $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$ for every $f \in Holo_c(\sigma(T))$.*

4. Meromorphic operators

An operator $T \in B(\mathcal{X})$ is *meromorphic*, $T \in (\mathcal{M})$, if all its non-zero spectral points are poles. It is clear that a meromorphic operator possesses at most countably many spectral points. Hence, *T meromorphic implies both T and T^* have SVEP*. Compact, more generally Riesz, operators are meromorphic. (Recall that Riesz operators are meromorphic operators all of whose translates by scalars, except for the translate by 0, are Fredholm operators; equivalently, Riesz operators are meromorphic operators such that the points of the spectrum of the operator, except for the point 0, are finite rank poles of the operator.) Some of the following properties of meromorphic operators are likely well known; we sketch a proof for the reader's convenience.

($P0$). In contrast to Riesz operators, the sum of a pair of commuting meromorphic operators may not be a meromorphic operator. Example: If we let $Q \in B(\mathcal{X})$ denote a quasinilpotent operator, then Q is meromorphic, the identity operator $I \in B(\mathcal{X})$ is meromorphic and commutes with Q, but $I + Q$ is not meromorphic. (This example shows also that the property of being quasinilpotent equivalent [20, p. 253] does not preserve the meromorphic property of an operator.) Again, the product of two commuting operators, one of which is meromorphic, may not be meromorphic. Example: The operators I and $I + Q$ above commute, I is meromorphic and $I + Q = I(I + Q)$ is not meromorphic. However: For $\lambda \in \mathbf{C}$ and natural numbers n,

$$(\lambda I - T)^n \in (\mathcal{M}) \Longleftrightarrow \lambda I - T \in (\mathcal{M}).$$

Proof. The Drazin spectrum being a regularity [21], $\sigma_D(f(A)) = f(\sigma_D(A))$ for all operators A and $f \in \mathrm{Holo}_c(A)$. The proof now follows since $\lambda I - T \in (\mathcal{M})$ if and only if $\sigma_D(\lambda I - T) \subseteq \{0\} \Longleftrightarrow \sigma_D((\lambda I - T)^n) \subseteq \{0\}$. ■

A *part of an operator* is its restriction to a closed invariant subspace. We say that $T \in B(\mathcal{X})$ *is polar* if it is polar at 0 (that is, if 0 is a pole of T).

($P1$). *A part of a meromorphic operator is again meromorphic. If a meromorphic operator T is such that T is polar, then ($\sigma(T)$ consists of a finite number of poles, and hence) T is algebraic.*

Proof. That the restriction of a meromorphic operator to an invariant subspace is again meromorphic is proved in [8, proposition 2.10]. If 0 is a pole of a meromorphic operator, then it has a finite spectrum consisting of poles of the resolvent. Hence, it is algebraic [1, theorem 3.83]. ■

($P2$). *$T \in (\mathcal{M})$ if and only if (the dual operator) $T^* \in (\mathcal{M})$.*

PROOF. Evident, since $\sigma(T) = \sigma(T^*)$ and T is polaroid if and only if T^* is polaroid. ■

($P3$). *Given $A, B \in B(\mathcal{X})$, $AB \in (\mathcal{M}) \Longleftrightarrow BA \in (\mathcal{M})$.*

PROOF. Follows since $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$, $\mathrm{asc}(\lambda I - AB) = \mathrm{asc}(\lambda I - BA)$ and $\mathrm{dsc}(\lambda I - AB) = \mathrm{dsc}(\lambda I - BA)$ for all non-zero λ [6]. ■

($P4$). *$T = \bigoplus_{i=1}^{n} T_i \in (\mathcal{M})$ if and only if $T_i \in (\mathcal{M})$ for all $1 \le i \le n$.*

PROOF. In this case, $\sigma(T) = \bigcup_{i=1}^{n} \sigma(T_i)$; since

$$\mathrm{asc}(\lambda I_i - T_i) \le \mathrm{asc}(\lambda I_i - T_i) \le \sum_{i=1}^{n} \mathrm{asc}(\lambda I_i - T_i) \text{ and}$$
$$\mathrm{dsc}(\lambda I_i - T_i) \le \mathrm{dsc}(\lambda I - T) \le \sum_{i=1}^{n} \mathrm{dsc}(\lambda I_i - T_i)$$

for all $1 \le i \le n$ and complex λ [23, exercise 7, p. 293], the proof follows. ■

($P5$). *$A, AB \in (\mathcal{M})$ does not imply $B \in (\mathcal{M})$.*

PROOF. To see this, take A to be the 0 operator. Then $AB \in (\mathcal{M})$ for every $B \in B(\mathcal{X})$. ■

Given Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{X} \overline{\otimes} \mathcal{Y}$ denote the completion, endowed with a reasonable cross norm, of the algebraic tensor product of $\mathcal{X}$ and $\mathcal{Y}$, and let $A \otimes B \in B(\mathcal{X} \overline{\otimes} \mathcal{Y})$ denote the tensor product of $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$. Let $L_A R_B \in B(B(\mathcal{Y}, \mathcal{X}))$ denote the left-right multiplication operator $L_A R_B(X) = AXB$ defined by $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$.

($P6$). *$A, B \in (\mathcal{M})$ implies $A \otimes B \in (\mathcal{M})$ and $L_A R_B \in (\mathcal{M})$. The converse fails.*

PROOF. The hypothesis $A, B \in (\mathcal{M})$ implies that A, B are polaroid operators such that $\sigma(A \otimes B)$ and $\sigma(L_A R_B)$ are countable sets with 0 as their only possible limit point. Since the polaroid property transfers from A, B to $A \otimes B$ and to $L_A R_B$ [11], the proof follows. To see that the converse fails, consider the example of the operators A and B such that A is quasinilpotent, the set iso$\sigma(B)$ properly contains $\{0\}$ and B is not polaroid, when it follows that A, $A\otimes B$ and $L_A R_B$ are meromorphic but B is not. ■

(*P*7). *If* $f(T) \in (\mathcal{M})$, $f \in Holo_c(\sigma(T))$, *then* $\sigma_D(T) = \sigma_x(T)$, *where* $\sigma_x = \sigma_{UBB}$ *or* σ_{LBB} *or* σ_{BB} *or* σ_{UBW} *or* σ_{LBW} *or* σ_{BW} *or* σ_{UBF} *or* σ_{LBF} *or* σ_{BF} *or* σ_{LD} *or* σ_{RD}.

Proof. Since $f(T)$ (resp., $f(T^*) = f(T)^*$) has SVEP at a point λ if and only if T (resp., T^*) has SVEP at every μ such that $f(\mu) = \lambda$ [1, theorem 2.39], $f(T) \in (\mathcal{M})$ implies both T and T^* have SVEP (everywhere). The proof now follows from Lemma 3.4. ■

(*P*8). *If* $A \in (\mathcal{M})$, $AX = XB$, X *is injective and* $\sigma(B)\setminus\{0\} \subseteq \sigma(A)\setminus\{0\}$, *then* $B \in (\mathcal{M})$.

Proof. If $0 \neq \lambda \in \sigma(B)$, λ is an isolated point of both $\sigma(A)$ and $\sigma(B)$. Letting $P_\lambda(A)$ and $P_\lambda(B)$ denote the associated spectral projections, it is seen that there exists an integer $p \geq 1$ such that $(\lambda I - A)^{-p}(0)$ coincides with the range of $P_\lambda(A)$, and then:

$$\begin{aligned} &0 = (\lambda I - A)^p P_\lambda(A)X = (\lambda I - A)^p X P_\lambda(B) = X(\lambda I - B)^p P_\lambda(B) \\ &\Longrightarrow (\lambda I - B)^p P_\lambda(B) = 0 \Longrightarrow H_0(\lambda I - B) \subseteq P_\lambda(B)(\mathcal{X}) \subseteq (\lambda I - B)^{-p}(0) \\ &\Longrightarrow H_0(\lambda I - B) = (\lambda I - B)^{-p}(0). \end{aligned}$$

Hence, B is polar at λ. ■

We say that an operator $T \in B(\mathcal{X})$ is *generalised meromorphic*, $T \in (\mathcal{G}M)$, if there exists a finite subset E of the set $\mathbf{C}$ of complex numbers such that T is polar at every $\lambda \in \mathbf{C}\setminus E$ (and then the points $\lambda \in \sigma(T)\setminus E$ have no points of accumulation, except possibly for the points of E). Thus, $T \in (\mathcal{G}M)$ implies both T and T^* have SVEP, and an operator $T \in (\mathcal{G}M)$ if and only if the set

$$\sigma_D(T) = \sigma_{LD}(T) = \sigma_{RD}(T) = \sigma_{BB}(T) = \sigma_{BW}(T) = \sigma_{BF}(T)$$

has finite cardinality. We remark here that generalised Riesz operators (of [22]) are generalised meromorphic.

Evidently, iso$f(\sigma(T))$ = iso$\sigma(f(T))$.

Lemma 4.1. *iso*$f(\sigma(T)) = f($*iso*$\sigma(T))$, $f \in Holo_c(\sigma(T))$.

Proof. Choose a $\lambda \in$ iso$\sigma(f(T))$, and let $\mu \in \sigma(T)$ such that $\lambda = f(\mu)$. Then $\mu \in \Omega$

for some connected component Ω of the domain of definition of f. If $\mu \notin \mathrm{iso}\sigma(T)$, then there exists a sequence $\{\mu_i\} \subseteq \sigma(T) \cap \Omega$ converging to μ. Since $f(\mu_i) = \lambda$ for at most a finite number of values μ_i, there exists an integer m such that $f(\mu_i) \neq \lambda$ for all $i \geq m$. But then $f(\mu_i)$ converges to $\lambda = f(\mu)$ implies $\lambda \notin \mathrm{iso}f(\sigma(T))$ – a contradiction. Hence, $\mu \in \mathrm{iso}\sigma(T)$ for every $\mu \in \sigma(T)$ such that $f(\mu) = \lambda$ (that is, $\mathrm{iso}f(\sigma(T)) \subseteq f(\mathrm{iso}\sigma(T))$). The reverse inclusion being evident, the proof is complete. ■

The following theorem proves that T inherits the polaroid property from $f(T)$, $f \in \mathrm{Holo}_c(\sigma(T)$.

Theorem 4.1. *T is polaroid if and only if $f(T)$ is polaroid for every $f \in Holo_c(\sigma(T))$.*

PROOF. Suppose to start with that T is polaroid and $\lambda \in \mathrm{iso}\sigma(f(T))$. Then (we may assume that) $\lambda - f(z)$ has finitely many zeros $\{\mu_1, ..., \mu_n\}$ in $\sigma(T)$, where $\mu_i \in \mathrm{iso}\sigma(T)$ for all $1 \leq i \leq n$. Evidently, $\lambda - f(z) = p(z)g(z)$, where $p(z) = \prod_{i=1}^n (\mu_i - z)^{a_i}$ is a polynomial and g is analytic (in the domain of definition of f) without zeros in $\sigma(T)$. This implies that $\lambda I - f(T) = p(T)g(T)$ with $g(T)$ invertible, $p(T) = \prod_{i=1}^n (\mu_i I - T)^{a_i}$ and

$$H_0(\lambda I - f(T)) = H_0(p(T)) = \bigoplus_{i=1}^n H_0(\mu_i I - T).$$

Since T is polaroid, there exists an integer $p \geq 1$ such that $\mathrm{asc}(\mu_i I - T) \leq p$, and then $H_0(\mu_i I - T) = (\mu_i I - T)^{-pa_i}(0)$, for all $1 \leq i \leq n$. Thus,

$$\begin{aligned} H_0(\lambda I - f(T)) &= \bigoplus_{i=1}^n H_0(\mu_i I - T) = \bigoplus_{i=1}^n (\mu_i I - T)^{-pa_i}(0) \\ &= \ker(\prod_{i=1}^n (\mu_i I - T)^{pa_i}) = p(T)^{-p}(0) = (\lambda I - f(T))^{-p}(0). \end{aligned}$$

This implies that $f(T)$ is polar at λ (see Lemma 2.1).

For the converse assume that $f(T)$ is polar for an $f \in \mathrm{Holo}_c(\sigma(T))$. We prove that every $\mu \in \mathrm{iso}\sigma(T)$ is a pole of T. Let $\mu \in \mathrm{iso}\sigma(T)$ and let $f(\mu) = \lambda$. Then $\lambda \in \mathrm{iso}\sigma(f(T))$, and so there exists an integer $p \geq 1$ such that $H_0(\lambda I - f(T)) = (\lambda I - f(T))^{-p}(0)$. Since $\lambda - f(z) = f(\mu) - f(z) = (\mu - z)^a p(z)g(z)$ for some integer $a \geq 1$, polynomial $p(z)$ such that $p(\mu) \neq 0$ and an analytic function $g(z)$, it follows from $H_0(\lambda I - f(T)) \cap p(T)^{-1}(0) = \{0\}$ that

$$\begin{aligned} H_0(\mu I - T) \subseteq H_0(\lambda I - f(T)) &= (\mu I - T)^{-pa}(0) \bigoplus p(T)^{-p}(0) \\ &= (\mu I - T)^{-pa}(0) \subseteq H_0(\mu I - T), \end{aligned}$$

that is, $H_0(\mu I - T) = (\mu I - T)^{-pa}(0)$. Since $\mu \in \mathrm{iso}\sigma(T)$, this implies that T is polar at μ. ■

Remark 4.1. *We are grateful to a referee for pointing out that a proof of Theorem 4.1 appears in [2], also that the results of our Theorem 4.2* infra *are related to the results of the reference* loc. cit..

The following corollaries are immediate from the above.

Corollary 4.1. *If $f(T) \in (\mathcal{M})$, $f \in Holo_c(\sigma(T))$, then:*

(i) For each μ such that $f(\mu) = 0$, either T is polar at μ or μ is a limit point of the poles of T. Consequently, if $f(T) \in (\mathcal{M})$, then $\sigma_{BF}(T) \subseteq \{\mu : f(\mu) = 0\}$.

(ii) $\mu I - T$ is B-Browder at every μ such that $f(\mu) \neq 0$.

Corollary 4.2. *$f(T) \in (\mathcal{M})$ for some $f \in Holo_c(\sigma(T))$ if and only if $T \in (\mathcal{G}M)$.*

Corollary 4.3. *If $f(T) \in (\mathcal{M})$, $f \in Holo_c(\sigma(T))$, then $\alpha(\mu I - T) < \infty \iff \alpha(f(\mu)I - f(T)) < \infty$. Consequently, if If $f(T) \in (\mathcal{M})$, $f \in Holo_c(\sigma(T))$, and $\alpha(\lambda I - f(T)) < \infty$ for all $\lambda \neq 0$, then T is generalised Riesz.*

Corollary 4.4. *If $f(T) \in (\mathcal{M})$ for some $f \in Holo_c(\sigma(T))$, then either T is algebraic or each $\mu \in \sigma(T)$ such that $f(\mu) = 0$ is either a limit point of the eigenvalues of T or $dim H_0(\mu I - T) = \infty$.*

Proof. If $f(\mu) = 0$, then either there exists a sequence $(\mu_n) \subset \text{iso}\sigma(T)$ such that $\mu_n \to \mu$ or $\mu \in \text{iso}\sigma(T)$. If $\mu \in \text{iso}\sigma(T)$ and T is polar at μ, then 0 is a pole of $f(T)$. This forces $f(T)$, hence also T, to be algebraic. If, instead, $\mu \in \text{iso}\sigma(T)$ and T is not polar at μ, then $\dim H_0(\mu I - T) = \infty$. (Observe that if $\dim H_0(\mu I - T) < \infty$, then there exists an integer $p \geq 1$ such that $H_0(\mu I - T) = (\mu I - T)^{-p}(0)$; consequently, T is polar at μ.) Assume finally that $\mu \notin \text{iso}\sigma(T)$. If $\sigma(T) \supset \{\mu_n\}$ and $\mu_n \to \mu$, then each μ_n is a pole, hence an eigenvalue, which implies that μ is a limit point of the eigenvalues of T. ■

Recall from [6] that for $A, B \in B(\mathcal{X})$, $\text{asc}(\lambda I - AB) = \text{asc}(\lambda I - BA)$ and $\text{dsc}(\lambda I - AB) = \text{dsc}(\lambda I - BA)$ for all $\lambda \neq 0$; furthermore, this follows from a slight modification of the argument of [6], if $\lambda \neq 0$ then $(\lambda I - BA)^d(\mathcal{X})$ is closed if and only if $(\lambda I - AB)^d(\mathcal{X})$ is closed. Hence: *for all $\lambda \neq 0$, $\lambda \notin \sigma_x(AB)$ if and only if $\lambda \notin \sigma_x(BA)$, where σ_x denotes either of σ_{LD} and σ_{RD}.*

Corollary 4.5. *$f(BA) \in (\mathcal{M})$, $f \in Holo_c(\sigma(BA))$, if and only if $g(AB) \in (\mathcal{M})$ for some $g \in Holo_c(\sigma(AB))$.*

Proof. Corollary 4.2 implies that $f(BA) \in (\mathcal{M}) \iff BA \in (\mathcal{G}M) \iff AB \in (\mathcal{G}M)$. Since $\sigma_D(AB) = \sigma_D(BA) \subseteq \{\mu_i, 1 \leq i \leq n : f(\mu_i) = 0\}$, $f(BA) \in (\mathcal{M})$ implies AB polynomially meromorphic. The same argument works for the reverse implication. ■

If $f(T) \in (\mathcal{M})$, $f \in \text{Holo}_c(\sigma(T))$, is such that $\sigma_D(f(T)) \neq \emptyset$, then $\sigma_D(f(T)) = \{0\}$, there exists a finite subset $\{\mu_1, ..., \mu_n\} \subset \mathbf{C}$ such that $f(\mu_i) = 0$ for all $1 \leq$

$i \leq n$, and there exist countable disjoint subsets $S_i = \{\mu_{i_m}\} \subset \mathbf{C}$ such that $\mu_i \in S_i$ and $S_1 \cup ... \cup S_n = \sigma(T)$. (Here, either of the sets S_i may consist of the singleton $\{\mu_i\}$; evidently, $H_0(\mu_i I - T)$ is then infinite dimensional.) Letting P_i denote the spectral projection associated with the spectral set S_i, we obtain closed T-invariant subspaces $\mathcal{X}_i$ of $\mathcal{X}$ and operators $T_i = T|_{\mathcal{X}_i}$ such that $\mathcal{X} = \bigoplus_{i=1}^n \mathcal{X}_i$ and $T = \bigoplus_{i=1}^n T_i$. Clearly, $\sigma_D(T_i) = \{\mu_i\}$ and $\mu_i I_i - T_i$ is meromorphic for all $1 \leq i \leq n$. (Here, as before, I_i is the identity $B(\mathcal{X}_i)$.) We have proved:

Proposition 4.1. *If $T \in B(\mathcal{X})$ and $f \in Holo_c(T)$ are such that $f(T)$ is meromorphic, then there exists a finite set $\{\mu_1, ..., \mu_n\} \subset f^{-1}(0)$ of zeros of f for which $T = \bigoplus_{i=1}^n T_i$ of parts T_i of T such that $\mu_i I_i - T_i$ is meromorphic for all $1 \leq i \leq n$.*

Proposition 4.1, first proved in [18], generalises a result of Gilfeather [13] on the structure of polynomially compact operators, a result of Han *et al.* [15] on the structure of polynomially Riesz operators on Hilbert spaces, and a result of Živkovic-Zlatanovič *et al.* [22] on the structure of polynomially Riesz operators on Banach spaces.

An operator T is polynomially meromorphic if there exists a non-trivial polynomial $p(.)$ such that $p(T)$ is meromorphic. Trivially, T polynomially meromorphic implies $f(T)$ meromorphic for some $f \in \mathrm{Holo}_c(\sigma(T))$. The following proposition says that this condition is if and only if. Recall from [3, proposition 3.2] that: (i) The product $\prod_{i=1}^n (\lambda_i I - T)^{\alpha_i}$ is B-Fredholm if and only if $(\lambda_i I - T)$ is B-Fredholm for all $1 \leq i \leq n$; (ii) if $f(T)$ is B-Fredholm and $g(T)$ is invertible, then $f(T)g(T)^{-1}$ is B-Fredholm.

Theorem 4.2. *For an operator $T \in B(\mathcal{X})$, the following statements are equivalent.*

(i) $f(T) \in (\mathcal{M})$ for some $f \in Holo_c(\sigma(T))$.
(ii) $p(T) \in (\mathcal{M})$ for some non-trivial polynomial $p(.)$.
(iii) There exists a finite sequence of scalars $\{\mu_i\}_{i=1}^n$ and a decomposition $\mathcal{X} = \bigoplus_{i=1}^n \mathcal{X}_i$ such that $T = \bigoplus_{i=1}^n T|_{\mathcal{X}_i} = \bigoplus_{i=1}^n T_i$, where $\mu_i I_i - T_i \in (\mathcal{M})$ for all $1 \leq i \leq n$.

PROOF. We have already proved $(ii) \Longrightarrow (iii)$, see Proposition 4.1, and $(ii) \Longrightarrow (i)$ is evident; we prove $(i) \Longrightarrow (ii)$ and $(iii) \Longrightarrow (ii)$.

$(i) \Longrightarrow (ii)$. Consider a point $\mu \neq 0$; then $\mu I - f(T)$ is B-Fredholm. Since $\mu - f(z)$ has at best a finite number of zeros on $\sigma(T)$,

$$\mu - f(z) = \prod_{i=1}^n (\lambda_i - z)^{\alpha_i} g(z)$$

for some non-vanishing analytic function $g(z)$. Hence,

$$(\mu I - f(T))g(T)^{-1} = \prod_{i=1}^{n} (\lambda_i I - T)^{\alpha_i} = p(T)$$

for some polynomial $p(.)$. Evidently, $p(T)$ is B-Fredholm. Observe that the operator $f(T)$ being meromorphic, both $f(T)$ and $f(T^*)$ have SVEP; hence, T and T^*, so also $p(T)$ and $p(T)^*$, have SVEP. Consequently, $p(T)$ is polaroid, and hence meromorphic.

$(iii) \Longrightarrow (ii)$. Letting $p_j(z)$ denote the polynomial $p_j(z) = \prod_{j=1}^{n} (\mu_j - z)$, it is seen that

$$\begin{aligned} p(T) &= \prod_{j=1}^{n} (\mu_j I - T) = \prod_{j=1}^{n} \{\bigoplus_{i=1}^{n} (\mu_j I_i - T_i)\} \\ &= \bigoplus_{i=1}^{n} \{\prod_{j=1}^{n} (\mu_j I_i - T_i)\} = \bigoplus_{i=1}^{n} p_j(T_i), \end{aligned}$$

where $\mu_i I_i - T_i \in (\mathcal{M})$ for all $1 \leq i \leq n$ (and $\mu_j I_i - T_i$ is invertible for all $1 \leq i \neq j \leq n$). Since $\sigma_D(\mu_i I_i - T_i) = \sigma_{BF}(\mu_i I_i - T_i) = \{0\}$, and since $p_j(T_i) = \prod_{j=1}^{n} (\mu_j I_i - T_i)$ is B-Fredholm if and only if $\mu_j I_i - T_i$ is B-Fredholm for all $1 \leq j \leq n$ [3], $p_j(T_i)$ is not B-Fredholm for all $1 \leq i \leq n$. It is clear that the spectrum $\sigma(p_j(T_i))$ is a countable set; hence, both $p_j(T_i)$ and $p_j(T_i)^*$ have SVEP (everywhere). The spectral mapping theorem for B-Fredholm operators [3, theorem 3.4] implies that

$$\sigma_{BF}(p_j(T_i)) = p_j(\sigma_{BF}(T_i)) = p_j(\mu_i) = \{0\}.$$

Consequently, $p_j(T_i) - \lambda$ is B-Fredholm for all complex $\lambda \neq 0$, and this coupled with SVEP for $p_j(T_i)$ and $p_j(T_i)^*$ implies that the points $\lambda \neq 0$ are poles of $p_j(T_i)$. Hence, $p_j(T_i)$, and so also $p(T) = \bigoplus_{i=1}^{n} (p_j(T_i)$, is meromorphic. ■

$T \in B(\mathcal{X})$ is normaloid if $||T||$ equals the spectral radius $r(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}}$. T is said to be *hereditarily normaloid*, $T \in (\mathcal{HN})$, if every part of T is normaloid, and T is *totally hereditarily normaolid*, $T \in (\mathcal{THN})$, if $T \in (\mathcal{HN})$ and every invertible part of T is normaloid. We say in the following that a subspace M of $\mathcal{X}$ is orthogonal (in the Birkhoff-James sense of orthogonality [12, p. 93]) to a subspace N of X, denoted $M \perp N$, if $||m|| \leq ||m + n||$ for all vectors $m \in M$ and $n \in N$. This asymmetric version of (Banach space) orthogonality coincides with the standard concept of orthogonality for Hilbert spaces. The following theorem considers $(\mathcal{THN})$-operators to prove that $(\mathcal{THN})$-operators T with countable spectrum such that 0 is the only (possible) limit point of $\sigma(T)$ are meromorphic.

Theorem 4.3. *If $T \in B(\mathcal{X}) \cap (\mathcal{THN})$ is such that $\sigma(T)$ is countable with (at best) a single limit point, then:*

(i) T is a translate of a (simply) polaroid meromorphic operator.

(ii) For every $\lambda \in iso\sigma(T)$, with corresponding spectral projection $P_\lambda(T)$, $||P_\lambda(T)|| = 1$ and $(\lambda I - T)^{-1}(0) \perp (\lambda I - T)(\mathcal{X})$.

PROOF. (i) Given a $\lambda \in \mathrm{iso}\sigma(T)$,

$$\mathcal{X} = H_0(\lambda I - T) \oplus K(\lambda I - T), \quad H_0(\lambda I - T) \neq \{0\}.$$

Defining the operators T_1 and T_2 by $T_1 = T|_{H_0(\lambda I - T)}$ and $T_2 = T|_{K(\lambda I - T)}$, it follows that $T = T_1 \oplus T_2$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. The operator T_1 being normaloid, if $\lambda = 0$, then $||T_1|| = 0$ and $T = 0 \bigoplus T_2$, where T_2 is an invertible $(\mathcal{THN})$-operator. Assume hence that $\lambda \neq 0$. The class of $(\mathcal{THN})$-operators being closed under multiplication by non-zero scalars, we may assume that $\lambda = 1$, and then $\sigma(T_1) = \{1\}$ and $\sup||T_1^n|| \leq 1$ (where the supremum is taken over all integers n). Recall from [20, theorem 1.5.14] that $\sigma(A) = \{1\}$ for a doubly power bounded operator A acting on a non-trivial Banach space if and only if A is the identity operator. Hence, $T_1 = I_1 = \lambda I|_{H_0(\lambda I - T)} = \lambda I|_{(\lambda I - T)^{-1}(0)}$. (Thus, λ is a simple pole of T.) Now let μ denote the (only) point of accumulation of $\sigma(T)$, and let $p_\mu(.)$ denote the polynomial $p_\mu(z) = \mu - z$. Then $\sigma(p_\mu(T)) = \sigma(T) - \{\mu\}$ is countable with 0 as its only point of accumulation. Since $p_\mu(T)$ is polaroid if and only if T is polaroid (see Theorem 4.1), $T = -p_\mu(T) + \mu I$ is a translate of a meromorphic operator.

(ii) We start by considering the case in which $\sigma(T)$ is contained in the boundary $\partial\mathbf{D}$ of a disc centered at 0. If $\sigma(T) \subseteq \partial\mathbf{D}$, then $T_0 = \frac{T}{||T||}$ is an invertible isometry such that the points $\mathrm{iso}\sigma(T_0)$ are all simple poles (hence, eigenvalues) of T_0. Since the eigenspaces corresponding to isolated eigenvalues of an invertible isometry have an invariant complement [19], (ii) follows. Consider now the case in which $\sigma(T) \not\subseteq \partial\mathbf{D}$. Let $\sigma_\pi(T)$ denote *the peripheral spectrum*

$$\sigma_\pi(T) = \{\lambda \in \sigma(T) : |\lambda| = r(T)\}$$

of T [16, p. 225]. Then $\sigma_\pi(T) = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ is a finite set such that

$$T^j P_{\lambda_i}(T) = \lambda_i^j P_{\lambda_i}(T), \quad j \geq 1 \text{ and } \lambda_i \in \sigma_\pi(T)$$

for all $1 \leq i \leq m$ (see the proof of (i) above). Let $\mathbf{C}$ be a circle centered at 0 of radius $\rho < r(T) = ||T||$, Then

$$\begin{aligned} T^j &= \sum_{i=1}^m \lambda_i^j P_{\lambda_i}(T) + \frac{1}{2\pi i}\int_{\mathbf{C}} \lambda^j(\lambda I - T)^{-1} d\lambda \\ &= \sum_{i=1}^m \lambda_i^j P_{\lambda_i}(T) + \Gamma_2. \end{aligned}$$

Now let

$$\max_{\lambda\in\mathbf{C}}||(\lambda I-T)^{-1}||=\frac{M}{\rho}.$$

Then

$$||\sum_{j=1}^{n}\frac{1}{\lambda_i^j}\Gamma_2||\leq M\sum_{j=1}^{n}(\frac{\rho}{|\lambda_i|})^j\leq M(1-\frac{\rho}{|\lambda_i|})^{-1}$$

for all integers $n\geq 1$. Again, if we let

$$\alpha=\max_{2\leq i\leq m}||P_{\lambda_i}(T)||,\quad \beta=\min_{2\leq i\leq m}|1-\frac{\lambda_i}{\lambda_1}| \text{ and } \Gamma_1=\sum_{j=1}^{n}\sum_{i=2}^{m}(\frac{\lambda_i}{\lambda_1})^jP_{\lambda_i}(T),$$

then

$$||\Gamma_1||=||\sum_{i=2}^{m}(\frac{\lambda_i}{\lambda_1})\frac{1-(\frac{\lambda_i}{\lambda_1})^n}{1-\frac{\lambda_i}{\lambda_1}}||\leq\frac{2m\alpha}{\beta}.$$

Combining this with the estimate

$$\frac{1}{n}||\sum_{j=1}^{n}(\frac{A}{\lambda_i})^j||\leq\frac{1}{n}.n=1,$$

we have

$$||P_{\lambda_1}(T)-\frac{1}{n}\sum_{j=1}^{n}(\frac{A}{\lambda_i})^j||=\frac{1}{n}||\Gamma_1+\Gamma_2||\longrightarrow 0 \text{ as } n\to\infty.$$

Hence, $||P_{\lambda_1}(T)||\leq 1$. Since $||P_{\lambda_1}(T)||=||P_{\lambda_1}(T)^2||\leq||P_{\lambda_1}(T)||^2$, $||P_{\lambda_1}(T)||=1$ and

$$||x||=||P_{\lambda_1}(T)x||=||P_{\lambda_1}(T)(x+y)||\leq||x+y||$$

for all $x\in(\lambda_1I-T)^{-1}(0)$ and $y\in(\lambda_1I-T)(\mathcal{X})$. Our choice of the point $\lambda_1\in\sigma_\pi(T)$ having been arbitrary, it follows that $||P_{\lambda_i}(T)||=1$ for all $1\leq i\leq m$, and $(\lambda_iI-T)^{-1}(0)\perp(\lambda_jI-T)^{-1}(0)$ for all $1\leq i\neq j\leq m$. Furthermore, we have the decompositions $\mathcal{X}=\mathcal{X}_{11}\bigoplus\mathcal{X}_{22}$ and $T=T|_{\mathcal{X}_{11}}\bigoplus T|_{\mathcal{X}_{22}}=T_{11}\bigoplus T_{22}$, where $\sigma(T_{11})=\sigma_\pi(T)$, $\sigma(T_{22})=\sigma(T)\setminus\sigma(T_{11})$ and $T_{22}\in(\mathcal{THN})$. To complete the proof, one now applies the argument above to T_{22} (and so on). ■

Theorem 4.3 implies $(\mathcal{THN})$ Hilbert space operators T, $T\in B(\mathcal{H})\cap(\mathcal{THN})$, with countable spectrum converging to a point are diagonal, hence normal, operators. The role played by the normaloid property of the invertible part of an operator in the proof of Theorem 4.3 is limited to proving the polaroid property of the operator; hence, Theorem 4.3 implies that meromorphic Hilbert space $(\mathcal{HN})$ operators are normal. More generally:

Corollary 4.6. *Hilbert space operators T such that $f(T) \in (\mathcal{M}) \cap (\mathcal{HN})$ for some $f \in Holo_c(\sigma(T))$ are normal.*

PROOF. If $f(T) \in (\mathcal{M}) \cap (\mathcal{HN})$, then there exist decompositions $\mathcal{H} = \bigoplus_{i=1}^m \mathcal{H}_i$ of $\mathcal{H}$ and $T = \bigoplus_{i=1}^m T|_{\mathcal{H}_i} = \bigoplus_{i=1}^m T_i$ of T, and a finite sequence of scalars $\{\mu_i\}_{i=1}^m$, such that $T_i \in (\mathcal{HN})$ and $\mu_i I_i - T_i \in B(\mathcal{H}_i) \cap (\mathcal{M})$ for all $1 \leq i \leq m$. The operator $\mu_i I_i - T_i$ being (meromorphic is) polaroid; hence, T_i is a polaroid $(\mathcal{HN})$-operator with countable spectrum and μ_i as its only limit point. Hence, T_i is normal for all1 $\leq i \leq m$. (We remark here that if $\sigma(T_i) = \{\mu_i\}$ for some i, then the hypothesis $T_i \in (\mathcal{HN})$ forces $H_0(\mu_i I_i - T_i) = (\mu_i I_i - T_i)^{-1}(0)$.) Hence, T is normal. ■

A number of the more commonly considered classes of Hilbert space operators satisfy the $(\mathcal{HN})$-property. Thus, the classes of operators $T \in B(\mathcal{H})$ such that T is:

hyponormal, that is, $TT^* \leq T^*T$;

paranormal, that is, $||Tx||^2 \leq ||T^2x||||x||$ for all $x \in \mathcal{H}$, and

$*$-paranormal, that is, $||T^*x||^2 \leq ||T^2x||||x||$ for all $x \in \mathcal{H}$

are $(\mathcal{HN})$-operators. Corollary 4.6 generalises a number of known results, amongst them [13, corollary 2] and [22, corollary 2.15], on the normality of holomorphically compact, more generally Riesz, operators. We remark that hyponormal, more generally paranormal, operators are known to be $(\mathcal{THN})$.

The $(\mathcal{HN})$ property in Corollary 4.6 is not essential to the normality of meromorphic Hilbert space operators. Consider, for example, the class of operators $T \in B(\mathcal{H})$ for which there exists a real number $M > 0$ such that $||(\lambda I - T)^*x|| \leq M||(\lambda I - T)x||$ for all $x \in \mathcal{H}$ and scalars λ. Such operators, referred to in the literature as M-hyponormal operators, are not normaloid. They are, however, known to be (generalised) sub-scalar operators satisfying the property that $H_0(\lambda I - T) = (\lambda I - T)^{-1}(0)$ for all complex λ [20, proposition 2.4.9]. The isolated points of the spectrum of an M-hyponormal operator are simple poles (and therefore, eigenvalues) of the operator, which satisfy $(\lambda I - T)^{-1}(0) = ((\lambda I - T)^*)^{-1}(0)$ [9, remark 3.2]. Hence, a meromorphic M-hyponormal operator (more generally, an M-hyponormal operator with countable spectrum) is normal.

Acknowledgements

The authors are supported by the Ministry of Science, Republic of Serbia.

References

[1] P. Aiena, *Fredholm and local spectral theory with applications to multipliers*, Kluwer, Dordrecht–Boston–London, 2004.

[2] P. Aiena, Algebraically paranormal operators on Banach spaces, *Banach Journal of Mathematical Analysis*, **7** (2013), 136–45.

[3] M. Berkani, On a class of quasi-Fredholm operators, *Integral Equations and Operator Theory*, **34** (1999), 244–9.
[4] M. Berkani and J.J. Koliha, Weyl type theorems for bounded linear operators, *Acta Scientiarum Mathematicarum*, **69** (2003), 359–76.
[5] M. Berkani, G.N. Castro-Gonzalez and S.V. Djordjević, The single valued extension property, quasiaffinity and the generalised Weyl's theorem, *Mathematica Bohemica*, **131** (2006), 29–38.
[6] J.J. Buoni and J.D. Faires, Ascent, descent, nullity and defect of products of operators, *Indiana University Mathematics Journal*, **25** (1976), 703–7.
[7] S.R. Caradus, W.E. Pfaffenberger and B. Yood, *Calkin algebras and algebras of operators on Banach spaces*, Marcel Dekker, New York, 1974.
[8] S.V. Djordjevič and B.P. Duggal, Drazin invertibility of the diagonal of an operator, *Linear and Multilinear Algebra*, **60** (2012), 65–71.
[9] B.P. Duggal, Riesz projections for a class of Hilbert space operators, *Linear Algebra and its Applications*, **407** (2005), 140–8.
[10] B.P. Duggal, Polaroid operators and generalized Browder-Weyl theorems, *Mathematical Proceedings of the Royal Irish Academy*, **108**A (2008), 149–63.
[11] B.P. Duggal, R.E. Harte and A.H. Kim, Weyl's theorem, tensor products and multiplication operators II, *Glasgow Mathematical Journal*, **52** (2010), 705–9.
[12] N. Dunford and J.T. Schwartz, *Linear operators part I*, Interscience, New York, 1964.
[13] F. Gilfeather, The structure and asymptotic behaviour of polynomially compact operators, *Proceedings of the American Mathematical Society*, **25** (1970), 127–34.
[14] S. Grabiner, Uniform ascent and descent of bounded operators, *Journal of the Mathematical Society of Japan*, **3**4 (1982), 317–37.
[15] Y.M. Han, S.H. Lee and W.Y. Lee, On the structure of polynomially compact operators, *Mathematische Zeitschrift*, **232** (1999), 257–63.
[16] H.G. Heuser, *Functional Analysis*, John Wiley and Sons, New York, 1982.
[17] A. Jeribi and N. Moalla, Fredholm operators and Riesz theory for polynomially compact operators, *Acta Applicandae Mathematica*, **90** (2006), 227–47.
[18] M.A. Kaashoek and M.R.F. Smyth, On operators T such that $f(T)$ is Riesz or meromorphic, *Proceedings of the Royal Irish Academy, Section A*, **72** (1972), 81–7.
[19] D. Koehler and P. Rosenthal, On isometries of normal spaces, *Studia Mathematica*, **36** (1970), 213–16.
[20] K.B. Laursen and M.M. Neumann, *Introduction to local spectral theory*, Clarendon Press, Oxford, 2000.
[21] M. Mbekhta and V. Müller, On the axiomatic theorey of the spectrum II, *Studia Mathematica* **119** (1996), 129–47.
[22] S.Č. Živkovic-Zlatanović, D.S. Djordjević, R.E. Harte and B.P. Duggal, On polynomially Riesz operators, *Filomat*, **28**(1) (2014), 197–205.
[23] A.E. Taylor and D.C. Lay, *Introduction to functional analysis*, Wiley, New York, 1980.