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Matrix transformations between the sequence spaces $w_0^p(\Lambda), v_0^p(\Lambda), c_0^p(\Lambda)$ $(1 < p < \infty)$ and certain BK spaces

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Abstract

In this paper, we determine the β-duals of the sets $w_0^p(\Lambda)$, $v_0^p(\Lambda)$ and $c_0^p(\Lambda)$ for *exponentially bounded sequences* Λ. Furthermore, we characterize matrix transformations between the sequence spaces $w_0^p(\Lambda), v_0^p(\Lambda), c_0^p(\Lambda)$ $(1 < p < \infty)$ and certain BK spaces. Finally, we apply the *Hausdorff measure of noncompactness* to give necessary and sufficient conditions for a linear operator between these spaces to be compact.

Keywords: Matrix transformations; Measure of noncompactness

1. Introduction

We write ω for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let ϕ, ℓ_∞, c and c_0 denote the sets of all finite, bounded, convergent and null sequences, and cs be the set of all convergent series. We write $\ell_p = \{x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \leqslant p < \infty$.

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By e and $e^{(n)}$ $(n \in \mathbb{N}_0)$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, \ldots$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$. For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ be its n-section.

Throughout, let $\mu = (\mu_n)_{n=0}^{\infty}$ be a nondecreasing sequence of positive reals tending to infinity. For $0 < p < \infty$, we write

$$w_0^p(\mu) = \left\{ x \in \omega : \lim_{n\to\infty} \frac{1}{\mu_n^p} \sum_{k=0}^{n} |x_k|^p = 0 \right\},$$

$$w_\infty^p(\mu) = \left\{ x \in \omega : \sup_n \frac{1}{\mu_n^p} \sum_{k=0}^{n} |x_k|^p < \infty \right\},$$

$$v_0^p(\mu) = \left\{ x \in \omega : \lim_{n\to\infty} \frac{1}{\mu_n^p} \sum_{k=0}^{n} |x_k - x_{k-1}|^p = 0 \right\},$$

$$v_\infty^p(\mu) = \left\{ x \in \omega : \sup_n \frac{1}{\mu_n^p} \sum_{k=0}^{n} |x_k - x_{k-1}|^p < \infty \right\},$$

$$c_0^p(\mu) = \left\{ x \in \omega : \lim_{n\to\infty} \frac{1}{\mu_n^p} \sum_{k=0}^{n} |\mu_k x_k - \mu_{k-1} x_{k-1}|^p = 0 \right\},$$

$$c_\infty^p(\mu) = \left\{ x \in \omega : \sup_n \frac{1}{\mu_n^p} \sum_{k=0}^{n} |\mu_k x_k - \mu_{k-1} x_{k-1}|^p < \infty \right\}$$

and $c^p(\mu) = \{x \in \omega : x - le \in c_0^p(\mu) \text{ for some } l \in \mathbb{C}\}$ (cf. [8]). If $p = 1$, we omit the index p in all cases; in particular, we write $c_0(\mu) = c_0^1(\mu)$, $c(\mu) = c^1(\mu)$ and $c_\infty(\mu) = c_\infty^1(\mu)$ for the sets of sequences that are μ-strongly convergent to zero, μ-strongly convergent and μ-strongly bounded (cf. [7,11]).

If $\mu_n = n + 1$ for $n = 0, 1, \ldots$ the sets $w_0^p(\mu)$ and $w_\infty^p(\mu)$ reduce to w_0^p and w_∞^p, the sets of sequences that are strongly summable and strongly bounded with index p by the C_1 method. These sets were introduced and studied by Maddox (cf. [4]).

For $0 < p \leqslant 1$, the β-duals were given in [7] $(p = 1)$ and [3] of the sets $c_0^p(\mu)$, $c^p(\mu)$ and $c_\infty^p(\mu)$, and some matrix transformations characterized on these spaces for *exponentially bounded* sequences μ. These results were extended in [9] to the case of $1 < p < \infty$ for the sets $c^p(\mu)$ and $c_\infty^p(\mu)$.

In this paper, we determine the β-duals, for exponentially bounded sequences μ, of the sets $v_0^p(\mu)$ and $c_0^p(\mu)$ for $p > 1$ which are different from the β-duals of $v_\infty^p(\mu)$ and $c^p(\mu)$ or $c_\infty^p(\mu)$; thus we complete the list of the β-duals of the sets $c_0^p(\mu)$, $c^p(\mu)$ and $c_\infty^p(\mu)$. Furthermore we characterize matrix transformations on the spaces $v_0^p(\mu)$ and $c_0^p(\mu)$ for $1 < p < \infty$ and apply the *Hausdorff measure of noncompactness* to give necessary and sufficient conditions for the entries of an infinite matrix to be a compact operator between these spaces and certain *BK spaces*. In particular, we extend the results involving the spaces $c_0(\mu)$, given in [10, Sections 3.7 and 3.8].

2. Notations, definitions and well-known results

In this section, we give some notations and recall some definitions and well-known results.

A sequence $(b^{(n)})_{n=0}^\infty$ in a linear metric space X is called *Schauder basis* if for every $x \in X$, there is a unique sequence $(\lambda_n)_{n=0}^\infty$ of scalars such that $x = \sum_{n=0}^\infty \lambda_n b^{(n)}$.

An *FK space* is a complete linear metric sequence space with the property that convergence implies coordinatewise convergence; a *BK space* is a normed FK space. An FK space $X \supset \phi$ is said to have AK if every sequence $x = (x_k)_{k=0}^\infty \in X$ has a unique representation $x = \sum_{k=0}^\infty x_k e^{(k)}$, that is $x = \lim_{n\to\infty} x^{[n]}$.

Let x and y be sequences, X and Y be subsets of ω and $A = (a_{nk})_{n,k=0}^\infty$ be an infinite matrix of complex numbers. We write $xy = (x_k y_k)_{k=0}^\infty$, $x^{-1} * Y = \{a \in \omega : ax \in Y\}$ and $M(X, Y) = \bigcap_{x\in X} x^{-1} * Y = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$ for the *multiplier space of X and Y*. In the special case of $Y = \text{cs}$, we write $x^\beta = x^{-1} * \text{cs}$ and $X^\beta = M(X, \text{cs})$ for the *β-dual of X*. By $A_n = (a_{nk})_{k=0}^\infty$ we denote the sequence in the nth row of A, and we write $A_n(x) = \sum_{k=0}^\infty a_{nk}x_k$ $(n = 0, 1, \ldots)$ and $A(x) = (A_n(x))_{n=0}^\infty$, provided $A_n \in x^\beta$ for all n. The set $X_A = \{x \in \omega : A(x) \in X\}$ is called the *matrix domain of A in X* and (X, Y) denotes the class of all matrices that map X into Y, that is $A \in (X, Y)$ if and only if $X \subset Y_A$, or equivalently $A_n \in X^\beta$ for all n and $A(x) \in Y$ for all $x \in X$.

Let X and Y be Banach spaces. Then $B(X, Y)$ is the set of all continuous linear operators $L : X \to Y$, a Banach space with the operator norm defined by $\|L\| = \sup\{\|L(x)\| : \|x\| \leqslant 1\}$ $(L \in B(X, Y))$. If $Y = \mathbb{C}$ then we write $X^* = B(X, \mathbb{C})$ for the space of continuous linear functionals on X with its norm defined by $\|f\| = \sup\{|(x)| : \|x\| \leqslant 1\}$ $(f \in X^*)$. We recall that a linear operator $L : X \to Y$ is called *compact* if $D(L) = X$ for the domain of L and if, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. It is well known (cf. [13, Theorem 4.2.8, p. 57]) that if X and Y are BK spaces and $A \in (X, Y)$ then $L_A \in B(X, Y)$ where L_A is defined by $L_A(x) = A(x)$ for all $x \in X$; we denote this by $(X, Y) \subset B(X, Y)$.

We define the matrices Σ, Δ and E by $\Sigma_{nk} = 1$ for $0 \leqslant k \leqslant n$, $\Sigma_{nk} = 0$ for $k > n$, $\Delta_{n,n-1} = -1$, $\Delta_{nn} = 1$, $\Delta_{nk} = 0$ otherwise, $E_{nk} = 0$ for $0 \leqslant k \leqslant n-1$ and $E_{nk} = 1$ for $k \geqslant n$, and use the convention that any term with a negative subscript is equal to zero. Then we can write $v_0^p(\mu) = (w_0^p(\mu))_\Delta$, $v_\infty^p(\mu) = (w_\infty^p(\mu))_\Delta$, $c_0^p(\mu) = \mu^{-1} * v_0^p(\mu)$ and $c_\infty^p(\mu) = \mu^{-1} * v_\infty^p(\mu)$.

Following the notations introduced in [7], we say that a nondecreasing sequence $\Lambda = (\lambda_k)_{k=0}^\infty$ of positive reals tending to infinity is *exponentially bounded* if there are reals s and t with $0 < s \leqslant t < 1$ such that for some subsequence

$(\lambda_{k(v)})_{v=0}^{\infty}$ of Λ, we have $s \leqslant \lambda_{k(v)}/\lambda_{k(v+1)} \leqslant t$ for all $v = 0, 1, \ldots$; such a subsequence $(\lambda_{k(v)})_{v=0}^{\infty}$ is called an *associated subsequence*. If $(k(v))_{v=0}^{\infty}$ is a strictly increasing sequence of nonnegative integers then we write $K^{\langle v\rangle}$ for the set of all integers k with $k(v) \leqslant k \leqslant k(v+1) - 1$, and $\sum_v$ and $\max_v$ for the sum and maximum taken over all k in $K^{\langle v\rangle}$.

Let $\Lambda = (\lambda_k)_{k=0}^{\infty}$ be an exponentially bounded sequence of positive reals and $(\lambda_{k(v)})_{v=0}^{\infty}$ be an associated subsequence throughout.

If $X^p(\Lambda)$ denotes any of the sets $w_0^p(\Lambda)$, $w_\infty^p(\Lambda)$, $v_0^p(\Lambda)$, $v_\infty^p(\Lambda)$, $c_0^p(\Lambda)$, $c^p(\Lambda)$ or $c_\infty^p(\Lambda)$ then we write $\tilde{X}^p(\Lambda)$ for the respective space with the sections $1/\lambda_k^p \sum_{j=0}^{k} \cdots$ replaced by the blocks $1/\lambda_{k(v+1)} \sum_v \cdots$ Furthermore, we define

$$\|x\|_{w_\infty^p(\Lambda)} = \sup_k \left(\frac{1}{\lambda_k^p} \sum_{j=0}^{k} |x_j|^p \right)^{1/p},$$

$$\|x\|_{\tilde{w}_\infty^p(\Lambda)} = \sup_v \left(\frac{1}{\lambda_{k(v+1)}^p} \sum\nolimits_v |x_k|^p \right)^{1/p},$$

$$\|x\|_{v_\infty^p(\Lambda)} = \|\Delta(x)\|_{w_\infty^p(\Lambda)}, \|x\|_{\tilde{v}_\infty^p(\Lambda)} = \|\Delta(x)\|_{\tilde{w}_\infty^p(\Lambda)},$$

$$\|x\|_{c_\infty^p(\Lambda)} = \|\Lambda x\|_{v_\infty^p(\Lambda)} = \|\Delta(\Lambda x)\|_{w_\infty^p(\Lambda)} \quad \text{and}$$

$$\|x\|_{\tilde{c}_\infty^p(\Lambda)} = \|\Lambda x\|_{\tilde{v}_\infty^p(\Lambda)} = \|\Delta(\Lambda x)\|_{\tilde{w}_\infty^p(\Lambda)}.$$

Proposition 2.1. *We have*:

(a) $w_0^p(\Lambda) = \tilde{w}_0^p(\Lambda)$, *the norms* $\|\cdot\|_{w_\infty^p(\Lambda)}$ *and* $\|\cdot\|_{\tilde{w}_\infty^p(\Lambda)}$ *are equivalent on* $w_0^p(\Lambda)$ *and* $w_0^p(\Lambda)$ *is a BK space with AK*;
(b) $v_0^p(\Lambda) = \tilde{v}_0^p(\Lambda)$, *the norms* $\|\cdot\|_{v_\infty^p(\Lambda)}$ *and* $\|\cdot\|_{\tilde{v}_\infty^p(\Lambda)}$ *are equivalent on* $v_0^p(\Lambda)$ *and the sequence* $(b^{(k)})_{k=0}^{\infty}$ *with* $b^{(k)} = \Sigma(e^{(k)})$, *that is* $b_j^{(k)} = 0$ *for* $j < k$ *and* $b_j^{(k)} = 1$ *for* $j \geqslant k$ $(k = 0, 1, \ldots)$, *is a Schauder basis of* $v_0^p(\Lambda)$;
(c) $c_0^p(\Lambda) = \tilde{c}_0^p(\Lambda)$, *the norms* $\|\cdot\|_{c_\infty^p(\Lambda)}$ *and* $\|\cdot\|_{\tilde{c}_\infty^p(\Lambda)}$ *are equivalent on* $c_0^p(\Lambda)$ *and the sequence* $(c^{(k)})_{k=0}^{\infty}$ *with* $c^{(k)} = (1/\Lambda)b^{(k)}$ $(k = 0, 1, \ldots)$ *is a Schauder basis of* $c_0^p(\Lambda)$.

Proof. Using the technique applied in the proof of [7, Theorem 1(a) and (b)], we can show the stated equality of the spaces and the equivalence of the norms.

(a) The space $w_0^p(\Lambda)$ is a BK space with AK by [6, Theorems 2 and 5].

Parts (b) and (c) follow from Part (a) and [10, Theorem 3.3, p. 178] and [8, Theorem 2.2]. □

3. The β-duals of the spaces $w_0^p(\Lambda)$, $v_0^p(\Lambda)$ and $c_0^p(\Lambda)$

In this section, we give the β-duals of the spaces $w_0^p(\Lambda)$, $v_0^p(\Lambda)$ and $c_0^p(\Lambda)$. If $X \supset \phi$ is a BK space and $a \in \omega$ then we write

$$\|a\|_X^* = \|a\|^* = \sup\left\{\left|\sum_{k=0}^{\infty} a_k x_k\right| : \|x\| = 1\right\},$$

provided the expression on the right is defined and finite which is the case whenever $a \in X^\beta$ (cf. [13, Theorem 7.2.9, p. 107]). We need the following results.

Proposition 3.1. *Let $1 < p < \infty$ and $q = p/(p-1)$.*

(a) *Then*

$$(w_0^p(\Lambda))^\beta = (w_\infty^p(\Lambda))^\beta = \mathscr{W}^p(\Lambda)$$

$$= \left\{a \in \omega : \sum_{v=0}^{\infty} \lambda_{k(v+1)} \left(\sum\nolimits_v |a_k|^q\right)^{1/q} < \infty\right\}.$$

Furthermore $(w_0^p(\Lambda))^\beta$ with

$$\|a\|_{\mathscr{W}^p(\Lambda)} = \sum_{v=0}^{\infty} \lambda_{k(v+1)} \left(\sum\nolimits_v |a_k|^q\right)^{1/q}$$

and $(w_0^p(\Lambda))^*$ are norm isomorphic when $w_0^p(\Lambda)$ has the norm $\|\cdot\|_{\tilde{w}_\infty^p(\Lambda)}$.

(b) *For each $n \in \mathbb{N}_0$, let $v(n)$ be the uniquely defined integer such that $n \in K^{\langle v(n)\rangle}$. We define the sequence d by*

$$d_n = \sum_{v=0}^{v(n)-1} \lambda_{k(v+1)}(k(v+1) - k(v))^{1/q} + \lambda_{k(v(n)+1)}(n + 1 - k(v(n)))^{1/q}$$

for $n = 0, 1, \ldots$ and put $M(v_0^p(\Lambda)) = d^{-1} * \ell_\infty$. Then

$$M(v_0^p(\Lambda), c_0) = M(v_0^p(\Lambda), c) = M(v_0^p(\Lambda)) \tag{3.1}$$

and

$$(v_0^p(\Lambda))^\beta = (\mathscr{W}^p(\Lambda) \cap M(v_0^p(\Lambda)))_E. \tag{3.2}$$

Furthermore, if $v_0^p(\Lambda)$ has the norm $\|\cdot\|_{\tilde{v}_\infty^p(\Lambda)}$ then

$$\|a\|_{v_0^p(\Lambda)}^* = \|E(a)\|_{\mathscr{W}^p(\Lambda)} \quad \textit{for all } a \in (v_0^p(\Lambda))^\beta. \tag{3.3}$$

Proof. (a) This is by [6, Theorems 4 and 6].

(b) The identity in (3.1) is [8, Lemma 3.2(a)]. From this, Part (a) and [8, Theorem 2.5], we obtain (3.2). Finally, we prove (3.3). Let $a \in (v_0^p(\Lambda))^\beta$ be given. We observe that $x \in v_0^p(\Lambda)$ if and only if $y = \Delta(x) \in w_0^p(\Lambda)$. Abel's summation by parts yields, with $R = E(a)$,

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n+1} R_k y_k - R_{n+1} x_{n+1} \quad (n = 0, 1, \ldots).$$

Since $a \in (v_0^p(\Lambda))^\beta$ implies $R \in M(v_0^p(\Lambda), c_0)$ by (3.1) and (3.2), it follows that $\sum_{k=0}^\infty a_k x_k = \sum_{k=0}^\infty R_k x_k$. Now

$$\|x\|_{\tilde{v}_\infty^p(\Lambda)} = \|y\|_{\tilde{w}_\infty^p(\Lambda)} \text{ implies } \|a\|^*_{\tilde{v}_\infty^p(\Lambda)} = \|R\|^*_{\tilde{w}_\infty^p(\Lambda)},$$

and (3.2) follows from Part (a). □

Theorem 3.1. *Let* $1 < p < \infty$ *and* $q = p/(p-1)$. *We define the sequence d as in Proposition* 3.1. *Then*

$$(c_0^p(\Lambda))^\beta = \left(\frac{1}{\Lambda}\right)^{-1} * ((w_0^p(\Lambda))^\beta \cap (d^{-1} * \ell_\infty))_E,$$

that is $a \in (c_0^p(\Lambda))^\beta$ *if and only if*

$$\sum_{v=0}^{\infty} \lambda_{k(v+1)} \left(\sum\nolimits_v \left| \sum_{j=k}^{\infty} \frac{a_j}{\lambda_j} \right|^q \right)^{1/q} < \infty$$

and

$$\sup_n \left| \sum_{k=n}^{\infty} \frac{a_j}{\lambda_j} \right| d_n < \infty.$$

Furthermore, if $c_0^p(\Lambda)$ *has the norm* $\|\cdot\|_{\tilde{c}_\infty^p(\Lambda)}$ *then*

$$\|a\|^*_{\tilde{c}_\infty^p(\Lambda)} = \|E(a/\Lambda)\|_{\mathscr{W}^p(\Lambda)} \quad \textit{for all } a \in (c_0^p(\Lambda))^\beta,$$

that is

$$\|a\|^*_{\tilde{c}_\infty^p(\Lambda)} = \sum_{v=0}^{\infty} \lambda_{k(v+1)} \left(\sum\nolimits_v \left| \sum_{j=k}^{\infty} \frac{a_j}{\lambda_j} \right|^q \right)^{1/q}. \tag{3.4}$$

Proof. The first part is [8, Theorem 3.2]. Furthermore, the equality of the norms on $(c_0^p(\Lambda))^\beta$ is an immediate consequence of (3.3) in Proposition 3.1 and the facts that $a \in (c_0^p(\Lambda))^\beta$ if and only if $a/\Lambda \in (v_0^p(\Lambda))^\beta$, and $\|x\|_{\tilde{c}_\infty^p(\Lambda)} = \|\Lambda x\|_{\tilde{v}_\infty^p(\Lambda)}$. □

4. Matrix transformations

In this section we characterize matrix transformations between the sequence space $w_0^p(\Lambda)$, $v_0^p(\Lambda)$, $c_0^p(\Lambda)$ $(1 < p < \infty)$ and certain BK spaces.

Throughout let $1 < p < \infty$, $q = p/(p-1)$ and the sequence d be defined as in Proposition 3.1. A subset X of ω is said to be *normal* if $x \in X$ and $y \in \omega$ with $|y_k| \leqslant |x_k|$ $(k = 0, 1, \dots)$ together imply $y \in X$. We need the following general results.

Proposition 4.1 [8, Theorem 2.7(a)]. *Let $X \supset \phi$ be a normal FK space with AK and Y be a linear space. If $M(X_\Delta, c) = M(X_\Delta, c_0)$ then $A \in (X_\Delta, Y)$ if and only if*

$$R^A \in (X, Y) \quad \text{where } r_{nk}^A = \sum_{j=k}^{\infty} a_{nk} (n, k = 0, 1, \dots) \tag{4.1}$$

and

$$R_n^A \in (X_\Delta, c) \quad \text{for all } n. \tag{4.2}$$

Proposition 4.2 (cf. [10, Theorem 1.23, p. 155]). *Let $X \supset \phi$ and Y be BK spaces.*

(a) *Then $A \in (X, \ell_\infty)$ if and only if*

$$\|A\|_X^* = \sup_n \|A_n\|_X^* < \infty. \tag{4.3}$$

Furthermore, if $A \in (X, \ell_\infty)$ then $\|L_A\| = \|A\|_X^*$.

(b) *If $(b^{(k)})_{k=0}^{\infty}$ is a Schauder basis of X and Y_1 is a closed BK space in Y, then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $A(b^{(k)}) \in Y_1$ for all k.*

We need the following result.

Proposition 4.3. *Let $X \supset \phi$ be a BK space. Then $A \in (X, \ell_1)$ if and only if*

$$\|A\|_{(X,\ell_1)} = \sup_{\substack{N \subset \mathbb{N}_0 \\ N \text{ finite}}} \left\| \sum_{n \in N} A_n \right\|_X^* < \infty \quad (cf.\ [5,\ Satz\ 1]).$$

Furthermore, if $A \in (X, \ell_1)$ then

$$\|A\|_{(X,\ell_1)} \leqslant \|L_A\| \leqslant 4 \cdot \|A\|_{(X,\ell_1)}. \tag{4.4}$$

Proof. We have to show (4.4). Let $A \in (X, \ell_1)$ and $m \in \mathbb{N}_0$ be given. Then, for all $N \subset \{0, \dots, m\}$ and for all $x \in X$ with $\|x\| = 1$,

$$\left|\sum_{n\in N} A_n(x)\right| \leqslant \sum_{n=0}^{m} |A_n(x)| \leqslant \|L_A\|,$$

and this implies

$$\|A\|_{(X,\ell_1)} \leqslant \|L_A\|. \tag{4.5}$$

Furthermore, given $\varepsilon > 0$, there is $x \in X$ with $\|x\| = 1$ such that

$$\|A(x)\|_1 = \sum_{n=0}^{\infty} |A_n(x)| \geqslant \|L_A\| - \frac{\varepsilon}{2},$$

and there is an integer $m(x)$ such that

$$\sum_{n=0}^{m(x)} |A_n(x)| \geqslant \|A(x)\|_1 - \frac{\varepsilon}{2}.$$

Consequently $\sum_{n=0}^{m(x)} |A_n(x)| \geqslant \|L_A\| - \varepsilon$. By [10, Lemma 3.9, p. 181],

$$4 \cdot \max_{N\subset\{0,\dots,m(x)\}} \left|\sum_{n\in N} A_n(x)\right| \geqslant \sum_{n=0}^{m(x)} |A_n(x)| \geqslant \|L_A\| - \varepsilon,$$

and so $4\|A\|_{(X,\ell_1)} \geqslant \|L_A\| - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $4\|A\|_{(X,\ell_1)} \geqslant \|L_A\|$, and together with (4.5) this yields (4.4). $\square$

A matrix T is called a *triangle* if $t_{nk} = 0$ $(k > n)$ and $t_{nn} \neq 0$ for all n.

Proposition 4.4 [10, Theorem 3.8, p. 180]. *Let T be a triangle. Then, for arbitrary subsets X and Y of ω, $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$. Furthermore, if X and Y are BK spaces and $A \in (X, Y_T)$ then*

$$\|L_A\| = \|L_B\|. \tag{4.6}$$

Now we characterize the classes $(w_0^p(\Lambda), \ell_\infty)$, $(w_0^p(\Lambda), c_0)$ and $(w_0^p(\Lambda), c)$.

Theorem 4.1. *We have*:

(a) $A \in (w_0^p(\Lambda), \ell_\infty)$ *if and only if*

$$\|A\|_{(w_\infty^p(\Lambda),\ell_\infty)} = \sup_n \sum_{v=0}^{\infty} \lambda_{k(v+1)} \left(\sum\nolimits_v |a_{nk}|^q\right)^{1/q} < \infty. \tag{4.7}$$

(b) $A \in (w_0^p(\Lambda), c_0)$ *if and only if condition* (4.7) *holds and*

$$\lim_{k\to\infty} a_{nk} = 0 \quad \textit{for each } k. \tag{4.8}$$

(c) $A \in (w_0^p(\Lambda), c)$ *if and only if condition* (4.7) *holds and*

$$\lim_{k\to\infty} a_{nk} = \alpha_k \quad \textit{for each } k. \tag{4.9}$$

(d) *If Y is any of the spaces ℓ_∞, c_0 or c, $w_0^p(\Lambda)$ has the norm $\|\cdot\|_{\tilde{w}_0^p(\Lambda)}$ and $A \in (w_0^p(\Lambda), Y)$ then*

$$\|L_A\| = \|A\|_{(w_0^p(\Lambda),\ell_\infty)}. \tag{4.10}$$

Proof. Part (a) follows from Proposition 3.1(a) and Proposition 4.2(a).

Parts (b) and (c) follow from Proposition 4.2(b), since c_0 and c are closed subspaces of ℓ_∞ and $w_0^p(\Lambda)$ has AK by Proposition 2.1(a).

Part (d) holds since $\mathscr{W}^p(\Lambda)$ with $\|\cdot\|_{\mathscr{W}^p(\Lambda)}$ and $(w_0^p(\Lambda))^*$ are norm isomorphic by Proposition 3.1(a) when $w_0^p(\Lambda)$ has the norm $\tilde{w}_\infty^p(\Lambda)$. □

Now we characterize the classes $(v_0^p(\Lambda), \ell_\infty)$, $(v_0^p(\Lambda), c_0)$ and $(v_0^p(\Lambda), c)$. As an immediate consequence of Proposition 4.1, Theorem 4.1 and Propositions 3.1(b) and 2.1(b), we obtain

Theorem 4.2. *We have*:

(a) $A \in (v_0^p(\Lambda), \ell_\infty)$ *if and only if*

$$\|A\|_{(v_\infty^p(\Lambda),\ell_\infty)} = \sup_n \sum_{v=0}^\infty \lambda_{k(v+1)} \left(\sum\nolimits_v \left| \sum_{j=k}^\infty a_{nj} \right|^q \right)^{1/q} < \infty \tag{4.11}$$

and

$$\sup_k d_k \left| \sum_{j=k}^\infty a_{nj} \right| < \infty \quad \textit{for each } n. \tag{4.12}$$

(b) $A \in (v_0^p(\Lambda), c_0)$ *if and only if conditions* (4.11) *and* (4.12) *hold and*

$$\lim_{n\to\infty} \sum_{j=k}^\infty a_{nj} = 0 \quad \textit{for each } k. \tag{4.13}$$

(c) $A \in (v_0^p(\Lambda), c)$ *if and only if conditions* (4.11) *and* (4.12) *hold and*

$$\lim_{n\to\infty} \sum_{j=k}^\infty a_{nj} = \alpha_k \quad \textit{for each } k. \tag{4.14}$$

(d) *If Y is any of the spaces ℓ_∞, c_0 or c, $v_0^p(\Lambda)$ has the norm $\|\cdot\|_{\tilde{v}_\infty^p(\Lambda)}$ and $A \in (v_0^p(\Lambda), Y)$ then*

$$\|L_A\| = \|A\|_{(v_\infty^p(\Lambda),\ell_\infty)}. \tag{4.15}$$

Now we characterize the classes $(c_0^p(\Lambda), \ell_\infty)$, $(c_0^p(\Lambda), c_0)$ and $(c_0^p(\Lambda), c)$.

Theorem 4.3. *We have*:

(a) $A \in (c_0^p(\Lambda), \ell_\infty)$ *if and only if*

$$\|A\|_{(c_0^p(\Lambda),\ell_\infty)} = \sup_n \sum_{v=0}^{\infty} \lambda_{k(v+1)} \left(\sum\nolimits_v \left| \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j} \right|^q \right)^{1/q} < \infty \tag{4.16}$$

and

$$\sup_k \left| \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j} \right| d_k < \infty \quad \textit{for each } n. \tag{4.17}$$

(b) $A \in (c_0^p(\Lambda), c_0)$ *if and only if conditions* (4.16), (4.17) *and* (4.13) *hold.*
(c) $A \in (c_0^p(\Lambda), c)$ *if and only if conditions* (4.16), (4.17) *and* (4.14) *hold.*
(d) *If Y is any of the spaces ℓ_∞, c_0 or c, $c_0^p(\Lambda)$ has the norm $\|\cdot\|_{\tilde{c}_\infty^p(\Lambda)}$ and $A \in (c_0^p(\Lambda), Y)$ then*

$$\|L_A\| = \|A\|_{(c_0^p(\Lambda),\ell_\infty)}. \tag{4.18}$$

Proof. Parts (a), (b) and (c) are [8, Theorem 3.4(1.)–(3.)].

(d) If $A \in (c_0^p(\Lambda), \ell_\infty)$ then $\|A\|^*_{\tilde{c}_\infty^p(\Lambda)} = \|L_A\|$ by Proposition 4.2(a). Since $\|A\|^*_{\tilde{c}_\infty^p(\Lambda)} = \sup_n \|A_n\|^*_{\tilde{c}_\infty^p(\Lambda)}$ for all n, the conclusion follows from (3.4) in Theorem 3.1. Finally, since $(c_0^p(\Lambda), c_0) \subset (c_0^p(\Lambda), c) \subset (c_0^p(\Lambda), \ell_\infty)$, the assertion also follows for $Y = c_0$ or $Y = c$, by what we have just shown and Parts (b) and (c). $\square$

Finally we determine the classes $(c_0^p(\Lambda), c_\infty(\mu))$, $(c_0^p(\Lambda), c_0(\mu))$ and $(c_0^p(\Lambda), c(\mu))$.

Theorem 4.4. *We have*:

(a) $A \in (c_0^p(\Lambda), c_\infty(\mu))$ *if and only if condition* (4.17) *holds and*

$$\begin{aligned} \|A\|_{(c_0^p(\Lambda),c_\infty(\mu))} = \sup_m \max_{N_m \subset \{0,\ldots,m\}} \sum_{v=0}^{\infty} \lambda_{k(v+1)} \\ \times \left(\sum\nolimits_v \left| \frac{1}{\mu_m} \sum_{n \in N_m} \left(\mu_n \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j} - \mu_{n-1} \sum_{j=k}^{\infty} \frac{a_{n-1,j}}{\lambda_j} \right) \right|^q \right)^{1/q} \\ < \infty. \end{aligned} \tag{4.19}$$

(b) $A \in (c_0^p(\Lambda), c_0(\mu))$ *if and only if conditions* (4.17) *and* (4.19) *hold and*

$$\lim_m \left(\frac{1}{\mu_m} \sum_{n=0}^{m} \left| \mu_n \sum_{j=k}^{\infty} a_{nj} - \mu_n \sum_{j=k}^{\infty} a_{n-1,j} \right| \right) = 0 \quad \textit{for each } k. \tag{4.20}$$

(c) $A \in (c_0^p(\Lambda), c(\mu))$ *if and only if conditions* (4.17) *and* (4.19) *hold and*

$$\lim_m \left(\frac{1}{\mu_m} \sum_{n=0}^{m} \left| \mu_n \sum_{j=k}^{\infty} a_{nj} - \mu_n \sum_{j=k}^{\infty} a_{n-1,j} \right| \right) = \alpha_k \quad \textit{for each } k. \tag{4.21}$$

(d) *If Y is any of the spaces $c_\infty(\mu)$, $c_0(\mu)$ or $c(\mu)$, $c_0^p(\Lambda)$ has the norm $\|\cdot\|_{\bar{c}_\infty^p(\Lambda)}$ and $A \in (c_0^p(\Lambda), Y)$ then*

$$\|A\|_{(c_0^p(\Lambda), c_\infty(\mu))} \leqslant \|L_A\| \leqslant 4 \cdot \|A\|_{(c_0^p(\Lambda), c_\infty(\mu))}. \tag{4.22}$$

Proof. Parts (a), (b) and (c) are [8, Theorem 3.4(4.)–(6.)].

(d) If X is a BK space and $A \in (X, Y)$, then by [10, Corollary 3.49 (3.109), p. 210], for

$$\|A\|_{(X,c_\infty)} = \sup_m \left(\max_{N_m \subset \{0,\dots,m\}} \left\| \frac{1}{\mu_m} \sum_{n \in N_m} (\mu_n A_n - \mu_{n-1} A_{n-1}) \right\|_X^* \right),$$

we have

$$\|A\|_{(X,c_\infty(\mu))} \leqslant \|L_A\| \leqslant 4 \cdot \|A\|_{(X,c_\infty)}.$$

Now (4.22) follows from (3.4) in Theorem 3.1. □

5. Measure of noncompactness and transformations

If X and Y are metric spaces, then $f : X \to Y$ is a compact map if $f(Q)$ is a relatively compact (i.e., if the closure of $f(Q)$ is a compact subset of Y) subset of Y for each bounded subset Q of X. In this section, among other things, we investigate when in some special cases the operator L_A is compact. Our investigations use the measure of noncompactness. Recall that if Q is a bounded subset of a metric space X, then the *Hausdorff measure of noncompactness* of Q is denoted by $\chi(Q)$, and

$$\chi(Q) = \inf\{\epsilon > 0 : Q \text{ has a finite } \epsilon - \text{net in } X\}.$$

The function χ is called the *Hausdorff measure of noncompactness*, and for its properties see [1,2,12]. Let us point out that the notation of the measure of noncompactness has proved useful in several areas of functional analysis, operator theory, fixed point theory, differential equations, etc. Denote by $\overline{Q}$ the closure of Q. For the convenience of the reader, let us mention that: If Q, Q_1 and Q_2 are bounded subsets of a metric space (X, d), then

$$\chi(Q) = 0 \iff Q \text{ is a totally bounded set},$$

$$\chi(Q) = \chi(\overline{Q}),$$

$$Q_1 \subset Q_2 \ \Rightarrow\ \chi(Q_1) \leqslant \chi(Q_2),$$

$$\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\},$$

$$\chi(Q_1 \cap Q_2) \leqslant \min\{\chi(Q_1), \chi(Q_2)\}.$$

If our space X is a normed space, then the function $\chi(Q)$ has some additional properties connected with the linear structure. We have e.g.

$$\chi(Q_1 + Q_2) \leqslant \chi(Q_1) + \chi(Q_2),$$
$$\chi(\lambda Q) = |\lambda|\chi(Q) \quad \text{for each } \lambda \in \mathbb{C}.$$

If X and Y are normed spaces, and $A \in B(X, Y)$, then the Hausdorff measure of noncompactness of A, denoted by $\|A\|_\chi$, is defined by $\|A\|_\chi = \chi(\mathrm{AK})$, where $K = \{x \in X : \|x\| \leqslant 1\}$ is the unit ball in X. Furthermore, A is compact if and only if $\|A\|_\chi = 0$; we also have $\|A\|_\chi \leqslant \|A\|$.

Recall the following well-known result (see e.g. [2, Theorem 6.1.1] or [1, 1.8.1]): Let X be a Banach space with a Schauder basis $\{e_1, e_2, \ldots\}$, Q be a bounded subset of X, and $P_n : X \to X$ be the projector onto the linear span of $\{e_1, e_2, \ldots, e_n\}$. Then

$$\frac{1}{a}\limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I - P_n)x\|\right) \leqslant \chi(Q)$$

$$\leqslant \inf_n \sup_{x\in Q}\|(I - P_n)x\| \leqslant \limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I - P_n)x\|\right), \tag{5.1}$$

where $a = \limsup_{n\to\infty}\|I - P_n\|$.

Theorem 5.1. *Let A be an infinite matrix, $1 < p < \infty$, $q = p/(p-1)$ and for any integers n and r with $n > r$, set*

$$\|A\|^{(r)}_{(w_0^p(\Lambda),\ell_\infty)} = \sup_{n>r}\sum_{v=0}^{\infty}\lambda_{k(v+1)}\left(\sum\nolimits_v |a_{nk}|^q\right)^{1/q}.$$

(a) *If $A \in (w_0^p(\Lambda), c_0)$, then*

$$\|L_A\|_\chi = \lim_{r\to\infty}\|A\|^{(r)}_{(w_0^p(\Lambda),\ell_\infty)}. \tag{5.2}$$

(b) *If $A \in (w_0^p(\Lambda), c)$, then*

$$\frac{1}{2}\lim_{r\to\infty}\|A\|^{(r)}_{(w_0^p(\Lambda),\ell_\infty)} \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty}\|A\|^{(r)}_{(w_0^p(\Lambda),\ell_\infty)}. \tag{5.3}$$

(c) *If* $A \in (w_0^p(\Lambda), \ell_\infty)$, *then*

$$0 \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty} \|A\|^{(r)}_{(w_0^p(\Lambda),\ell_\infty)}. \tag{5.4}$$

Proof. Let us remark that the limits in (5.2)–(5.4) exist. Set $K = \{x \in w_0^p(\Lambda) : \|x\|_{\tilde{w}_0^p(\Lambda)} \leqslant 1\}$. (a) By inequality (5.1), we have

$$\|L_A\|_\chi = \chi(\mathrm{AK}) = \lim_{r\to\infty}\left[\sup_{x\in K}\|(I-P_r)Ax\|\right], \tag{5.5}$$

where $P_r : c_0 \to c_0$ $(r = 0, 1, \ldots)$ is the projector on the first $r+1$ coordinates, i.e., $P_r(x) = (x_0, \ldots, x_r, 0, 0, \ldots)$, for $x = (x_k) \in c_0$; let us remark that $\|I - P_r\| = 1$ for $r = 1, 2, \ldots$ Let $A_{(r)} = (\tilde{a}_{nk})$ be the infinite matrix defined by $\tilde{a}_{nk} = 0$ if $0 \leqslant n \leqslant r$ and $\tilde{a}_{nk} = a_{nk}$ if $r < n$. Now, by Theorem 4.1(d) we have

$$\sup_{x\in K}\|(I-P_r)Ax\| = \|L_{A_{(r)}}\| = \|A_{(r)}\|_{(w^p(\Lambda),\ell_\infty)} = \|A\|^{(r)}_{(w^p(\Lambda),\ell_\infty)}. \tag{5.6}$$

Clearly, by (5.5) and (5.6) we get (5.2).

(b) Let us remark that every sequence $x = (x_k)_{k=0}^\infty \in c$ has a unique representation $x = le + \sum_{k=0}^\infty (x_k - l)e^{(k)}$ where $l \in \mathbb{C}$ is such that $x - le \in c_0$. Let us define $P_r : c \to c$ by $P_r(x) = le + \sum_{k=0}^m (x_k - l)e^{(k)}$ $(r = 0, 1, \ldots)$. It is known that $\|I - P_r\| = 2$ $(r = 0, 1, \ldots)$. Now the proof of (b) is similar as in the case (a), and we omit it (it should be borne in mind that now a in (5.1) is 2).

(c) Let us prove (5.4). Now define $P_r : \ell_\infty \to \ell_\infty$, by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, \ldots)$, $x = (x_k) \in \ell_\infty$ $(r = 0, 1, \ldots)$. It is clear that

$$\mathrm{AK} \subset P_r(\mathrm{AK}) + (I - P_r)(\mathrm{AK}).$$

Now, by the elementary properties of the function χ we have

$$\begin{aligned}\chi(\mathrm{AK}) &\leqslant \chi(P_r(\mathrm{AK})) + \chi((I-P_r)(\mathrm{AK}))\\ &= \chi((I-P_r)(\mathrm{AK}) \leqslant \sup_{x\in K}\|(I-P_r)Ax\| = \|L_{A_{(r)}}\|.\end{aligned} \tag{5.7}$$

By (5.7) and Theorem 4.1(d) we get (5.4). □

Now as a corollary of the above theorem we have

Corollary 5.1
(a) *If* $A \in (w_0^p(\Lambda), c_0)$ *or* $A \in (w_0^p(\Lambda), c)$, *then*

$$L_A \text{ is compact if and only if } \lim_{r\to\infty}\|A\|^{(r)}_{(w_0^p(\Lambda),\ell_\infty)} = 0. \tag{5.8}$$

(b) *If* $A \in (w_0^p(\Lambda), \ell_\infty)$, *then*

$$L_A \text{ is compact if } \lim_{r\to\infty}\|A\|^{(r)}_{(w_0^p(\Lambda),l_\infty)} = 0. \tag{5.9}$$

The following example will show that it is possible for L_A in (5.9) to be compact in the case $\lim_{r\to\infty}\|A\|^{(r)}_{(w_0^p(\Lambda),l_\infty)} > 0$, and hence in general we have just "if" in (5.9).

Example 5.1. Let the matrix A be defined by $A_n = e^{(k(0))}$ for $n = 0, 1, \ldots$ Then $A \in (w_0^p(\Lambda), \ell_\infty)$ and

$$\|A\|^{(r)}_{(w_0^p(\Lambda),\ell_\infty)} = \lambda_{k(1)} \quad \text{for each } r.$$

Hence $\lim_{r\to\infty}\|A\|^{(r)}_{(w_0^p(\Lambda),\ell_\infty)} = \lambda_{k(1)} > 0$. Since $L_A(x) = x_{k(0)}e$ for each $x \in w_0^p(\Lambda)$, L_A is a compact operator.

The proof of the following theorem follows from Theorem 4.2 by the method of Theorem 5.1.

Theorem 5.2. *Let $1 < p < \infty$, $q = p/(p-1)$, and for any integers n and r with $n > r$, set*

$$\|A\|^{(r)}_{(v_0^p(\Lambda),\ell_\infty)} = \sup_{n>r}\sum_{v=0}^{\infty}\lambda_{k(v+1)}\left(\sum\nolimits_v\left|\sum_{j=k}^{\infty}a_{nj}\right|^q\right)^{1/q}.$$

If $A \in (v_0^p(\Lambda), c_0)$, then

$$\|L_A\|_\chi = \lim_{r\to\infty}\|A\|^{(r)}_{(v_0^p(\Lambda),\ell_\infty)}.$$

If $A \in (v_0^p(\Lambda), c)$, then

$$\frac{1}{2}\lim_{r\to\infty}\|A\|^{(r)}_{(v_0^p(\Lambda),\ell_\infty)} \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty}\|A\|^{(r)}_{(v_0^p(\Lambda),\ell_\infty)}.$$

If $A \in (v_0^p(\Lambda), \ell_\infty)$, then

$$0 \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty}\|A\|^{(r)}_{(v_0^p(\Lambda),\ell_\infty)}.$$

Corollary 5.2

(a) *If $A \in (v_0^p(\Lambda), c_0)$ or $A \in (v_0^p(\Lambda), c)$, then*

$$L_A \text{ is compact if and only if } \lim_{r\to\infty}\|A\|^{(r)}_{(v_0^p(\Lambda),\ell_\infty)} = 0.$$

(b) *If $A \in (v_0^p(\Lambda), \ell_\infty)$, then*

$$L_A \text{ is compact if } \lim_{r\to\infty}\|A\|^{(r)}_{(v_0^p(\Lambda),\ell_\infty)} = 0. \tag{5.10}$$

The following example will show that it is possible for L_A in (5.10) to be compact in the case $\lim_{r\to\infty}\|A\|^{(r)}_{(w_0^p(\Lambda),\ell_\infty)} > 0$, and hence in general we have just "if" in (5.10).

Example 5.2. Let the matrix A be as in Example 5.1. Then $A \in (v_0^p(\Lambda), \ell_\infty)$ by Theorem 4.2(a), L_A is compact and

$$\begin{aligned}\|A\|^{(r)}_{(v_0^p(\Lambda),\ell_\infty)} &= \sup_{n>r} \sum_{v=0}^{\infty} \lambda_{k(v+1)} \left(\sum_{k=k(v)}^{k(v+1)-1} \left| \sum_{j=k}^{\infty} a_{nj} \right|^q \right)^{1/q} \\ &= \sup_{n>r} \lambda_{k(1)} \left(\sum_{k=k(0)}^{k(1)-1} \left| \sum_{j=k}^{\infty} a_{nj} \right|^q \right)^{1/q} \\ &= \sup_{n>r} \lambda_{k(1)} = \lambda_{k(1)} \quad \text{for } r = 0, 1, \ldots\end{aligned}$$

Hence

$$\lim_{r\to\infty} \|A\|^{(r)}_{(v_0^p(\Lambda),\ell_\infty)} = \lambda_{k(1)} > 0.$$

The following theorem follows from Theorem 4.3 analogously as in the proof of Theorem 5.1.

Theorem 5.3. *Let* $1 < p < \infty$, $q = p/(p-1)$, *and for any integers n and r with* $n > r$, *set*

$$\|A\|^{(r)}_{(c_0^p(\Lambda),\ell_\infty)} = \sup_{n>r} \sum_{v=0}^{\infty} \lambda_{k(v+1)} \left(\sum\nolimits_v \left| \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j} \right|^q \right)^{1/q}.$$

(a) *If* $A \in (c_0^p(\Lambda), c_0)$, *then*

$$\|L_A\|_\chi = \lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),\ell_\infty)}.$$

(b) *If* $A \in (c_0^p(\Lambda), c)$, *then*

$$\frac{1}{2} \lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),\ell_\infty)} \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),\ell_\infty)}.$$

(c) *If* $A \in (c_0^p(\Lambda), \ell_\infty)$, *then*

$$0 \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),\ell_\infty)}.$$

As a corollary we have

Corollary 5.3
(a) *If* $A \in (c_0^p(\Lambda), c_0)$ *or* $A \in (c_0^p(\Lambda), c)$, *then*

$$L_A \text{ compact if and only if } \lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),\ell_\infty)} = 0.$$

(b) *If* $A \in (c_0^p(\Lambda), \ell_\infty)$, *then*

$$L_A \text{ compact if } \lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),\ell_\infty)} = 0. \tag{5.11}$$

The following example will show that it is possible for L_A in (5.11) to be compact in the case $\lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),\ell_\infty)} > 0$, and hence in general we have just "if" in (5.11).

Example 5.3. Let the matrix A be as in Example 5.1. Then $A \in (c_0^p(\Lambda), \ell_\infty)$ by Theorem 4.3(a), L_A is compact and

$$\begin{aligned}\|A\|^{(r)}_{(c_0^p(\Lambda),\ell_\infty)} &= \sup_{n>r} \sum_{v=0}^{\infty} \lambda_{k(v+1)} \left(\sum_{k=k(v)}^{k(v+1)-1} \left| \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j} \right|^q \right)^{1/q} \\ &= \sup_{n>r} \lambda_{k(1)} \left(\sum_{k=k(0)}^{k(1)-1} \left| \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j} \right|^q \right)^{1/q} \\ &= \sup_{n>r} \frac{\lambda_{k(1)}}{\lambda_{k(0)}} = \frac{\lambda_{k(1)}}{\lambda_{k(0)}} \quad \text{for } r = 0, 1, \ldots\end{aligned}$$

Hence

$$\lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),\ell_\infty)} = \frac{\lambda_{k(1)}}{\lambda_{k(0)}} > 0.$$

Theorem 5.4. *Let A be an infinite matrix*, $1 < p < \infty$, $q = p/(p-1)$ *and for any integers n and r with $n > r$, set*

$$\begin{aligned}\|A\|^{(r)}_{(c_0^p(\Lambda),c_\infty(\mu))} = \sup_{m>r} \max_{N_{r,m}\subset\{r+1,\ldots,m\}} &\sum_{v=0}^{\infty} \lambda_{k(v+1)} \\ &\times \left(\sum\nolimits_v \left| \frac{1}{\mu_m} \sum_{n\in N_{r,m}} \left(\mu_n \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j} - \mu_{n-1} \sum_{j=k}^{\infty} \frac{a_{n-1,j}}{\lambda_j} \right) \right|^q \right)^{1/q} \\ &< \infty.\end{aligned} \tag{5.12}$$

(a) *If* $A \in (c_0^p(\Lambda), c_0(\mu))$, *then*

$$\lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),c_\infty(\mu))} \leqslant \|L_A\|_\chi \leqslant 4 \lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),c_\infty(\mu))}. \tag{5.13}$$

(b) *If* $A \in (c_0^p(\Lambda), c(\mu))$, *then*

$$\frac{1}{2}\lim_{r\to\infty}\|A\|^{(r)}_{(c_0^p(\Lambda),c_\infty(\mu))} \leqslant \|L_A\|_\chi \leqslant 4\lim_{r\to\infty}\|A\|^{(r)}_{(c_0^p(\Lambda),c_\infty(\mu))}. \tag{5.14}$$

(c) *If* $A \in (c_0^p(\Lambda), c_\infty(\mu))$, *then*

$$0 \leqslant \|L_A\|_\chi \leqslant 4\lim_{r\to\infty}\|A\|^{(r)}_{(c_0^p(\Lambda),c_\infty(\mu))}. \tag{5.15}$$

Proof. Since $c_0(\mu)$ has AK [5, Theorem 2(c)] every sequence $x = (x_k)_{k=0}^\infty \in c_0(\mu)$ has a unique representation $x = \sum_{k=0}^\infty x_k e^{(k)}$. Let $P_r : c_0(\mu) \to c_0(\mu)$ $(r = 0, 1, \ldots)$ be the projector on the first $r+1$ coordinates, that is $P_r(x) = (x_0, \ldots, x_r, 0, 0, \ldots)$ for $x = (x_k)_{k=0}^\infty \in c_0(\mu)$. It follows that

$$\begin{aligned}\|(I-P_r)(x)\| &= \|(\underbrace{0,\ldots,0}_{r+1}, x_{r+1}, x_{r+2}, \ldots)\| \\ &= \sup\left\{\frac{1}{\mu_{r+k}}\left(|\mu_{r+1}x_{r+1}| + \sum_{j=r+2}^{r+k}|\mu_j x_j - \mu_{j-1}x_{j-1}|\right) : k = 1, 2, \ldots\right\}.\end{aligned} \tag{5.16}$$

From

$$\begin{aligned}|\mu_{r+1}x_{r+1}| \leqslant{}& |\mu_{r+1}x_{r+1} - \mu_r x_r| + |\mu_r x_r - \mu_{r-1}x_{r-1}| \\ &+ \cdots + |\mu_1 x_1 - \mu_0 x_0| + |\mu_0 x_0|,\end{aligned}$$

we have

$$|\mu_{r+1}x_{r+1}| + \sum_{j=r+2}^{r+k}|\mu_j x_j - \mu_{j-1}x_{j-1}| \leqslant \sum_{j=0}^{r+k}|\mu_j x_j - \mu_{j-1}x_{j-1}| \quad \text{for } k = 1, 2, \ldots \tag{5.17}$$

From (5.16) and (5.17) it follows that

$$\|(I-P_r)(x)\| = \sup_{k=1,2,\ldots}\frac{1}{\mu_{r+k}}\sum_{j=0}^{r+k}|\mu_j x_j - \mu_{j-1}x_{j-1}| \leqslant \|x\|. \tag{5.18}$$

Therefore, $\|I - P_r\| \leqslant 1$. Since $I - P_r$ is a projector, we have $\|I - P_r\| \geqslant 1$. Hence $\|I - P_r\| = 1$, $r = 0, 1, \ldots$ Now the proof of (5.13) follows from Theorem 4.4(b).

(b) Every sequence $x = (x_k)_{k=0}^\infty \in c(\mu)$ has a unique representation $x = le + \sum_{k=0}^\infty (x_k - l)e^{(k)}$ where $l \in \mathbb{C}$ is such that $x - le \in c_0(\mu)$ [3, Theorem 2(c)]. Let us define $Q_r : c(\mu) \mapsto c(\mu)$ by $Q_r(x) = le + \sum_{k=0}^r (x_k - l)e^{(k)}$, that is $Q_r(x) = (x_0, \ldots, x_r, l, l, \ldots)$ for $r = 0, 1, \ldots$ Since

$$\begin{aligned}
&\|(I-Q_r)(x)\| \\
&= \|(\underbrace{0,\ldots,0}_{r+1}, x_{r+1}-l, x_{r+2}-l,\ldots)\| \\
&= \sup\left\{ \frac{1}{\mu_{r+k}} \left(|\mu_{r+1}(x_{r+1}-l)| + \sum_{j=r+2}^{r+k} |\mu_j(x_j-l) - \mu_{j-1}(x_{j-1}-l)| \right) : k=1,2,\ldots \right\} \\
&\leqslant \sup\left\{ |l| + \frac{1}{\mu_{r+k}} \left(|\mu_{r+1}x_{r+1}| + \sum_{j=r+2}^{r+k} |\mu_j x_j - \mu_{j-1}x_{j-1}| \right) : k=1,2,\ldots \right\} \\
&= |l| + \sup\left\{ \frac{1}{\mu_{r+k}} \left(|\mu_{r+1}x_{r+1}| + \sum_{j=r+2}^{r+k} |\mu_j x_j - \mu_{j-1}x_{j-1}| \right) : k=1,2,\ldots \right\},
\end{aligned}$$

we obtain from (5.15) and (5.17) we get

$$\|(I-Q_r)x\| \leqslant |l| + \|x\|. \tag{5.19}$$

Let us prove that $|l| \leqslant \|x\|$. From $x - le \in c_0(\mu)$ it follows that

$$\lim_{n\to\infty} \frac{1}{\mu_n} \sum_{k=0}^{n} |\mu_k(x_k - l) - \mu_{k-1}(x_{k-1} - l)| = 0. \tag{5.20}$$

Since μ is a nondecreasing sequence, we have

$$\begin{aligned}
|l| &= \frac{1}{\mu_n} \sum_{k=0}^{n} |\mu_{k-1} l - \mu_k l| \\
&= \frac{1}{\mu_n} \sum_{k=0}^{n} |\mu_k x_k - \mu_k l - \mu_{k-1}x_{k-1} + \mu_{k-1} l - \mu_k x_k + \mu_{k-1}x_{k-1}| \\
&= \frac{1}{\mu_n} \sum_{k=0}^{n} |\mu_k(x_k - l) - \mu_{k-1}(x_{k-1} - l)| + \frac{1}{\mu_n} \sum_{k=0}^{n} |\mu_k x_k - \mu_{k-1}x_{k-1}|,
\end{aligned}$$

that is

$$|l| \leqslant \frac{1}{\mu_n} \sum_{k=0}^{n} |\mu_k(x_k - l) - \mu_{k-1}(x_{k-1} - l)| + \|x\|. \tag{5.21}$$

By (5.20) and (5.21) we get

$$|l| \leqslant \|x\|. \tag{5.22}$$

From (5.19) and (5.22) it follows $\|(I-Q_r)x\| \leqslant 2\|x\|$, that is $\|I - Q_r\| \leqslant 2$. Finally by Theorem 4.4. we get (5.13).

(c) Inequality (5.14) could be proved similarly as inequality (5.4). □

Now as a corollary of Theorem 5.4 we have

Corollary 5.4
(a) *If* $A \in (c_0^p(\Lambda), c_0(\mu))$ *or* $A \in (c_0^p(\Lambda), c(\mu))$, *then*

$$L_A \text{ compact if and only if } \lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),c_\infty(\mu))} = 0.$$

(b) *If* $A \in (c_0^p(\Lambda), c_\infty(\mu))$, *then*

$$L_A \text{ compact if } \lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),c_\infty(\mu))} = 0. \tag{5.23}$$

The following example will show that it is possible for L_A in (5.23) to be compact in the case $\lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),c_\infty(\mu))} > 0$, and hence in general in (5.23) we have just "if".

Example 5.4. Let the matrix A be as in Example 5.1. Then $A \in (c_0^p(\Lambda), c_\infty(\mu))$ by Theorem 4.4(a), L_A is compact and

$$\begin{aligned}
\|A\|^{(r)}_{(c_0^p(\Lambda),c_\infty(\mu))} &= \sup_{m>r} \max_{N_{r,m}\subset\{r+1,\dots,m\}} \lambda_{k(1)} \left| \frac{1}{\mu_m} \sum_{n\in N_{r,m}} \left(\mu_n \frac{1}{\lambda_{k(0)}} - \mu_{n-1} \frac{1}{\lambda_{k(0)}} \right) \right| \\
&= \sup_{m>r} \frac{\lambda_{k(1)}}{\lambda_{k(0)}} \frac{\mu_m - \mu_{r+1}}{\mu_m} = \frac{\lambda_{k(1)}}{\lambda_{k(0)}} \sup_{m>r} \left(1 - \frac{\mu_{r+1}}{\mu_m} \right) \\
&= \frac{\lambda_{k(1)}}{\lambda_{k(0)}} \quad \text{for each } r.
\end{aligned}$$

Thus

$$\lim_{r\to\infty} \|A\|^{(r)}_{(c_0^p(\Lambda),c_\infty(\mu))} = \frac{\lambda_{k(1)}}{\lambda_{k(0)}} > 0.$$

In the proof of Theorem 5.4 we investigate the projector Q_r. Now we would like to finish this paper with the following inequality for the norm of the projector $I - Q_r$.

Lemma 5.1. *Let* Q_r $(r = 0, 1, \dots)$ *be the projector considered in the proof of Theorem* 5.4. *Then*

$$2 \limsup_{n\to\infty} \frac{\mu_{n-1}}{\mu_n} \leqslant \|I - Q_r\| \leqslant 2 \quad \textit{for all } r = 0, 1, \dots \tag{5.24}$$

Proof. From the proof of Theorem 5.4 we know that $\|I - Q_r\| \leqslant 2$ for all r. Hence, it is enough to prove $2 \limsup_{n\to\infty} (\mu_{n-1}/\mu_n) l_e \|I - Q_r\|$. To prove this let us consider the sequence $b^{(n)}$, $n > r + 1$, from Proposition 2.1(b). Now

$$b^{(n)} - e = (\underbrace{-1, \ldots, -1}_{n}, 0, 0, \ldots) \in c_0(\mu)$$

implies $b^{(n)} \in c(\mu)$. Since $\|b^{(n)}\| = 1$, it follows that

$$\begin{aligned}\|I - Q_r\| &\geqslant \|(I - Q_r)(b^{(n)})\\ &= \|(\underbrace{0, \ldots, 0}_{r+1}, \underbrace{-1, \ldots, -1}_{n-(r+1)}, 0, 0, \ldots)\|\\ &= \sup\left\{1, 2\frac{\mu_{n-1}}{\mu_n}, 2\frac{\mu_{n-1}}{\mu_{n+1}}, 2\frac{\mu_{n-1}}{\mu_{n+2}}, \ldots\right\}\\ &\geqslant 2\frac{\mu_{n-1}}{\mu_n}.\end{aligned}$$

Since it is true for each $n > r + 1$, we get (5.24). □

Let us remark that if $\mu_n = n + 1$ for $n = 0, 1, \ldots,$ then $\|I - Q_r\| = 2$ for $r = 0, 1, \ldots$

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