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POLYNOMIALLY RIESZ ELEMENTS

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ABSTRACT. A Banach algebra element $a \in A$ is said to be "polynomially Riesz", relative to the homomorphism $T : A \to B$, if there exists a nonzero complex polynomial $p(z)$ such that the image $Tp(a) \in B$ is quasinilpotent.

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1. Introduction. Let $\mathbb{C}$ denote the set of all complex numbers. If A is a complex Banach algebra with identity 1 and invertible group $A^{-1} = A^{-1}_{left} \cap A^{-1}_{right}$, intersection of the semigroups of the left and of the right invertibles, then we write

$$\sigma(a) \equiv \sigma_A(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin A^{-1}\}$$

for the *spectrum* of $a \in A$, and similarly for the left and the right spectrum.

For $K \subset \mathbb{C}$, ∂K denotes the topological boundary of K. We recall

$$\partial\sigma(a) \subseteq \sigma^{left}(a) \cap \sigma^{right}(a) \subseteq \sigma(a). \tag{1.1}$$

If K is a compact set, $K \subseteq \mathbb{C}$, we shall write ηK for the *connected hull* of K, where the complement $\mathbb{C} \setminus \eta K$ is the unique unbounded component of the complement $\mathbb{C} \setminus K$ ([5]; [3], Definition 7.10.1). A hole of K is a component of $\eta K \setminus K$. Generally ([5], Theorem 1.2, Theorem 1.3; [3], Theorem 7.10.3), for compact subsets $H, K \subseteq \mathbb{C}$,

$$\partial H \subseteq K \subseteq H \Longrightarrow \partial H \subseteq \partial K \subseteq K \subseteq H \subseteq \eta K = \eta H, \tag{1.2}$$

and H can be obtained from K by filling in some holes of K.

Evidently, if $K \subseteq \mathbb{C}$ is finite, then $\eta K = K$.

Therefore, for compact subsets $H, K \subseteq \mathbb{C}$,

$$\eta K = \eta H \Longrightarrow (H \text{ is finite} \Longleftrightarrow K \text{ is finite}), \tag{1.3}$$

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and in that case $H = K$.

The quasinilpotents of A form the set

$$\begin{aligned} QN(A) &= \{a \in A : \|a^n\|^{1/n} \to 0\} = \{a \in A : \sigma(a) = \{0\}\} \\ &= \{a \in A : 1 - \mathbb{C}a \subset A^{-1}\}. \end{aligned}$$

Recall that

$$a,\ b \in QN(A),\ ab = ba \Longrightarrow a + b \in QN(A), \tag{1.4}$$

$$a \in QN(A),\ b \in A,\ ab = ba \Longrightarrow ab \in QN(A). \tag{1.5}$$

The radical of A is the set

$$\mathrm{Rad}(A) = \{d \in A : 1 - Ad \subseteq A^{-1}\} = \{d \in A : 1 - dA \subseteq A^{-1}\}.$$

It is well-known that $\mathrm{Rad}(A)$ is a two-sided ideal.

A map $T : A \to B$ is a *homomorphism* if T is linear and satisfies $T(xy) = TxTy$, $x,\ y \in A$, and $T1 = 1$. The homomorphism T has the *Riesz property* if 0 is the only one possible point of accumulation of $\sigma(a)$ for every $a \in T^{-1}(0)$, that is, if $Ta = 0$, then $\sigma(a)$ is either finite or a sequence converging to 0 [1]. The homomorphism T has the *strong Riesz property* if

$$\forall\ a \in A\ : \partial\sigma(a) \subset \sigma(Ta) \cup \mathrm{iso}\,\sigma(a), \tag{1.6}$$

where $\mathrm{iso}\,\sigma(a)$ denotes the set of the isolated points of $\sigma(a)$. By the *essential boundary-hull theorem* ([5], Theorem 4.2) the strong Riesz property can be rewritten

$$\forall\ a \in A\ :\ \sigma(a) \subseteq \eta\sigma(Ta) \cup \mathrm{iso}\ \sigma(a). \tag{1.7}$$

From (1.7) it is clear that the strong Riesz property implies the Riesz property. In [4], Theorem 4 it was shown that if $T : A \to B$ is bounded homomorphism with closed range, then the Riesz property implies the strong Riesz property. For unbounded T this was shown in [6], Corollary 7.9.

Let $T : A \to B$ be a homomorphism of complex Banach algebras which is not necessarily bounded.

We shall describe $a \in A$ as T *Riesz* if

$$T(a) \in \mathrm{QN}(B).$$

We shall say that $a \in A$ is *left (right)* T *Fredholm* if it has a left (right) invertible image:

$$a \in T^{-1}(B^{-1}_{left}) \quad (a \in T^{-1}(B^{-1}_{right})).$$

An element $a \in A$ is T *Fredholm* if it has an invertible image ([1], [3]),

$$a \in T^{-1}(B^{-1}),$$

and T *Weyl* if it is the sum of an invertible and one whose image is zero:

$$a = c + d \text{ with } c \in A^{-1},\ Td = 0.$$

If the previous sum is commutative, then $a \in A$ is T *Browder*:

$$a = c + d \text{ with } c \in A^{-1},\ Td = 0,\ cd = dc.$$

The induced left T Fredholm, right T Fredholm, T Fredholm, T Weyl and T Browder spectra are given by

$$\sigma_T^{left}(a) = \sigma_B^{left}(Ta); \qquad \sigma_T^{right}(a) = \sigma_B^{right}(Ta); \quad \sigma_T(a) = \sigma_B(Ta);$$
$$\omega_T(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not } T \text{ Weyl}\};$$
$$\beta_T(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not } T \text{ Browder}\}.$$

If we say that $a \in A$ is *almost invertible* whenever

$$0 \notin \text{acc } \sigma(a),$$

then there is implication, for arbitrary $a \in A$

$$a\ T \textit{ Fredholm and almost invertible} \Longrightarrow a\ T \textit{ Browder}.$$

This implication was shown in [1], Theorem 1 for bounded homomorphism and it was extended to unbounded T in [7], Corollary 2.5.

Conversely if and only if the homomorphism $T : A \to B$ has the Riesz property, then T Browder elements are almost invertible. This was first shown ([1]; [2]; [3], Theorem 7.7.4) for bounded homomorphisms and extended ([7], Theorem 3.4 and the remark following this theorem) to arbitrary homomorphisms between Banach algebras. Therefore,

$$\beta_T(a) = \sigma_T(a) \cup \text{acc } \sigma(a), \tag{1.8}$$

if and only if T has the Riesz property.

Commuting products of almost invertibles remain almost invertible [3], Theorem 7.5.4. Products of T Fredholm elements remain T Fredholm. Therefore, if T has the Riesz property, then commuting products of T Browder elements are also T Browder. In [2], Theorem 3.8 it was proved for a bounded homomorphism T with the Riesz property that if the product of two commuting elements is T Browder, then they both are T Browder, while H. du T. Mouton, S. Mouton and H. Raubenheimer proved that the boundedness of T can be omitted ([6], Theorem 8.10):

THEOREM 1.1. *Let A and B be Banach algebras, $T : A \to B$ a homomorphism satisfying the Riesz property and $a_1,\ a_2 \in A$ such that $a_1a_2 = a_2a_1$. If a_1a_2 is T Browder, then both a_1 and a_2 are T Browder.*

The following result ([8], Theorem 10.3) holds also for homomorphisms T which are not necessarily bounded.

THEOREM 1.2. *If $T : A \to B$ has the strong Riesz property, then for $d \in A$ each of the following are equivalent:*

$$d \text{ is } T \text{ Riesz}. \tag{1.9}$$

$$(\forall a)\ (ad = da \Longrightarrow \beta_T(a + d) = \beta_T(a)). \tag{1.10}$$

$$\beta_T(d) = \{0\}. \tag{1.11}$$

If $K \subset \mathbb{C}$ is compact, then we write $f \in \text{Holo}(K)$ if f is a complex function which is holomorphic in a neighbourhood of K. In particular we write $f \in \text{Holo}_1(K)$ if f is a complex function which is holomorphic and non-constant on each connected component of a neighbourhood U of K.

If $a \in A$, $f \in \text{Holo}(\sigma_A(a))$ and $T : A \to B$ has the Riesz property, then ([1], Theorem 2)

$$\beta_T(f(a)) = f(\beta_T(a)). \tag{1.12}$$

We note that for this assertion it is not necessary for T to be bounded.

In this paper we study polynomially quasinilpotent and polynomially Riesz elements in Banach algebras and give some spectral characterizations and perturbations properties of these elements. A Banach algebra element $a \in A$ is said to be polynomially quasinilpotent if there exists a nonzero complex polynomial $p(z)$ such that $p(a)$ is quasinilpotent, and an element $a \in A$ is said to be polynomially Riesz with respect to Banach algebra homomorphism T if there exists a nonzero complex polynomial $p(z)$ such that $Tp(a)$ is quasinilpotent. The assertions in the present paper which refer to Banach algebra homomorphism T are proved without the assumption about the boudedness of T.

The paper is divided into three sections. In Section 2 the polynomially Riesz elements relative to some homomorphism are characterized (Theorem 2.4) in terms of their Fredholm spectra, and, if the homomorphism has the strong Riesz property, in terms of their Browder spectra. This second characterization is a consequence of the fact that the connected hulls of the Fredholm and Browder spectra are equal (and hence they are equal to the connected hull of Weyl spectrum) under the assumption that T has the strong Riesz property (Theorem 2.3, Corollaries 2.1, 2.2). Otherwise the equality of the connected hulls of the Fredholm, Weyl and Browder spectra is proved in [6] under the assumption that T has closed range and the Riesz property [6, Corolaries 7.6, 7.8] which is by [6, Corollary 7.9] a stronger assumption than the assumption that T has the strong Riesz property. In Section 3 we cosider perturbation properties of T Browder and left (right) T Fredholm elements. From Theorem 1.2 it follows that if $T : A \to B$ has the strong Riesz property, $a,\ d \in A$ such that a is T Browder, d is T Riesz and $ad = da$, then $a - d$ is T Browder. This perturbation property of Browder elements is generalised in Theorem 3.1 by replacing "a is T Browder" by "$f(a)$ is T Browder" and "d is T Riesz" by "$f(d)$ is T Riesz" where $f \in \text{Holo}(\sigma(a) \cup \sigma(d))$.

In the following sections, unless we say otherwise, $T : A \to B$ is arbitrary Banach algebra homomorphism which is not necessarily bounded.

2. Polynomially quasinilpotent and polynomially Riesz elements. We shall write

$$H +_{comm} K = \{c + d : (c, d) \in H \times K,\ cd = dc\}$$

for the commuting sum and

$$H \cdot_{comm} K = \{cd : (c, d) \in H \times K,\ cd = dc\}$$

for the commuting product of subsets $H, K \subseteq A$.

We say that $S \subseteq A$ is a *commutative ideal* if

$$S +_{comm} S \subseteq S, \; A \cdot_{comm} S \subseteq S.$$

We shall write Poly $= \mathbf{C}[z]$ for the algebra of complex polynomials. If $S \subseteq A$ is an arbitrary set we shall write that $a \in \text{Poly}^{-1}(S)$ if there exists a nonzero complex polynomial $p(z)$ such that $p(a) \in S$. If $S \subseteq A$ is a commutative ideal, the set

$$\mathcal{P}_a^S = \{p \in \text{Poly} : p(a) \in S\}$$

of polynomials p for which $p(a) \in S$ will be an ideal of the algebra Poly. Since the natural numbers are well ordered there will be a unique polynomial p of minimal degree with leading coefficient 1 contained in $\mathcal{P}_a^S$ which we call the minimal polynomial of a; we shall write $p = \pi_a \equiv \pi_a^S$. Then $\mathcal{P}_a^S$ is generated by $p = \pi_a$, i.e. $\mathcal{P}_a^S = \pi_a \cdot \text{Poly}$.

Evidently, every ideal is a commutative ideal. According to (1.4) and (1.5) we conclude that $QN(A)$ is a commutative ideal in the algebra A. Also, $T^{-1}(QN(B))$ is a commutative ideal in the algebra A.

We shall say that an element $a \in A$ is *polynomially quasinilpotent* and write $a \in \text{Poly}^{-1}\text{QN}(A)$ if there exists a nonzero complex polynomial $p(z)$ such that $p(a) \in QN(A)$.

The following theorem characterizes polynomially quasinilpotent elements of algebra A.

THEOREM 2.1. *Let $a \in A$. Then $a \in \text{Poly}^{-1}\text{QN}(A)$ if and only if $\sigma(a)$ is finite if and only if $\sigma^{left}(a)$ $(\sigma^{right}(a))$ is finite and in that case*

$$\sigma^{left}(a) = \sigma^{right}(a) = \sigma(a) = \pi_a^{-1}(0), \tag{2.1}$$

where π_a is the minimal polynomial of a.

Proof. Suppose that $a \in \text{Poly}^{-1}\text{QN}(A)$. From $\pi_a(a) \in QN(A)$ it follows that

$$\pi_a(\sigma(a)) = \sigma(\pi_a(a)) = \{0\}.$$

Therefore,

$$\sigma(a) \subset \pi_a^{-1}(0), \tag{2.2}$$

and hence, $\sigma(a)$ is finite.

Conversely, suppose that $\sigma(a)$ is finite and let $\sigma(a) = \{\lambda_1, \ldots, \lambda_n\}$. For $p(z) = (z - \lambda_1) \cdot \cdots \cdot (z - \lambda_n)$ we have $\{0\} = p(\sigma(a)) = \sigma(p(a))$, and so, $p(a) \in \text{QN}(A)$.

Let $a \in \text{Poly}^{-1}\text{QN}(A)$ and let λ be a zero of the minimal polynomial π_a. Then $\pi_a(z) = (z - \lambda)q(z)$ and therefore,

$$\pi_a(a) = (a - \lambda)q(a) = q(a)(a - \lambda) \in QN(A). \tag{2.3}$$

We show that $\lambda \in \sigma(a)$. If $\lambda \notin \sigma(a)$, then $a - \lambda$ is an invertible element which commutes with $q(a)$ and hence $(a-\lambda)^{-1}$ commutes with $\pi_a(a)$. From (2.3) and (1.5) it follows that $q(a) \in QN(A)$ which contradicts the fact that the polynomial π_a is minimal. Therefore, $\pi_a^{-1}(0) \subset \sigma(a)$, which together with (2.2) gives $\sigma(a) = \pi_a^{-1}(0)$.

From (1.1), (1.2) and (1.3) it follows that $\sigma(a)$ is finite if and only if $\sigma^{left}(a)$ ($\sigma^{right}(a)$) is finite and in that case there is equality $\sigma^{left}(a) = \sigma^{right}(a) = \sigma(a)$. This completes the proof. □

Recall that from Jacobson's lemma, i.e.

$$1 - ab \in A^{-1} \Longleftrightarrow 1 - ba \in A^{-1}, \quad a,\ b \in A,$$

follows that

$$\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}. \tag{2.4}$$

Theorem 2.2. *Let a, $b \in A$. Then $ab \in \mathrm{Poly}^{-1}\mathrm{QN}(A)$ if and only if $ba \in \mathrm{Poly}^{-1}\mathrm{QN}(A)$ and in that case*

$$\pi_{ab}^{-1}(0) \cup \{0\} = \pi_{ba}^{-1}(0) \cup \{0\}.$$

Proof. Follows from Theorem 2.1 and (2.4). □

If $T : A \to B$ is a homomorphism with closed range which satisfies the Riesz property and which is not necessarily bounded, then for every $a \in A$, $\eta\beta_T(a) = \eta\sigma_T(a)$ ([6], Corollary 7.8). We can improve on this:

Theorem 2.3. *Let $T : A \to B$ be a homomorphism with the strong Riesz property. Then*

$$\partial\beta_T(a) \subset \partial\sigma_T(a) \subset \sigma_T(a) \subset \beta_T(a) \subset \eta\sigma_T(a) = \eta\beta_T(a), \tag{2.5}$$

and $\beta_T(a)$ consists of $\sigma_T(a)$ and possibly some holes in $\sigma_T(a)$.

Proof. According to (1.2) it is enough to prove the inclusion

$$\partial\beta_T(a) \subset \sigma_T(a). \tag{2.6}$$

From (1.8) it follows that $\mathrm{int}\sigma(a) = \mathrm{int}\beta_T(a)$ and hence

$$\partial\beta_T(a) \subset \partial\sigma(a). \tag{2.7}$$

From (2.7), (1.6) and (1.8) it follows

$$\begin{aligned} \partial\beta_T(a) &\subset \partial\sigma(a) \cap \beta_T(a) \\ &\subset (\sigma_T(a) \cup \mathrm{iso}\sigma(a)) \cap (\sigma_T(a) \cup \mathrm{acc}\sigma(a)) \\ &\subset \sigma_T(a). \end{aligned}$$

□

We remark that Theorem 2.3, as well as each of the following assertions of this section, is true without the assumption about the boundedness of T.

COROLLARY 2.1. *Let $T : A \to B$ be a homomorphism with the strong Riesz property. Then*

$$\partial\beta_T(a) \subset \partial\omega_T(a) \subset \omega_T(a) \subset \beta_T(a) \subset \eta\omega_T(a) = \eta\beta_T(a),$$

and $\beta_T(a)$ consists of $\omega_T(a)$ and possibly some holes in $\omega_T(a)$.

Proof. Since $\sigma_T(a) \subset \omega_T(a) \subset \beta_T(a)$, the assertion follows from the inclusion (2.6) and (1.2). □

The following corollary is an improvement of Corollary 7.6 in [6].

COROLLARY 2.2. *Let $T : A \to B$ be a homomorphism with the strong Riesz property. Then*

$$\omega_T(a) \subset \eta\sigma_T(a) = \eta\omega_T(a) = \eta\beta_T(a).$$

Proof. Follows from Theorem 2.3 and Corollary 2.1. □

COROLLARY 2.3. *Let $T : A \to B$ be a homomorphism with the strong Riesz property. Then $\beta_T(a)$ is finite if and only if $\omega_T(a)$ is finite if and only if $\sigma_T(a)$ is finite and in that case these sets are equal.*

Proof. Follows from Theorem 2.3, Corollary 2.2 and (1.3). □

We shall say that an element $a \in A$ is *polynomially T Riesz* and write $a \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN}(B)$ if there exists a nonzero complex polynomial $p(z)$ such that $p(a) \in T^{-1}QN(B)$.

The following result characterizes polynomially Riesz elements and it is an improvement of Theorem 11.1 in [8].

THEOREM 2.4. *Let $T : A \to B$ be a homomorphism. Then a is a polynomially T Riesz element if and only if $\sigma_T(a)$ is finite if and only if $\sigma_T^{left}(a)$ ($\sigma_T^{right}(a)$) is finite and in that case*

$$\sigma_T^{left}(a) = \sigma_T^{right}(a) = \sigma_T(a) = \pi_a^{-1}(0), \tag{2.8}$$

where π_a is the minimal polynomial of a.

If in particular T has the strong Riesz property, then a is a polynomially T Riesz element if and only if $\beta_T(a)$ is finite, and in that case also

$$\beta_T(a) = \pi_a^{-1}(0). \tag{2.9}$$

Proof. Since $a \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN}(B)$ is equivalent to $Ta \in \mathrm{Poly}^{-1}\mathrm{QN}(B)$, the first part of the assertions follows from Theorem 2.1 applied to Ta.

Suppose that T has the strong Riesz property. Then, as we have already proved, a is a polynomially T Riesz element if and only if $\sigma_T(a)$ is finite, and according to Corollary 2.3 this is equivalent to finiteness of $\beta_T(a)$ and in that case

$\beta_T(a) = \sigma_T(a) = \pi_a^{-1}(0)$. □

Let us mention that, in the case when T has the strong Riesz property, the implication:

$$a \in \text{Poly}^{-1}T^{-1}\text{QN}(B) \Longrightarrow \beta_T(a) = \pi_a^{-1}(0)$$

was shown in [8] (Theorem 11.1) in a different way.

Corollary 2.4. *Let $T : A \to B$ be a homomorphism with the strong Riesz property. If $a \in \text{Poly}^{-1}T^{-1}\text{QN}(B)$, then $\sigma(a)$ is at most countable.*

Proof. From (1.8) it follows that the set $\sigma(a) \setminus \beta_T(a)$ consists of isolated points of $\sigma(a)$ and therefore it is at most countable. Since $\beta_T(a)$ is finite by (2.9), we conclude that $\sigma(a)$ is at most countable. □

Corollary 2.5. *Let $T : A \to B$ be a homomorphism and $a,\ b \in A$. Then*

$$ab \in \text{Poly}^{-1}T^{-1}\text{QN}(B) \Longleftrightarrow ba \in \text{Poly}^{-1}T^{-1}\text{QN}(B),$$

and in that case

$$\sigma_T(ab) \cup \{0\} = \pi_{ab}^{-1}(0) \cup \{0\} = \pi_{ba}^{-1}(0) \cup \{0\} = \sigma_T(ba) \cup \{0\}.$$

Proof. Follows from Theorem 2.2 applied to Ta and Tb, and Theorem 2.4. □

In [8, Theorem 12.1] it is proved that if $S \subset A$ is a commutative ideal, $a \in A$ and $f \in \text{Holo}(\sigma(a))$, then there is implication:

$$a \in \text{Poly}^{-1}(S) \Longrightarrow f(a) \in \text{Poly}^{-1}(S).$$

If in particular $f \in \text{Holo}_1(\sigma(a))$, then there is implication:

$$f(a) \in \text{Poly}^{-1}(S) \Longrightarrow a \in \text{Poly}^{-1}(S).$$

Therefore, for every $f \in \text{Holo}_1(\sigma(a))$ it holds:

$$a \in \text{Poly}^{-1}(S) \Longleftrightarrow f(a) \in \text{Poly}^{-1}(S). \tag{2.10}$$

The following corollary is an improvement of Theorem 12.2 in [8].

Corollary 2.6. *Let $T : A \to B$ be a homomorphism, $a \in A$ and $f \in \text{Holo}_1(\sigma(a))$.*

(a) Then $a \in \text{Poly}^{-1}(\text{Rad}(B))$ if and only if $f(a) \in \text{Poly}^{-1}(\text{Rad}(B))$.

(b) More generally, the following conditions are equivalent:

(i) a is a polynomially T Riesz element.

(ii) $\sigma_T(a)$ (or $\sigma_T^{left}(a)$, or $\sigma_T^{right}(a)$) is finite.

(iii) $f(a)$ *is a polynomially* T *Riesz element.*

(iv) $\sigma_T(f(a))$ *(or* $\sigma_T^{left}(f(a))$*, or* $\sigma_T^{right}(f(a))$*) is finite,*
and in that case

$$\begin{aligned}\sigma_T^{left}(a) = \sigma_T^{right}(a) = \sigma_T(a) = \pi_a^{-1}(0),\\ \sigma_T^{left}(f(a)) = \sigma_T^{right}(a) = \sigma_T(f(a)) = f(\pi_a^{-1}(0)),\end{aligned} \tag{2.11}$$

where π_a *is the minimal polynomial of* a*.*

(c) If in particular T *has the strong Riesz property, then the following conditions are equivalent:*

(i) a *is a polynomially* T *Riesz element.*

(ii) $\beta_T(a)$ *is finite.*

(iii) $f(a)$ *is a polynomially* T *Riesz element.*

(iv) $\beta_T(f(a))$ *is finite,*
and in that case also

$$\begin{aligned}\beta_T(a) = \pi_a^{-1}(0),\\ \beta_T(f(a)) = f(\pi_a^{-1}(0)).\end{aligned} \tag{2.12}$$

Proof. The assertion (a) follows from the equivalence (2.10) if we put $S = T^{-1}(\mathrm{Rad}(B))$.

The assertions (b) and (c) follow from Theorem 2.4 and the equivalence (2.10) if we take $S = T^{-1}(\mathrm{QN}(B))$. For the equalities (2.11) and (2.12) see [8, Theorem 12.2] ((2.11) follows also from spectral mapping theorem for Fredholm spectrum and (2.12) follows also from (1.12)). □

3. Perturbations. We shall say that $a \in A$ is holomorphically Riesz if there exists an $f \in \mathrm{Holo}(\sigma(a))$ such that $f(a)$ is Riesz. From the inclusions (12.3) in Theorem 12.1 in [8] it follows that $a \in A$ is a polynomially T Riesz element if and only if there exists a function $f \in \mathrm{Holo}_1(\sigma(a))$ such that $f(a)$ is T Riesz. Since $\mathrm{Holo}_1(\sigma(a)) \subset \mathrm{Holo}(\sigma(a))$, the concept of hollomorphically Riesz element is a little more general than the concept of polynomially Riesz element.

From Theorem 1.2 it follows that if $T : A \to B$ has the strong Riesz property, $a,\ d \in A$ such that a is T Browder, d T Riesz and $ad = da$, then $a - d$ is T Browder. The following result shows that it holds more general, also for T which is not necessarily bounded.

Theorem 3.1. *Let* $a,\ d \in A$ *and let* $T : A \to B$ *be a homomorphism with the strong Riesz property. For* $f \in \mathrm{Holo}(\sigma(a) \cup \sigma(d))$*,*

$$ad = da \quad \text{and} \quad f(d) \in T^{-1}\mathrm{QN}(B)$$

implies

$$f(a) \text{ is } T \text{ Browder} \Longrightarrow a - d \text{ is } \ T \text{ Browder}.$$

Proof. Let f be a nonzero holomorphic function in a neighbourhood U of $\sigma(a) \cup \sigma(d)$, $f(d) \in T^{-1}\mathrm{QN}(B)$, $ad = da$ and let $f(a)$ be T Browder. If Ω is an open set such that $\sigma(a) \cup \sigma(d) \subset \Omega \subset \overline{\Omega} \subset U$ and whose boundary $\partial\Omega$ consists of a finite numbers of simple closed rectifiable curves which do not intersect, then $f(a) = \frac{1}{2\pi i}\int_{\partial\Omega}(\lambda - a)^{-1} f(\lambda) d\lambda$ and $f(d) = \frac{1}{2\pi i}\int_{\partial\Omega}(\lambda - d)^{-1} f(\lambda) d\lambda$. Since $ad = da$ it follows that $(\lambda - a)^{-1}$ and $(\mu - d)^{-1}$ commute for every $\lambda,\ \mu \in \partial\Omega$ and therefore, $f(a)$ and $f(d)$ commute. Hence, $\beta_T(f(a) - f(d)) = \beta_T(f(a))$ by Theorem 1.2, and from $0 \notin \beta_T(f(a))$ it follows that $0 \notin \beta_T(f(a) - f(d))$, i.e.

$$f(a) - f(d) \text{ is } T \text{ Browder.} \tag{3.1}$$

Since

$$f(a) - f(d) = \frac{1}{2\pi i}\int_{\partial\Omega}((\lambda - a)^{-1} - (\lambda - d)^{-1}) f(\lambda) d\lambda$$

and

$$\begin{aligned}
(\lambda - a)^{-1} - (\lambda - d)^{-1} &= (\lambda - a)^{-1}(a - d)(\lambda - d)^{-1} \\
&= (\lambda - a)^{-1}(\lambda - d)^{-1}(a - d) \\
&= (a - d)(\lambda - a)^{-1}(\lambda - d)^{-1},
\end{aligned}$$

we have

$$f(a) - f(d) = a_1(a - d) = (a - d)a_1, \tag{3.2}$$

where $a_1 = \frac{1}{2\pi i}\int_{\partial\Omega}((\lambda - a)^{-1}(\lambda - d)^{-1}) f(\lambda) d\lambda \in A$.

From (3.1), (3.2) and Theorem 1.1 we obtain that $a - d$ is T Browder. □

THEOREM 3.2. *Let $a,\ d \in A$ and let $T : A \to B$ be a homomorphism. For $f \in \mathrm{Holo}(\sigma(a) \cup \sigma(d))$,*

$$ad = da \quad \text{and} \quad f(d) \in T^{-1}\mathrm{QN}(B)$$

implies

$$f(a) \text{ is left (right) } T \text{ Fredholm} \Longrightarrow a - d \text{ is left (right) } T \text{ Fredholm.}$$

Proof. It is similar to the proof of Theorem 3.1 and follows from [8, Theorem 10.1] and the well-known fact that if a_1 and a_2 commute and $a_1 a_2$ is left (right) T Fredholm, then both a_1 and a_2 are left (right) T Fredholm. □

If we add in the assumptions of the previous theorem that $T : A \to B$ is bounded, then we can weaken the commutativity condition putting $ad - da \in T^{-1}(\mathrm{Rad}(B))$. Mention that the proof of the following theorem is similar to the proof of Theorem 2.1 in [10] for bounded linear operators. For the sake of completeness we give the proof.

THEOREM 3.3. *Let $a,\ d \in A$ and let $T : A \to B$ be a bounded homomorphism. For $f \in \mathrm{Holo}(\sigma(a) \cup \sigma(d))$,*

$$ad - da \in T^{-1}(\mathrm{Rad}(B)) \quad \text{and} \quad f(d) \in T^{-1}\mathrm{QN}(B)$$

implies

$$f(a) \text{ is left (right) } T \text{ Fredholm} \Longrightarrow a - d \text{ is left (right) } T \text{ Fredholm.}$$

Proof. Let Ω be as in the proof of Theorem 3.1. Since $T^{-1}(\mathrm{Rad}(B))$ is a two-sided ideal, from $ad - da \in T^{-1}(\mathrm{Rad}(B))$ we obtain that

$$(\lambda - a)^{-1}(\mu - d)^{-1} - (\mu - d)^{-1}(\lambda - a)^{-1} \in T^{-1}(\mathrm{Rad}(B)), \quad \lambda,\ \mu \in \partial\Omega. \tag{3.3}$$

Since $\mathrm{Rad}(B)$ is closed and T is continuous, it follows that $T^{-1}(\mathrm{Rad}(B))$ is closed, and from (3.3) we conclude that

$$f(a)f(d) - f(d)f(a) \in T^{-1}(\mathrm{Rad}(B)).$$

By [8, Theorem10.1] we conclude that $\sigma_T^*(f(a)-f(d)) = \sigma_T^*(f(a))$ where σ_T^* denotes σ_T^{left} (σ_T^{right}). As $f(a)$ is left (right) T Fredholm, it follows that $f(a) - f(d)$ is left (right) T Fredholm. Since $T^{-1}(\mathrm{Rad}(B))$ is a two-sided ideal, for every $\lambda \in \partial\Omega$ we get

$$\begin{aligned}(\lambda - a)^{-1} - (\lambda - d)^{-1} &= (\lambda - a)^{-1}(a - d)(\lambda - d)^{-1} \\ &= (\lambda - a)^{-1}(\lambda - d)^{-1}(a - d) + p_1(\lambda) \\ &= (a - d)(\lambda - a)^{-1}(\lambda - d)^{-1} + p_2(\lambda),\end{aligned}$$

where $p_1(\lambda),\ p_2(\lambda) \in T^{-1}(\mathrm{Rad}(B))$, and since $T^{-1}(\mathrm{Rad}(B))$ is closed we get

$$\begin{aligned}f(a) - f(d) &= \frac{1}{2\pi i}\int_{\partial\Omega}((\lambda - a)^{-1} - (\lambda - d)^{-1})f(\lambda)d\lambda \\ &= a_1(a - d) + b_1 = (a - d)a_1 + b_2,\end{aligned}$$

where $a_1 = \frac{1}{2\pi i}\int_{\partial\Omega}((\lambda - a)^{-1}(\lambda - d)^{-1})f(\lambda)d\lambda \in A$ and

$$\begin{aligned}b_1 &= \frac{1}{2\pi i}\int_{\partial\Omega} p_1(\lambda)f(\lambda)d\lambda \in T^{-1}(\mathrm{Rad}(B)), \\ b_2 &= \frac{1}{2\pi i}\int_{\partial\Omega} p_2(\lambda)f(\lambda)d\lambda \in T^{-1}(\mathrm{Rad}(B)).\end{aligned}$$

Thus $a_1(a - d),\ (a - d)a_1$ is left (right) T Fredholm and hence, $a - d$ is left (right) T Fredholm. □

The assertion of Theorem 3.3 was proved in [8, Theorem 12.3] in a different way for the case when T is a homomorphism which is not necessarily bounded, but $f \in \mathrm{Holo}_1(\sigma(a) \cup \sigma(d))$.

Theorem 3.4. *Let $a,\ d \in A$ and let $T : A \to B$ be a homomorphism with the strong Riesz property. For $f \in \mathrm{Holo}(\sigma(a) \cup \sigma(d))$,*

$$ad = da \quad \text{and} \quad f(d) \in T^{-1}\mathrm{QN}(B)$$

implies

$$\beta_T(a) \cap f^{-1}(0) = \emptyset \Longrightarrow a - d \text{ is } \ T \text{ Browder.}$$

Proof. Let $ad = da$, $f(d) \in T^{-1}\mathrm{QN}(B)$ and let $\beta_T(a) \cap f^{-1}(0) = \emptyset$. Then $0 \notin f(\beta_T(a))$ and since $f(\beta_T(a)) = \beta_T(f(a))$ by (1.12), we get $f(a)$ is T Browder. From Theorem 3.1 it follows that $a - d$ is T Browder. □

Similarly, the following assertions refer to perturbation properties of left and right T Fredholm elements.

Theorem 3.5. *Let $a,\ d \in A$ and let $T : A \to B$ be a homomorphism.*

(i) For $f \in \mathrm{Holo}(\sigma(a) \cup \sigma(d))$,

$$ad = da \quad \text{and} \quad f(d) \in T^{-1}\mathrm{QN}(B)$$

implies

$$\sigma_T^{left}(a) \cap f^{-1}(0) = \emptyset \Longrightarrow a - d \text{ is } \ T \text{ left } T \text{ Fredholm,} \tag{3.4}$$

and

$$\sigma_T^{right}(a) \cap f^{-1}(0) = \emptyset \Longrightarrow a - d \text{ is } \ T \text{ right } T \text{ Fredholm.} \tag{3.5}$$

(ii) For $f \in \mathrm{Holo}_1(\sigma(a) \cup \sigma(d))$,

$$ad - da \in T^{-1}(\mathrm{Rad}(B)) \quad \text{and} \quad f(d) \in T^{-1}\mathrm{QN}(B)$$

implies (3.4) *and* (3.5).

(iii) If T is a bounded homomorphism and $f \in \mathrm{Holo}(\sigma(a) \cup \sigma(d))$, then

$$ad - da \in T^{-1}(\mathrm{Rad}(B)) \quad \text{and} \quad f(d) \in T^{-1}\mathrm{QN}(B)$$

implies (3.4) *and* (3.5).

Proof. Similarly to the proof of Theorem 3.4, (i) follows from Theorem 3.2, (ii) follows from [8, Theorem 12.3] and (iii) follows from Theorem 3.3. □

Remark that the assertion (ii) in Theorem 3.5 is equivalent to Theorem 2.3 i [9] (see the comment at the beginning of this section).

Corollary 3.1. *Let $a,\ d \in A$ and let $T : A \to B$ be a homomorphism with the strong Riesz property. Then*

$$ad = da \quad \text{and} \quad d \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN}(B)$$

implies

$$\beta_T(a) \cap \pi_d^{-1}(0) = \emptyset \Longrightarrow a - d \text{ is } \ T \text{ Browder.}$$

Proof. Follows from Theorem 3.4. □

Corollary 3.2. *Let $a,\ d \in A$ and $T : A \to B$ be a homomorphism. Then*

$$ad - da \in T^{-1}(Rad(B)) \quad \text{and} \quad a,\ d \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN}(B)$$

implies

$$\pi_a^{-1}(0) \cap \pi_d^{-1}(0) = \emptyset \Longrightarrow a - d \text{ is } T \text{ Fredholm.}$$

Particular, if $T : A \to B$ is a homomorphism with the strong Riesz property, then

$$ad = da \quad \text{and} \quad a,\ d \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN}(B)$$

implies

$$\pi_a^{-1}(0) \cap \pi_d^{-1}(0) = \emptyset \Longrightarrow a - d \text{ is } T \text{ Browder.}$$

Proof. Let $a,\ d \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN}(B)$, $ad - da \in T^{-1}(Rad(B))$ and $\pi_a^{-1}(0) \cap \pi_d^{-1}(0) = \emptyset$. From (2.8) it follows that $\sigma_T(a) \cap \pi_d^{-1}(0) = \emptyset$, and from Theorem 3.5 (ii) we get that $a - d$ is T Fredholm.

Similarly, if T has the strong Riesz property, $ad = da$ and $\pi_a^{-1}(0) \cap \pi_d^{-1}(0) = \emptyset$, then $\beta_T(a) \cap \pi_d^{-1}(0) = \emptyset$ according to (2.9), and from Theorem 3.4 it follows that $a - d$ is T Browder. □

Corollary 3.3. *Let $d \in A$ and let $T : A \to B$ be a homomorphism with the strong Riesz property. For $f \in \mathrm{Holo}(\sigma(d))$ and $\lambda \in \mathbb{C}$,*

$$f(d) \in T^{-1}\mathrm{QN}(B) \quad \text{and} \quad f(\lambda) \neq 0$$

implies

$$d - \lambda \ \text{ is } \ T \text{ Browder.}$$

Proof. Suppose that $f(d) \in T^{-1}\mathrm{QN}(B)$ and $f(\lambda) \neq 0$. From Theorem 1.2 it follows that $\beta_T(f(d)) = \{0\}$ and hence, according to (1.12), we get

$$f(\beta_T(d)) = \{0\}. \tag{3.6}$$

Since $f(\lambda) \neq 0$, from (3.6) it follows that $\lambda \notin \beta_T(d)$, that is $d-\lambda$ is T Browder. □

Remark that Corollary 3.3 follows also from Theorem 3.4 for $a = \lambda 1$.

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