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Fredholm theory in irreducible C*-algebras

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Abstract. We use the C* algebra "spectral permanence" for the Moore-Penrose inverse to offer an update of Section 6.3 of the well-known book by Caradus, Pfaffenberger and Yood [2], as well as an extension of Theorem 5.1 in [4].

1. Introduction and preliminaries

If H is a Hilbert space we shall write $B_{00}(H) \subseteq B(H)$ for the two-sided ideal of *finite rank operators*, and $B_0(H)$ for the larger ideal of *compact* operators, and recall that $B_0(H)$ is actually the norm closure of $B_{00}(H)$. In a general Banach algebra A we shall write $A^{-1} = A^{-1}_{left} \cap A^{-1}_{right}$, for the (multiplicative) group of all invertible elements of A, which is indeed the intersection of the larger semigroups A^{-1}_{left} and A^{-1}_{right} of left and of right invertibles.

Let A be a C^* algebra with the unit 1. The well-known Gelfand-Naimark-Segal theorem states that there is a Hilbert space H and a *-isometric homomorphism

$$\Gamma : A \to B(H) ,$$

from A to the algebra $B = B(H)$ of bounded operators on H. This homomorphism Γ is called *Gelfand-Naimark-Segal* (GNS) representation of the algebra A. The effect of the GNS representation is to reduce general C^* algebras to closed *-subalgebras of the algebras $B(H)$: certainly the image $\Gamma(A)$ which C* isomorphic to A really is such a subalgebra, and the GNS theorem itself was responsible for a change in the meaning of the expression "C* algebra" to be synonymous with what used to be call "B* algebras" [10]. The usual terminology is that A is "represented over" H.

When $a \in A$ we shall sometimes write $\widehat{a}$ for the image $\Gamma(a) \in B(H)$. Evidently, since Γ is a *-homomorphism,

$$\widehat{(a^*)} = (\widehat{a})^* = \widehat{a}^* \in B(H)\ (a \in A) .$$

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If $\Gamma : A \to B(H)$ and if there is a closed subspace $V \subseteq H$ for which, for each $T = \widehat{a} \in \Gamma(A)$ we have *reduction*

$$\widehat{a}(V) \subseteq V \; ; \; \widehat{a}(V^\perp) \subseteq V^\perp \; ,$$

then $V \subseteq H$ is said to be a *reducing subspace* for the C* algebra A. Now the algebra A is said to be *irreducible* if the only reducing subspace for A are $\{0\}$ and H.

Irreducible algebras have an important property ([1] Theorem 6.3.3) relating their image in $B(H)$ to the compact ideal $B_0(H)$.

Theorem 1.1. *Let A be an irreducible C^* algebra. If the GNS image of A, $\Gamma(A)$, contains even one non zero compact operator then it contains them all, i.e. there is implication*

1.1 $$\Gamma(A) \cap B_0(H) \neq \{0\} \Longrightarrow B_0(H) \subseteq \Gamma(A) \; .$$

We recall that the finite rank operators $B_{00}(H)$ are generated by the *rank one* operators $H \otimes H$ where if $x, y \in H$ we write

$$(x \otimes y)(w) = \langle w; x\rangle y \; (w \in H) \; .$$

The following shows that if the image of a two-sided ideal of a C^* algebra A contains any non trivial rank one operator then it contains all the finite rank operators. If in particular the ideal is closed then it follows that its image contains all the compact operators.

Theorem 1.2. *Let A be a C^* algebra which is represented over H. If $\Gamma(A)$ contains rank-one operators, and if M is a non-zero two-sided ideal of A, then $B_{00}(H) \subset \Gamma(M)$.*

Moreover, if M is closed, then $B_0(H) = \Gamma(M)$.

Proof. The first part is actually true if H is Babach space, and $\Gamma(A)$ is a subalgebra of $B(H)$ containing rank-one operators. Just go trough the standard proof. This is Theorem 5.2.1 in [2].

The second part follows from $\mathrm{cl}\, B_{00}(H) = B_0(H)$ if H is a Hilbert space. $\square$

For $T \in B(H)$ we use the following notations: $T^{-1}(0)$ - the null space, $T(H)$ - the range, $\mathrm{null}(T) = \dim T^{-1}(0)$, $\mathrm{def}(T) = \mathrm{codim}\, T(H)$, $\mathrm{asc}(T)$ - the ascent, $\mathrm{dsc}(T)$ the descent of T.

If $a \in A$ and A is C^* algebra with the unit 1, then the *spectrum* of a in A is

$$\sigma_A(a) = \sigma_A^{left}(a) \cup \sigma_A^{right}(a) \; ,$$

where

$$\sigma_A^{left}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin A^{-1}_{left}\} \; ,$$

and

$$\sigma_A^{right}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin A^{-1}_{right}\} \; .$$

Obviously, if A is represented over H, $\sigma_A(a) = \sigma_{\Gamma(A)}(\widehat{a})$, and similarly for the left and the right spectrum. It is known that if B is C^* algebra with the unit 1 and if A is C^* subalgebra of B also with the unit 1, then $\sigma_A(a) = \sigma_B(a)$ for all $a \in A$. Therefore, $\sigma_{B(H)}(\widehat{a}) = \sigma_{\Gamma(A)}(\widehat{a}) = \sigma_A(a)$ for all $a \in A$. Hence, if $a \in A$ and $\widehat{a}$ is invertible in $B(H)$, then $a \in A^{-1}$ and $a^{-1} \in A$.

The analogous assertion holds for the left and the right spectrum.

Lemma 1.1. *Let A be a C^* algebra which is represented over H, and let $a \in A$. Then $\widehat{a}$ is left (right) invertible in $B(H)$ if and only if a is left (right) invertible in A.*

In other words

$$\sigma_A^{left}(a) = \sigma_{B(H)}^{left}(\widehat{a}) \text{ and } \sigma_A^{right}(a) = \sigma_{B(H)}^{right}(\widehat{a}).$$

Proof. Clearly, if a is left invertible in A, then $\widehat{a}$ is left invertible in $B(H)$.

To prove the converse, suppose that $\widehat{a}$ is left invertible in $B(H)$. Then $\widehat{a}^{-1}(0) = \{0\}$ and $\widehat{a}(H)$ is closed. Hence $\widehat{a^*}(H)$ is closed and $H = \widehat{a}(H) \oplus (\widehat{a^*})^{-1}(0)$ and so, $\widehat{a^*}(H) = \widehat{a^*}\widehat{a}(H)$ and the range $\widehat{a^*}\widehat{a}(H)$ is closed. Since $(\widehat{a^*}\widehat{a})^{-1}(0) = \widehat{a}^{-1}(0) = \{0\}$, from selfadjointness of $\widehat{a^*}\widehat{a}$ we get that $\widehat{a^*}\widehat{a} = \widehat{a^*a}$ is invertible in $B(H)$. From $a \in A$ it follows that $a^*, a^*a \in A$. Consequently, $(a^*a)^{-1} \in A$. Since $(a^*a)^{-1}a^*a = 1$ and $(a^*a)^{-1}a^* \in A$, we obtain that a is left invertible in A. □

A *generalized inverse* for an element $a \in A$ of a Banach algebra is an element $b \in A$ for which

$$a = aba\ . \tag{1}$$

Evidently necessary and sufficient for $a \in A$ to have as generalized inverse is the condition $a \in aAa$. If $b \in A$ satisfies (1) then so also does $c = bab$, in which case also $c = cac$, so that the relationship between an element and its generalized inverse can be made symmetric. Further if (1) holds then each of the elements $ba = (ba)^2$ and $ab = (ab)^2$ are *idempotent*. When A is a C* algebra then it is possible for the generalized inverse $b \in A$ to also generate self adjoint idempotents ab and ba, so that

$$a = aba\ ;\ b = bab\ ;\ (ab)^* = a\ ;\ (ba)^* = ba\ .$$

It turns out that such a generalized inverse $b \in A$ is determined uniquely if it exists, in which case it referred to as the *Moore Penrose inverse* of $a \in A$, and written $a^\dagger$.

In general it is rather obvious that if $T \in B(X)$ has a generalized inverse $S \in B(X)$ that it must also have closed range: indeed

$$T(X) = TS(X) = (I - TS)^{-1}(0)\ ,$$

since the range of such an operator is the null space of another. For Hilbert spaces the reverse implication is also true: If we specialize to the algebra $B = B(H)$ then elements, i.e. operators have generalized inverses if and only if they have closed range: if $T \in B(H)$ is arbitrary there is equivalence

$$\exists T^\dagger \in B(H) \Longleftrightarrow T \in B(H)^\cap \Longleftrightarrow T(H) = \mathrm{cl}(TH)\ .$$

Notice that if $T^\dagger$ exists, then $T^\dagger T$ is the orthogonal projection from H onto $T^*(H) = T^*T(H)$, and $I - T^\dagger T$ is the orthogonal projection from H onto $T^{-1}(0) = (T^*T)^{-1}(0)$.

If A is a C^* algebra, we write

$$A^\cap = \{a \in A : a \in aAa\}\ ,$$

for the set of "relatively regular" elements and $A^\dagger$ for the set of elements for which there exists Moore-Penrose inverse. Now what is interesting for C^* algebras that Harte and Mbekhta ([6], Theorem 6) have shown that whenever $a \in A$ is relatively regular, then it necessarily has a Moore Penrose inverse, i.e.

$$A^\cap = A^\dagger. \tag{2}$$

Recall that $a \in A$ is relatively regular if and only if the ideal aA is closed ([6], Theorems 2 and 8). In [7] the reduced minimum modulus of the operator L_a, the left multiplication by a, is considered:

$$\gamma(a) \equiv \gamma_A(a) = \gamma(L_a) = \inf\{\|ax\| : \mathrm{dist}(x, L_a^{-1}(0)) \geq 1,\ x \in A\}.$$

From [8], Theorem IV.5.2 it follows that $\gamma(a) > 0$ iff $L_a(A) = aA$ is closed. Therefore,

$$\gamma(a) > 0 \iff a \in A^\cap. \tag{3}$$

We recall that if $A \subset B$ for C^* algebra B, then ([7], Theorem 4(4.4))

$$\gamma_A(a) = \gamma_B(a). \tag{4}$$

From (3) and (4) we conclude that $a \in A$ is relatively regular in A iff it is relatively regular in B:

$$A^\cap = A \cap B^\cap. \tag{5}$$

Now from (2) and (5) it follows

$$A^{\dagger} = A^{\cap} = A \cap B^{\cap}. \tag{6}$$

The assertion (6) is equivalent to the following assertion which is proved in [4], Theorem 4.2 by using Drazin inverse: If A, B are C^* algebras and $T : A \to B$ is an isometric C^* homomorphism, then

$$a \in A^{\dagger} \Longleftrightarrow Ta \in B^{\cap}.$$

It is not difficult to see that the previous assertions are also equivalent to the following. For the convenience of the reader we give a complete proof via Drazin inverse.

Theorem 1.3. *Let A be a C^*-algebra, which is represented over H, and let $t \in A$ satisfy $\widehat{t}(H) = \mathrm{cl}\,\widehat{t}(H)$. Then there exists $t^{\dagger} \in A$.*

Proof. Since $\widehat{t}(H) = \mathrm{cl}\,\widehat{t}(H)$, we conclude that $(\widehat{t^*t})(H) = \mathrm{cl}(\widehat{t^*t})(H)$ and $\widehat{t}^{\dagger} = (\widehat{t^*t})^{\dagger}\widehat{t^*}$. Since $t \in A$ we get that $\widehat{t^*}, \widehat{t^*t} \in A$. Also, $\sigma^A(t^*t) = \sigma^{B(H)}(\widehat{t^*t})$. According to the proof of Lemma 1.1, the range of $\widehat{t^*t}$ is closed and from the selfadjointness of $\widehat{t^*t}$ we obtain that $\mathrm{asc}(\widehat{t^*t}) = \mathrm{dsc}(\widehat{t^*t}) = 1$. Hence $0 \notin \mathrm{acc}\,\sigma^{B(H)}(\widehat{t^*t}) = \mathrm{acc}\,\sigma^{\widehat{A}}(\widehat{t^*t})$. The inverse $(\widehat{t^*t})^{\dagger}$ is given by the functional calculus, since the Moore-Penrose inverse of a selfadjoint operator $\widehat{t^*t}$ coincides with its Drazin inverse [3]. Precisely,

$$(\widehat{t^*t})^{\dagger} = \frac{1}{2\pi i}\int_{\gamma} \frac{1}{\lambda}(\lambda I - \widehat{t^*t})^{-1} d\lambda,$$

where γ is a Jordan curve around $\sigma^{B(H)}(\widehat{t^*t}) \setminus \{0\}$, separating $\sigma^{B(H)}(\widehat{t^*t}) \setminus \{0\}$ from $\{0\}$. For all $\lambda \in \gamma$ we have that $(\lambda I - \widehat{t^*t})^{-1} \in \widehat{A}$. Consequently, $(\widehat{t^*t})^{\dagger} \in \widehat{A}$ and $(\widehat{t})^{\dagger} \in \widehat{A}$. The mapping $\Gamma : A \to \widehat{A}$ is an $*$-isometric isomorphism, so there exists $t^{\dagger} \in A$ satisfying $\widehat{(t^{\dagger})} = (\widehat{t})^{\dagger}$ $(= \widehat{t}^{\dagger})$. $\square$

2. Fredholm theory

An operator $T \in B(H)$ is called upper semi-Fredholm if $\mathrm{null}(T) < \infty$ and $T(H)$ is closed, while $T \in B(H)$ is called lower semi-Fredholm if $\mathrm{def}(T) < \infty$. An operator $T \in B(H)$ is called semi-Fredholm if it is upper or lower semi-Fredholm. For such an operator the index is given by $i(T) = \mathrm{null}(T) - \mathrm{def}(T)$, and if it is finite then T is called Fredholm, or $T \in \Phi(H)$. When $i(T) = 0$, then T is called Weyl, or $T \in \mathcal{W}(H)$.

An operator $T \in B(H)$ is called left Fredholm, or $T \in \Phi_l(H)$, if T is relatively regular upper semi-Fredholm, while T is called right Fredholm, or $T \in \Phi_r(H)$, if T is relatively regular lower semi-Fredholm. Clearly, $\Phi(H) = \Phi_l(H) \cap \Phi_r(H)$.

The set of left Weyl operators, $\mathcal{W}_l(H)$, is the set of left Fredholm operators with non-positive index, while the set of right Weyl operators, $\mathcal{W}_r(H)$, is the set of right Fredholm operators with non-negative index. Evidently, $\mathcal{W}(H) = \mathcal{W}_l(H) \cap \mathcal{W}_r(H)$.

An operator $T \in B(H)$ is called left Browder, or $T \in \mathcal{B}_l(H)$, if T is left Fredholm with finite ascent, while T is called right Browder, or $T \in \mathcal{B}_r(H)$, if T is right Fredholm with finite descent ([11]). The set of Browder operators, $\mathcal{B}(H)$, is the set of Fredholm operators with finite ascent and descent. Clearly, $\mathcal{B}(H) = \mathcal{B}_l(H) \cap \mathcal{B}_r(H)$.

Let $\pi : B(H) \to B(H)/B_0(H)$ be the natural homomorphism.

Recall that $T \in B(H)$ is, respectively, left Fredholm, right Fredholm, Fredholm in $B(H)$, provided that $\pi(T)$ is: left invertible, right invertible, invertible ([1], Theorem 5.1.5; [2], Theorem 4.3.2, 4.3.3; [2], Theorem 3.2.8; [5], Theorem 6.4.3; [9], Theorem 16.13).

An operator $T \in B(H)$ is, respectively, left Weyl, right Weyl, Weyl, if and only if $T \in B(H)^{-1}_l + B_0(H)$, $T \in B(H)^{-1}_r + B_0(H)$, $T \in B(H)^{-1} + B_0(H)$ ([9], Theorem 19.7).

We write

$$U +_{comm} V = \{C + D : (C, D) \in U \times V\ ,\ CD = DC\}$$

for the commuting sum of subsets $U, V \subseteq B(H)$.

An operator $T \in B(H)$ is, respectively, left Browder, right Browder, Browder, if and only if $T \in B(H)_l^{-1} +_{comm} B_0(H)$, $T \in B(H)_r^{-1} +_{comm} B_0(H)$, $T \in B(H)^{-1} +_{comm} B_0(H)$ ([11], Theorem 5 and 6; [2], Theorem 1.4.5).

These definitions and assertions enable us to consider the same notion in C^*-algebras with respect to proper ideals. Let M be a closed two-sided ideal of A, and let $\rho : \widehat{A} \to \widehat{A}/(\widehat{M} \cap B_0(H))$ be the natural homomorphism. Now it is easy to define left-, right-, and Fredholm elements in A with respect to $\widehat{M} \cap B_0(H)$. Moreover, $t \in A$ is left (right) Weyl in A with respect to ρ, provided that $\widehat{t} \in \widehat{A}_l^{-1} + (\widehat{M} \cap B_0(H))$ ($\widehat{t} \in \widehat{A}_r^{-1} + (\widehat{M} \cap B_0(H))$), and $t \in A$ is Weyl in A with respect to ρ, provided that $\widehat{t} \in \widehat{A}^{-1} + (\widehat{M} \cap B_0(H))$.

Also, $t \in A$ is left (right) Browder in A with respect to ρ, provided that $\widehat{t} \in \widehat{A}_l^{-1} +_{comm} (\widehat{M} \cap B_0(H))$ ($\widehat{t} \in \widehat{A}_r^{-1} +_{comm} (\widehat{M} \cap B_0(H))$), and $t \in A$ is Browder in A with respect to ρ, provided that $\widehat{t} \in \widehat{A}^{-1} +_{comm} (\widehat{M} \cap B_0(H))$.

Now we prove (again) Theorem 6.3.4 in [1] in a different way using Theorem 1.3. Notice that the original formulation in [2] does not require that A is irreducible, but this general situation is not supported by a correct proof in [2]. Hence, we use the additional assumption. Also the following theorem is an extension of Theorem 6.3.4 in [1] and Theorem 5.1 in [4].

Theorem 2.1. *Let A be an irreducible C^*-algebra, which is represented over H, and let M be a non-zero closed two-sided ideal of A. Then for $t \in A$ the following hold:*

(a) $\hat{t}$ is a left Fredholm operator in $B(H) \iff \rho(\hat{t})$ is left invertible;

(b) $\hat{t}$ is a right Fredholm operator in $B(H) \iff \rho(\hat{t})$ is right invertible;

(c) $\widehat{t}$ is a Fredholm operator in $B(H) \iff \rho(\widehat{t})$ is invertible in $\widehat{A}/(\widehat{M} \cap B_0(H))$.

(d) $\widehat{t}$ is left Weyl in $B(H) \iff \widehat{t}$ is left Weyl in $\widehat{A}$ with respect to $\widehat{M} \cap B_0(H)$.

(e) $\widehat{t}$ is right Weyl in $B(H) \iff \widehat{t}$ is right Weyl in $\widehat{A}$ with respect to $\widehat{M} \cap B_0(H)$.

(f) $\widehat{t}$ is Weyl in $B(H) \iff \widehat{t}$ is Weyl in $\widehat{A}$ with respect to $\widehat{M} \cap B_0(H)$.

Proof. (a) Suppose that $\widehat{t}$ is left Fredholm in $B(H)$. Then $\widehat{t}(H)$ is closed, and by Theorem 1.3 there exists $\widehat{t}^\dagger \in \widehat{A}$. Now we consider $I - \widehat{t}^\dagger\widehat{t}$. If $I - \widehat{t}^\dagger\widehat{t} = 0$, then $\widehat{t}$ is left invertible and therefore $\rho(\widehat{t})$ is left invertible. If $I - \widehat{t}^\dagger\widehat{t} \neq 0$, since $I - \widehat{t}^\dagger\widehat{t}$ is the orthogonal projection of H onto $\widehat{t}^{-1}(0)$ (which is finite dimensional subspace), so $I - \widehat{t}^\dagger\widehat{t} \in B_{00}(H) \cap \widehat{A} \neq \{0\}$. Since A is irreducible, by Theorem 1.1 it follows that $B_0(H) \subset \widehat{A}$, and by Theorem 1.2, $B_0(H) \subset \widehat{M}$. Now, from $(\widehat{t^*t})^{-1}(0) = \widehat{t}^{-1}(0)$, it follows $\mathrm{null}(\widehat{t^*t}) < \infty$. Since $\widehat{t^*t}$ is selfadjoint and has the closed range (see the proof of Lemma 1.1), it follows that $\mathrm{null}(\widehat{t^*t}) = \dim(\widehat{t^*t}(H))^\perp = \mathrm{codim}\,\widehat{t^*t}(H) = \mathrm{def}(\widehat{t^*t})$, and so, $\widehat{t^*t}$ is Weyl in $B(H)$. Hence there exist $S \in B(H)^{-1}$ and $K \in B_0(H)$ such that $\widehat{t^*t} = S + K$. Since $\widehat{t^*t}$, $K \in \widehat{A}$, we get $S \in \widehat{A}$ and so $\widehat{S}^{-1} \in \widehat{A}$. It follows that $S^{-1}\widehat{t^*t} = I + S^{-1}K$, $S^{-1}\widehat{t^*} \in \widehat{A}$ and $S^{-1}K \in B_0(H) \subset \widehat{M}$ which implies that $\rho(\widehat{t})$ is left invertible.

On the other hand, if $\rho(\widehat{t})$ is left invertible, then obviously $\pi(\widehat{t})$ is left invertible and $\widehat{t}$ is left Fredholm in $B(H)$. Notice that we do not need the irreducibility of A in this direction.

(b) Suppose that $\widehat{t}$ is right Fredholm in $B(H)$. Then $\widehat{t}(H)$ is closed, and by Theorem 1.3 there exists $\widehat{t}^\dagger \in \widehat{A}$. If $I - \widehat{tt}^\dagger = 0$, then $\widehat{t}$ is right invertible and therefore $\rho(\widehat{t})$ is right invertible. Suppose that $I - \widehat{tt}^\dagger \neq 0$. Since $(I - \widehat{tt}^\dagger)(H) = (\widehat{tt}^\dagger)^{-1}(0) = (\widehat{t}^\dagger)^{-1}(0) = \widehat{t}(H)^\perp$ and $\dim \widehat{t}(H)^\perp = \mathrm{codim}\,\widehat{t}(H) < \infty$ because $\widehat{t}$ is right Fredholm in $B(H)$, we get that $I - \widehat{tt}^\dagger \in B_{00}(H) \cap \widehat{A} \neq \{0\}$. It implies $B_0(H) \subset \widehat{M}$. For the rest of the proof change the order of $\widehat{t^*}$ and $\widehat{t}$ in the previous proof.

The converse is clear.

(c) Follows from (a) and (b).

(d) Suppose that $\widehat{t}$ is left Weyl in $B(H)$. Then $\widehat{t}(H)$ is closed, and there exists $\widehat{t}^\dagger \in \widehat{A}$. If $I - \widehat{t}^\dagger\widehat{t} = 0$, then $\widehat{t}$ is left invertible and therefore $\hat{t}$ is left Weyl in $\widehat{A}$ with respect to $\widehat{M} \cap B_0(H)$. If $I - \widehat{t}^\dagger\widehat{t} \neq 0$, in the same way as in the proof of (a) we conclude that $B_0(H) \subset \widehat{M}$. There exist $S \in B(H)_l^{-1}$ and $K \in B_0(H)$ such that $\widehat{t} = S + K$. Since $\widehat{t}$, $K \in \widehat{A}$, it follows that $S \in \widehat{A}$. By Lemma 1.1 it follows that $S \in \widehat{A}_l^{-1}$, and we obtain that $\widehat{t}$ is left Weyl in A with respect to $\widehat{M} \cap B_0(H)$.

On the other hand, if $\widehat{t} = U + V$, where $U \in \widehat{A}_l^{-1}$ and $V \in \widehat{M} \cap B_0(H)$, then obviously $\widehat{t}$ is left Weyl in $B(H)$.

(e) Similarly to (d).

(f) Suppose that $\widehat{t}$ is Weyl in $B(H)$. As in the previous we conclude that there exists $\widehat{t}^{\dagger} \in \widehat{A}$. If both $I - \widehat{t}^{\dagger}\widehat{t}$ and $I - \widehat{t}\widehat{t}^{\dagger}$ are equal to zero, then $\widehat{t}$ is invertible in $\widehat{A}$, and therefore, it is Weyl in $\widehat{A}$ with respect to $\widehat{M} \cap B_0(H)$.

If $I - \widehat{t}^{\dagger}\widehat{t} \neq 0$ or $I - \widehat{t}\widehat{t}^{\dagger} \neq 0$, then as in the proof for (a) and (b) we conclude that $B_0(H) \subset \widehat{M}$. Now, there exist $S \in B(H)^{-1}$ and $K \in B_0(H)$ such that $\widehat{t} = S + K$. Since $\widehat{t},\ K \in \widehat{A}$, it follows that $S \in \widehat{A}$. Hence $S \in \widehat{A}^{-1}$, and we obtain that $\widehat{t}$ is Weyl in A with respect to $\widehat{M} \cap B_0(H)$.

The converse is clear. □

We remark that, analogously to (d), (e) and (f) respectively, it can be proved with the same assumptions as in Theorem 2.1 that, for $t \in A$, $\widehat{t}$ is left (right) Browder in $B(H)$ iff $\widehat{t}$ is left (right) Browder in $\widehat{A}$ with respect to $\widehat{M} \cap B_0(H)$, as well as, $\widehat{t}$ is Browder in $B(H)$ iff $\widehat{t}$ is Browder in $\widehat{A}$ with respect to $\widehat{M} \cap B_0(H)$.

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