Ruston, Riesz and perturbation classes

S. Č. Živković-Zlatanović, Dragan S. Djordjević and R. Harte[1]

Abstract

We determine the perturbation classes of Fredholm and Weyl elements, as well the "commuting perturbation classes" of Fredholm, Weyl and Browder elements with respect to unbounded Banach algebra homomorphism T. Among other things we use Ruston elements of Mouton, Mouton and Raubenheimer. Also, we investigate the class of polynomially almost T null and the class of polynomially T Riesz elements.

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1 Introduction

A homomorphism of rings $T : A \to B$ gives rise to a "Fredholm theory" in the departure ring A: we can distinguish *Fredholm*, *Weyl* and *Browder* elements

$$A^{-1} \subseteq A^{-1} +_{comm} T^{-1}(0) \subseteq A^{-1} + T^{-1}(0) \subseteq T^{-1}(B^{-1}) \subseteq A\ ;$$

if we replace invertibles by either left or right invertibles throughout we get "left" and "right" Fredholm, Weyl and Browder elements of A. In this note we are interested in the *perturbation classes* of these semigroups, in the sense of Lebow and Schechter [18], and the analagous *commutative perturbation classes*. Our efforts at their determination takes us through certain intermediate semigroups: the *Ruston* and *almost Ruston* elements of Mouton, Mouton and Raubenheimer [20].

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2 Radical

Suppose A is a complex Banach algebra, with identity 1 and invertible group A^{-1}: much of what we say will apply to rings or even additive categories. The *radical* of A is the set

$$\mathrm{Rad}(A) = \{d \in A : 1 - Ad \subseteq A^{-1}\} = \{d \in A : 1 - dA \subseteq A^{-1}\} \ : \tag{2.1}$$

the equivalence is due to the Jacobson lemma. If $x, y \in A$ are arbitrary then

$$1 - xy \in A^{-1} \Longleftrightarrow 1 - yx \in A^{-1} \ :$$

one quarter of the argument is the implication

$$w(1 - xy) = 1 \Longrightarrow (1 + ywx)(1 - yx) = 1 \ .$$

The radical is unchanged if the invertible group A^{-1} is replaced by either the semigroup A^{-1}_{left} of left invertible elements, or the semigroup A^{-1}_{right} of right invertible elements: for if $1 - Aa \subseteq A^{-1}_{left}$ then for arbitrary $a' \in A$ there is $a'' \in A$ for which $a''(1 - a'a) = 1$, giving

$$a'' = 1 + (a''a')a \in A^{-1}_{left} \implies a'' \in A^{-1} \implies 1 - a'a \in A^{-1} \ .$$

$\mathrm{Rad}(A)$ can also be realised ([11] Theorem 7.2.3) as the intersection of all maximal proper left ideals, similarly right ideals. Invertibility in a sense bypasses the radical, which can be recognised ([18] Theorems 2.5, 2.6) as the *perturbation class* of the invertible group:

Theorem 2.1. *If A is a Banach algebra with identity then there is equality*

$$\mathrm{Rad}(A) = \mathrm{Ptrb}(A^{-1}) = \mathrm{Ptrb}(A^{-1}_{left}) = \mathrm{Ptrb}(A^{-1}_{right}) \ , \tag{2.2}$$

where if $S \subseteq A$ then

$$\mathrm{Ptrb}(S) = \{a \in A : S + a \subseteq S\} \ ,$$

and equivalence, for each $H(A) \in \{A^{-1}, A^{-1}_{left}, A^{-1}_{right}\}$,

$$a \in H(A) \Longleftrightarrow a + \mathrm{Rad}(A) \in H(A/\mathrm{Rad}(A)) \ . \tag{2.3}$$

Proof. It is immediately clear that the radical is a subset of the perturbation group of the left invertibles:

$$a \in \mathrm{Rad}(A) \Longrightarrow A^{-1}_{left} + a \subseteq (1 + aA)A^{-1}_{left} \subseteq A^{-1}A^{-1}_{left} = A^{-1}_{left} \ .$$

Conversely, since A is a Banach algebra, there is equality

$$A^{-1} + A^{-1} = A \ .$$

Now if $a \in \mathrm{Ptrb}(A^{-1}_{left})$ we argue

$$1 + A^{-1}a \subset A^{-1}(A^{-1} + a) \subset A^{-1}(A^{-1}_{left} + a) \subset A^{-1}A^{-1}_{left} = A^{-1}_{left},$$

and therefore

$$1 + Aa = 1 + (A^{-1} + A^{-1})a \subseteq (1 + A^{-1}a) + A^{-1}a \subseteq A^{-1}_{left} + A^{-1}a \subseteq$$
$$\subseteq A^{-1}(A^{-1}_{left} + a) \subseteq A^{-1}A^{-1}_{left} = A^{-1}_{left} \ .$$

This identifies the radical with the perturbation class of the left invertibles, and similarly (formally: reverse products) of the right invertibles, and therefore also of the invertibles.

Forward implication in (2.3) is automatic; conversely there is implication, for arbitrary $a, a' \in A$,

$$1 - a'a \in \mathrm{Rad}(A) \Longrightarrow a'a = 1 - (1 - a'a) \in 1 - A(1 - a'a) \subseteq A^{-1} \Longrightarrow a \in A^{-1}_{left} \ .$$

Thus if $a \in A$ has a left invertible coset, then a is left invertible, giving (2.3) for left, and therefore also right, and therefore also two sided, invertibility. $\square$

3 Quasinilpotent

The concept of the radical makes sense in an arbitrary ring; by contrast *quasinilpotents* involve the norm. When A is a complex Banach algebra then the quasinilpotents of A form the set

$$\mathrm{QN}(A) = \{d \in A : \|d^n\|^{1/n} \to 0\} = \{d \in A : 1 - \mathbf{C}d \subseteq A^{-1}\} \ . \tag{3.1}$$

The equivalence of these two conditions ([11], Theorem 9.5.2 and Theorem 9.5.3) is not trivial, and relies on Liouville's theorem from complex analysis. It follows easily from (3.1) that also, writing $\mathrm{comm}_A(S)$ and $\mathrm{comm}^2_{\mathrm{A}}(\mathrm{S})$, respectively, for the *commutant* and *double commutant* of $S \subseteq A$,

$$\begin{aligned} \mathrm{QN}(A) &= \{d \in A : 1 - \mathrm{comm}_A(d)d \subseteq A^{-1}\} \\ &= \{d \in A : 1 - \mathrm{comm}^2_A(d)d \subseteq A^{-1}\} \ , \end{aligned} \tag{3.2}$$

since each of these conditions is intermediate between the conditions of (3.1).

We claim that the quasinilpotents form a commutative analogue of the perturbation class of the invertibles:

Theorem 3.1. *If A is a Banach algebra, then*

$$a \in \mathrm{QN}(A) \iff a + \mathrm{Rad}(A) \in \mathrm{QN}(A/\mathrm{Rad}(A)), \tag{3.3}$$

$$\mathrm{Rad}(A) \subseteq \mathrm{QN}(A), \tag{3.4}$$

and the radical is the largest left, and the largest right ideal of A contained in the quasinilpotents.

Also

$$\mathrm{QN}(A) = \mathrm{Ptrb}_{comm}(A^{-1}) \ , \tag{3.5}$$

where if $S \subseteq A$

$$\mathrm{Ptrb}_{comm}(S) = \{a \in A : S +_{comm} \{a\} \subseteq S\} \ ,$$

writing

$$H +_{comm} K = \{c + d : (c, d) \in H \times K \ , \ cd = dc\} \tag{3.6}$$

for the commuting sum of subsets $H, K \subseteq A$.

Proof. (3.3) follows from (2.3), and (3.4) is clear from a comparison of (2.1) and the second part of (3.1). If $AJ \subseteq J \subseteq \mathrm{QN}(A)$, then there is implication

$$x \in J \Longrightarrow 1 - Ax \subseteq 1 + \mathrm{QN}(A) \subseteq A^{-1} \Longrightarrow x \in \mathrm{Rad}(A) \ ,$$

which is the third assertion for left ideals. Towards (3.5) we can "commutatively" follow the argument of Theorem 2.1. Inclusion

$$\mathrm{QN}(A) \subseteq \mathrm{Ptrb}_{comm}(A^{-1})$$

follows from (3.2): if $a \in A^{-1}$ commutes with $d \in \mathrm{QN}(A)$, then also the inverse a^{-1} commutes with d, giving

$$a + d = a(1 + a^{-1}d) \in A^{-1}(1 + \mathrm{comm}(d)d) \subseteq A^{-1} \ .$$

Conversely if $d \in \mathrm{Ptrb}_{comm}(A^{-1})$, then

$$0 \neq \lambda \in \mathbf{C} \Longrightarrow \lambda^{-1} + d \in A^{-1} \Longrightarrow 1 + \lambda d \in A^{-1} \ ,$$

giving the second part of (3.1). □

It is not immediately clear how to extend this argument to left and right invertibles, since neither left nor right inverses of an element in general commute, or double commute, with it. The quasinilpotents do just a little bit more than act as the commuting perturbation class of the invertible group: necessary and sufficient for $d \in \mathrm{QN}(A)$ is implication, for arbitrary $a \in A$,

$$\big(ad - da \in \mathrm{Rad}(A)\ ,\ a \in A^{-1}\big) \Longrightarrow a + d \in A^{-1}\ . \tag{3.7}$$

According to (2.3) and (3.3), this follows from (3.5), applied to the quotient $A/\mathrm{Rad}(A)$.

Of course the implication (3.7), by its nature, holds with a and $a + d$ interchanged.

4 Spectrum

We recall the *spectrum* of $a \in A$,

$$\sigma(a) \equiv \sigma_A(a) = \sigma^{left}(a) \cup \sigma^{right}(a)\ ,$$

where

$$\begin{aligned}\sigma^{left}(a) \equiv \sigma_A^{left}(a) &= \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}_{left}\}\ ,\\ \sigma^{right}(a) \equiv \sigma_A^{right}(a) &= \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}_{right}\}\ .\end{aligned}$$

For $K \subset \mathbf{C}$, ∂K denotes the boundary of K. Here ηK is ([15]; [11] Definition 7.10.1) the *connected hull* of a compact set $K \subseteq \mathbf{C}$, where the complement $\mathbf{C} \setminus \eta K$ is the unique unbounded component of the complement $\mathbf{C} \setminus K$. Generally ([15], Theorem 1.2, Theorem 1.3; [11], Theorem 7.10.3), for compact subsets $H, K \subseteq \mathbf{C}$,

$$\partial H \subseteq K \subseteq H \Longrightarrow \partial H \subseteq \partial K \subseteq K \subseteq H \subseteq \eta K = \eta H\ . \tag{4.1}$$

For $a \in A$, it is well known that $\partial\sigma(a) \subseteq \sigma^{left}(a)$ and $\partial\sigma(a) \subseteq \sigma^{right}(a)$. Hence, by (4.1)

$$\eta\sigma(a) = \eta\sigma^{left}(a) = \eta\sigma^{right}(a). \tag{4.2}$$

Consequently,

$$\sigma(a) = \{0\} \Longleftrightarrow \sigma^{left}(a) = \{0\} \Longleftrightarrow \sigma^{right}(a) = \{0\}. \tag{4.3}$$

If $\omega \in \{\sigma, \sigma^{left}, \sigma^{right}\}$, then from (3.1) and (4.3) it follows

$$a \in \mathrm{QN}(A) \Longleftrightarrow \omega(a) = \{0\}. \tag{4.4}$$

From (2.3) it is clear that if $a \in A$ and $\omega \in \{\sigma, \sigma^{left}, \sigma^{right}\}$, then

$$\omega(a) = \omega(a + \mathrm{Rad}(A)). \tag{4.5}$$

With the help of the *spectral mapping theorem in two variables* ([11] Theorem 11.3.4; [21] Theorem 8.3) there is implication, for arbitrary $a, b \in A$ and polynomials $p : \mathbf{C}^2 \to \mathbf{C}$ in two variables, and each $\omega \in \{\sigma, \sigma^{left}, \sigma^{right}\}$,

$$ab = ba \Longrightarrow \omega(p(a,b)) \subseteq p(\omega(a) \times \omega(b)) \ . \tag{4.6}$$

Lemma 4.1. *If ω is one of σ, σ^{left} and σ^{right} and $a, b \in A$ such that $ab - ba \in \mathrm{Rad}(A)$, then*

$$\omega(a+b) \subset \omega(a) + \omega(b).$$

Proof. Let $S : A \to A/\mathrm{Rad}(A)$ denote the quotient map. Since $S(a)$ and $S(b)$ commute, by (4.5) and (4.6) we have $\omega(a+b) = \omega(S(a+b)) \subset \omega(S(a)) + \omega(S(b)) = \omega(a) + \omega(b)$. □

While (3.7) and Theorem 3.1 can trivially be rewritten in terms of the spectrum, the spectral theory enables them to be extended to the left and the right spectrum.

Theorem 4.1. *If A is a Banach algebra with identity and if $\omega \in \{\sigma, \sigma^{left}, \sigma^{right}\}$, then for arbitrary $c, d \in A$ there is implication*

$$d \in \mathrm{Rad}(A) \Longrightarrow \omega(c) = \omega(c+d) \ . \tag{4.7}$$

Also if $d \in A$, then the following are equivalent:

$$d \in \mathrm{QN}(A) \ ; \tag{4.8}$$

$$(\forall a \in A) \ (ad - da \in \mathrm{Rad}(A) \Longrightarrow \omega(a) = \omega(a+d)) \ ; \tag{4.9}$$

$$(\forall a \in A) \ (ad - da = 0 \Longrightarrow \omega(a) = \omega(a+d)) \ . \tag{4.10}$$

In particular

$$\mathrm{QN}(A) = \mathrm{Ptrb}_{comm}(A^{-1}_{left}) = \mathrm{Ptrb}_{comm}(A^{-1}_{right}) \ . \tag{4.11}$$

Proof. Implication (4.7) is the translation of (2.3) into the language of the spectrum. If $d \in A$ is quasinilpotent, then $\omega(d) = \{0\}$ by (4.4), and for $a \in A$, such that $ad - da \in \mathrm{Rad}(A)$, Lemma 4.1 gives

$$\omega(a+d) \subseteq \omega(a) + \omega(d) = \omega(a) \ ;$$

applying this with $(a+d, -d)$ in place of (a, d) gives also

$$\omega(a) = \omega(a+d-d) \subseteq \omega(a+d) + \omega(-d) = \omega(a+d) \ .$$

Thus (4.8) implies (4.9);(4.9) trivially implies (4.10). Conversely (4.10), with $a = 0$, gives $\omega(d) = \{0\}$, which by (4.4) implies (4.8). (4.11) follows from the equivalence (4.8)$\Longleftrightarrow$(4.10). □

Let Poly denote the algebra of complex polynomials, Poly $= \mathbf{C}[z]$. As spinoff, we offer a curious perturbation result:

Theorem 4.2. *Suppose $a,\ d \in A$, and that $p \in$ Poly is a polynomial, and that ω is one of σ, σ^{left} and σ^{right}: then if*

$$ad - da \in \mathrm{Rad}(A) \text{ and } p(d) \in \mathrm{QN}(A)\ , \tag{4.12}$$

there is implication

$$0 \in \omega(a - d) \Longrightarrow 0 \in \omega(p(a))\ . \tag{4.13}$$

Proof. If $ad = da$, then by a remainder theorem argument

$$A(p(a) - p(d)) \subseteq A(a - d)\ .$$

This gives (4.13) with $\omega = \sigma^{left}$ and $p(a) - p(d)$ in place of $p(a)$. Now if also $\omega(p(d)) = \{0\}$, then by (4.6) there is inclusion

$$\omega(p(a) - p(d)) \subseteq \omega(p(a)) \subseteq \omega(p(a) - p(d))\ .$$

Finally if we assume only (4.12), we transfer the argument to the quotient $A/\mathrm{Rad}(A)$. □

5 Polar

An element $a \in A$ in a ring with identity (more generally, a semigroup) is said to be *simply polar* if there is $b \in A$ for which

$$a = aba\ ,\ \text{with } ab = ba\ .$$

If in addition

$$b = bab,$$

then such an element $b = a^{\times}$ is necessarily unique, and commutes with everything that commutes with a, and is sometimes known as the *group inverse* of a. More generally if there is $n \in \mathbf{N}$ for which a^n is simply polar, we shall say that $a \in A$ is *polar*. Equivalently $a \in A$ is polar (or *Drazin invertible*), in the sense that there is $b \in A$ for which

$$ab = ba = p = p^2\ ,\ bab = b\ ,\ \text{with } 0 \in \{a^n(1 - p) : n \in \mathbf{N}\}\ .$$

When A is a Banach algebra then, more generally still, we shall say that $a \in A$ is *quasi polar*, written $a \in \mathrm{QP}(A)$, if there is $b \in A$ for which

$$ab = ba = p = p^2 \text{ with } a(1-p) \in \mathrm{QN}(A) \ . \tag{5.1}$$

Equivalently [17] there is $q = q^2 \in A$ for which

$$aq = qa \ ; \ a + q \in A^{-1} \ ; \ aq \in \mathrm{QN}(A) \ . \tag{5.2}$$

The element $b = a^\times$ is still unique and still double commutes with a, now referred to as the *Koliha-Drazin inverse*. Notice that the relationship $p = 1 - q$ connects p from (5.1) with q from (5.2). Necessary and sufficient for (5.1) is that $a \in A$ is *almost invertible*, in the sense that $0 \in \mathbf{C}$ is at worst an isolated point of spectrum:

$$0 \notin \mathrm{acc}\ \sigma_A(a) \ . \tag{5.3}$$

Implication (5.1) $\Longrightarrow$ (5.3) is easy: the element $a \in A$ is the direct sum of $ap \in pAp$ which is invertible and $aq \in qAq$ which is quasinilpotent. Conversely if (5.3) holds, then the projection $q = 1-p$ and the Koliha-Drazin inverse $a^\times$ can be constructed as Cauchy integrals. Commuting products of quasipolars, and commuting sums of quasipolars and quasinilpotents, remain quasipolar:

$$\mathrm{QP}(A) \cdot_{comm} \mathrm{QP}(A) \subseteq \mathrm{QP}(A) \tag{5.4}$$

and

$$\mathrm{QP}(A) +_{comm} \mathrm{QN}(A) \subseteq \mathrm{QP}(A) \ :$$

it is not hard ([11] Theorem 7.5.4; [17]) to write out formulae for the Koliha-Drazin inverse in each case. Retha Heymann ([16] Lemma 2.1.15) derives (5.4) from (5.3) by observing, for compact subsets K, H of $\mathbf{C}$,

$$0 \notin \mathrm{acc}(K)_\cup \mathrm{acc}(H) \Longrightarrow 0 \notin \mathrm{acc}(KH) \ .$$

6 Fredholm

If $T : A \to B$ is a homomorphism of rings (usually complex Banach algebras), in the sense that for arbitrary a, a' in A

$$T(a'a) - T(a')T(a) = 0 = T(1) - 1 \ ,$$

then $T(A^{-1}) \subseteq B^{-1}$, and hence

$$A^{-1} \subseteq A^{-1} + T^{-1}(0) \subseteq T^{-1}B^{-1} \ .$$

We shall describe $d \in A$ as T *null* if $Td = 0$.

We shall say that $a \in A$ is *left T Fredholm* iff it has a left invertible image:

$$a \in T^{-1}(B^{-1}_{left}) \ , \tag{6.1}$$

and $a \in A$ is *left T Weyl* if it splits into the sum of a left invertible and a T null element:

$$a \in A^{-1}_{left} + T^{-1}(0) \ . \tag{6.2}$$

When the sum (6.2) is commutative, then $a \in A$ is *left T Browder*: in the notation of (3.6)

$$a \in A^{-1}_{left} +_{comm} T^{-1}(0) \ . \tag{6.3}$$

Right and two-sided T Fredholm, T Weyl and T Browder elements are defined analagously. The induced left T Fredholm, T Weyl and T Browder spectra are given by

$$\sigma^{left}_{f,T}(a) = \sigma^{left}_{B}(Ta) \quad ; \quad \sigma^{left}_{\omega,T}(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}_{left} + T^{-1}(0)\} \ ;$$
$$\sigma^{left}_{b,T}(a) \quad = \quad \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}_{left} +_{comm} T^{-1}(0)\} \ .$$

The corresponding right and two-sided spectra are clear. Evidently

left invertible $\Longrightarrow$ left T Browder $\Longrightarrow$ left T Weyl $\Longrightarrow$ left T Fredholm,

and similarly for right and two-sided. In terms of spectra there is inclusion for arbitrary $a \in A$

$$\sigma^{left}_{f,T}(a) \subseteq \sigma^{left}_{\omega,T}(a) \subseteq \sigma^{left}_{b,T}(a) \subseteq \sigma^{left}(a) \ .$$

More generally quasipolar T Fredholm elements are T Browder:

$$T^{-1}(B^{-1})_\cap \mathrm{QP}(A) \subseteq A^{-1} +_{comm} T^{-1}(0) \ ; \tag{6.4}$$

thus also

$$\sigma_{b,T}(a) \subseteq \sigma_{f,T}(a)_\cup \mathrm{acc} \ \sigma(a) \ . \tag{6.5}$$

We remark that (6.4) and (6.5) were shown, with "polar" in place of "quasipolar", by Harte ([9]; [10]; [11] Theorem 7.7.4) for bounded homomorphism T. Mouton and Raubenheimer [19] Corollary 2.5 extended this to unbounded T using Grobler and Raubenheimer's result [8] Proposition 2.1. Notice also that in (6.3) we have a decomposition $a = c + d$ in which invertible c and T null d each commute with a; when a is almost invertible Fredholm they even double commute.

We remark that the T Fredholm property involves the target algebra B, while the Weyl and Browder properties depend only on the null space $T^{-1}(0)$: thus if we write $S : A \to D = A/T^{-1}(0)$ for the natural quotient, then S Weyl and T Weyl are equivalent, as are S Browder and T Browder.

We also remark that it is not immediately obvious that if for example $a \in A$ is both left and right T Weyl, then it is necessarily two-sided T Weyl.

Corollary 6.1. *Let $a,\ b \in A$ and $ab - ba \in T^{-1}(\mathrm{Rad}B)$. If ω is one of $\sigma_{f,T}, \sigma_{f,T}^{left}, \sigma_{f,T}^{right}$, then*

$$\omega(a + b) \subset \omega(a) + \omega(b).$$

Proof. Lemma 4.1, applied to $T(a)$ and $T(b)$ in B. □

7 Riesz

If $T : A \to B$ is a homomorphism of rings, then we shall describe $d \in A$ as *almost T null* if $Td \in \mathrm{Rad}(B)$, and for Banach algebras as *T Riesz* if

$$T(d) \in \mathrm{QN}(B)\ .$$

This would make sense in more general rings if we knew how to define "quasinilpotents". Observe that, whether or not the homomorphism $T : A \to B$ is bounded, there is inclusion

$$T\ \mathrm{QN}(A) \subseteq \mathrm{QN}(B) \tag{7.1}$$

and hence

$$T\ \mathrm{QP}(A) \subseteq \mathrm{QP}(B)\ . \tag{7.2}$$

Indeed ([8] Proposition 2.1) Grobler and Raubenheimer show, for (7.2), that if $q \in A$ is derived from $a \in A$ by the usual Cauchy integral, then $Tq \in B$ is derived from $Ta \in B$ in the same way. Alternatively it is clear that, like ordinary invertibility, the preservation of Koliha-Drazin invertibility has no need of boundedness of T. In fact (7.1) implies (7.2), and is itself clear from the second of the two equivalent conditions of (3.1).

Corollary 7.1. *If $T : A \to B$ is onto, then the set of almost T null elements is the largest left and the largest right ideal of A contained in the set of T Riesz elements.*

Proof. Let J be a left ideal contained in the set $T^{-1}\mathrm{QN}(B)$. Since T is onto, then $T(J)$ is a left ideal and since $T(J)$ is contained in $\mathrm{QN}(B)$, from Theorem 3.1 it follows $T(J) \subset \mathrm{Rad}(B)$ and therefore $J \subset T^{-1}\mathrm{Rad}(B)$. This proves the assertion for left ideals. $\square$

The Fredholm theory associated with a homomorphism $T : A \to B$ becomes sharper if the homomorphism behaves itself:

Definition 7.1. We shall say that the homomorphism $T : A \to B$ has the Riesz property if there is implication, for arbitrary $a \in A$,

$$a \in T^{-1}(0) \Longrightarrow \mathrm{acc}\ \sigma_A(a) \subseteq \{0\}\ , \tag{7.3}$$

and the strong Riesz property if there is inclusion, for arbitrary $a \in A$,

$$\partial\sigma_A(a) \subseteq \sigma_{f,T}(a)_\cup \mathrm{iso}\ \sigma_A(a)\ . \tag{7.4}$$

Equivalently, T is Riesz when the ideal $T^{-1}(0)$ is [3] "inessential". In words, the Riesz property says that T null elements of A are almost invertible, hence quasipolar. If in particular everything in the null space $T^{-1}(0)$ is actually polar we shall say that T is *finitely Riesz*. By the *essential boundary-hull theorem* ([15] Theorem 4.2) the strong Riesz property can be rewritten

$$\forall\ a \in A\ :\ \sigma_A(a) \subseteq \eta\sigma_{f,T}(a)_\cup \mathrm{iso}\ \sigma_A(a)\ . \tag{7.5}$$

where $\eta\sigma_{f,T}(a)$ is the connected hull of $\sigma_{f,T}(a)$. From (7.5) it is clear that the strong Riesz property implies the Riesz property; conversely if $T : A \to B$ has closed range, then the two are equivalent. For the Calkin homomorphism this is a consequence of the *punctured neighbourhood theorem*, and for more general onto homomorphisms is due to Aupetit [3]; the cosmetic extension to closed range is [12]. For unbounded T this was shown in [20] Corollary 7.9.

Generally (6.4) almost invertible T Fredholm elements are T Browder: conversely if and only if the homomorphism $T : A \to B$ has the Riesz property, then T Browder elements are almost invertible. This was first shown (with "finitely Riesz" in place of "Riesz") ([9]; [10]; [11] Theorem 7.7.4) for bounded homomorphisms and extended ([19] Theorem 3.4 and the remark following this theorem) to arbitrary homomorphisms between Banach algebras. In terms of spectra there is equality in (6.5) iff T has the Riesz property.

8 Ruston

Intermediate between Weyl and Browder properties would be various "Ruston" conditions [19], [20], [23]:

Definition 8.1. We shall say that $a \in A$ is *left T Ruston* provided

$$a = c + d \text{ with } c \in A^{-1}_{left}\,,\ cd - dc = 0\,,\ T(d) \in \mathrm{QN}(B)\,, \tag{8.1}$$

almost left T Ruston provided

$$a = c + d \text{ with } c \in A^{-1}_{left}\,,\ cd - dc \in T^{-1}(0)\,,\ T(d) \in \mathrm{QN}(B)\,, \tag{8.2}$$

and *almost essentially left T Ruston* provided

$$a = c + d \text{ with } c \in A^{-1}_{left}\,,\ cd - dc \in T^{-1}\mathrm{Rad}(B)\,,\ T(d) \in \mathrm{QN}(B)\,. \tag{8.3}$$

We shall also describe $a \in A$ as *left T Raubenheimer* provided

$$a \in A^{-1}_{left} + T^{-1}\mathrm{Rad}(B)\,, \tag{8.4}$$

and as *commutatively left T Raubenheimer* provided

$$a \in A^{-1}_{left} +_{comm} T^{-1}\mathrm{Rad}(B)\,. \tag{8.5}$$

Right, almost right and almost essentially right Ruston, and right Raubenheimer, and commutatively right Raubenheimer, elements are obtained by replacing $c \in A^{-1}_{left}$ by $c \in A^{-1}_{right}$. Two-sided Ruston and Raubenheimer elements are obtained by taking $c \in A^{-1}$. Evidently [19], [20]:

Theorem 8.1. *If $T : A \to B$ there is implication*

$$(6.3) \Longrightarrow (8.5) \Longrightarrow (8.1) \Longrightarrow (8.2) \Longrightarrow (8.3) \Longrightarrow (6.1)\ ,$$

$$(6.3) \Longrightarrow (6.2) \Longrightarrow (8.2)\ ,$$

and also

$$(6.2) \Longrightarrow (8.4) \Longrightarrow (8.3)\ .$$

and similarly for right and two-sided Ruston and Raubenheimer elements.

Proof. (6.3)$\Longrightarrow$ (8.5)$\Longrightarrow$(8.1)$\Longrightarrow$ (8.2)$\Longrightarrow$(8.3) is clear. The argument for the implication (8.3)$\Longrightarrow$(6.1) is a simple extension of the argument [23] for two-sided Fredholm and Ruston: writing $S : B \to D = B/\mathrm{Rad}(B)$ for the canonical homomorphism, suppose that $a = c + d$ with $c \in A^{-1}_{left}$,

$Td \in \mathrm{QN}(B)$ and $T(cd-dc) \in \mathrm{Rad}(B)$. From (2.3) and (3.3) it follows that STc and STd in D are respectively left invertible, and quasinilpotent, and since they commute, by (4.11) the sum $STa \in D^{-1}_{left}$ is left invertible. By (2.3) it follows that $Ta \in B^{-1}_{left}$ is left invertible.

The implications (6.3)$\Longrightarrow$ (6.2)$\Longrightarrow$ (8.2) and (6.2) $\Longrightarrow$ (8.4)$\Longrightarrow$ (8.3) are clear. □

When the homomorphism is onto, then we have three conditions equivalent to Fredholmness:

Theorem 8.2. *If $T : A \to B$ is onto, $T(A) = B$, then each of the following conditions is equivalent to $a \in A$ left T Fredholm:*

$$\exists\ a' \in A\ :\ 1 - a'a \in T^{-1}(0)\ ; \tag{8.6}$$

$$\exists\ a' \in A\ :\ T(1 - a'a) \in \mathrm{Rad}(B)\ ; \tag{8.7}$$

$$\exists\ a' \in A\ :\ T(1 - a'a) \in \mathrm{QN}(B)\ . \tag{8.8}$$

Proof. If T is onto and $Ta \in B^{-1}_{left}$, then there must be $b' = T(a') \in T(A) = B$ for which $T(a')T(a) = T(1)$, giving (8.6), which visibly implies (8.7) and (8.8). Conversely if $a'a = 1 - d$ with $T(d) \in \mathrm{QN}(B)$, then

$$T(a')T(a) = T(1) - T(d) \in T(1) + \mathrm{QN}(B) \subseteq B^{-1}\ ,$$

giving $T(a) \in B^{-1}_{left}$. □

If $T : A \to B$ has closed range and the Riesz property, then ([20] Theorem 6.6) almost Ruston elements are Weyl, and Ruston elements are Browder. We can improve on this:

Theorem 8.3. *If $T : A \to B$ has the strong Riesz property, then almost essentially T Ruston elements are T Weyl. Also, T Ruston elements are T Browder.*

Proof. Suppose $a = c+d$ where c is invertible and $T(d)$ is quasinilpotent and commuting with Tc modulo the radical of B, hence also with $T(c^{-1})$. Then, going in and out of the quotient $D = B/\mathrm{Rad}(B)$, we get that $T(c^{-1}d) \in \mathrm{QN}(B)$, and hence

$$T(c^{-1}a) = T(1) + T(c^{-1}d) \in T(1) + \mathrm{QN}(B) \subseteq B^{-1}.$$

Hence, $c^{-1}a$ is T Fredholm and $\sigma_{f,T}(c^{-1}a) = \sigma_B(T(c^{-1}a)) = \sigma_B(T(1)) = \{1\}$ by (4.6). Using with (7.5), we get:

$$\sigma_A(c^{-1}a) \subseteq \eta\{1\}_\cup \mathrm{iso}\ \sigma_A(c^{-1}a) = \{1\}_\cup \mathrm{iso}\ \sigma_A(c^{-1}a)\ .$$

It follows that $c^{-1}a$ is almost invertible T Fredholm, therefore T Browder. This says that $a = cc^{-1}a$ is T Weyl, giving the first part. If in particular also $cd = dc$, then $ac = ca$ and hence, c and $c^{-1}a$ commute. Thus a is the commuting product of an invertible and an almost invertible T Fredholm, therefore (5.4) almost invertible T Fredholm and hence T Browder. $\square$

Corollary 8.1. *If $T : A \to B$ has the strong Riesz property, then Weyl, Raubenheimer, almost Russton and almost essentially Ruston elements are the same, as are Browder, commutatively Raubenheimer and Ruston elements.*

Proof. It follows from Theorem 8.3 and the implications (6.2) $\Longrightarrow$ (8.2) $\Longrightarrow$ (8.3), (6.2) $\Longrightarrow$ (8.4) $\Longrightarrow$ (8.3) and (6.3) $\Longrightarrow$ (8.5) $\Longrightarrow$ (8.1). $\square$

9 Perturbation classes

We turn to the perturbation classes in A for various kinds of T Fredholmness. Since left T Fredholm, and left T Weyl, elements constitute open semigroups invariant under multiplication by the invertible group, it is clear ([18] Theorem 2.4) that their perturbation classes are two-sided ideals. When T is bounded, then the Fredholm and the Weyl elements form open sets, and hence ([18] Lemma 2.1) their perturbation classes are closed. Generally there is implication, for $K \subseteq B$,

$$T^{-1}\mathrm{Ptrb}(K) \subseteq \mathrm{Ptrb}(T^{-1}K) \subseteq T^{-1}\mathrm{Ptrb}(K \cap TA) \ . \tag{9.1}$$

Indeed if $Td \in \mathrm{Ptrb}(K)$ and $Ta \in K$, then $T(a+d) \in K$; conversely if $d \in \mathrm{Ptrb}(T^{-1}K)$ and $Ta = b \in K \cap T(A)$, then $T(a) + T(d) = b + Td \in K \cap T(A)$ giving $T(d) \in \mathrm{Ptrb}(K \cap TA)$.

When T is onto, then there is equality throughout (9.1). Theorem 2.1 has an extension from invertible to Fredholm:

Theorem 9.1. *If $T : A \to B$ is a homomorphism of Banach algebras, then for each $H(A)$ of A^{-1}_{left}, A^{-1}_{right} and A^{-1}*

$$T^{-1}\mathrm{Rad}(B) \subseteq \mathrm{Ptrb}(T^{-1}H(B)) \subseteq T^{-1}\mathrm{QN}(B) \tag{9.2}$$

and

$$\mathrm{Ptrb}(H(A) + T^{-1}(0)) \subseteq T^{-1}\mathrm{QN}(B) \ . \tag{9.3}$$

If in particular T is onto, then also

$$\mathrm{Ptrb}(H(A) + T^{-1}(0)) \subseteq T^{-1}\mathrm{Rad}(B) \tag{9.4}$$

and

$$T^{-1}\mathrm{Rad}(B) = \mathrm{Ptrb}(T^{-1}H(B)) \ . \tag{9.5}$$

Proof. The first inclusion of (9.2) is the first part of (9.1), with $K = H(B)$. Conversely if $d \in \mathrm{Ptrb}(T^{-1}B^{-1}_{left})$, then $T(d-\lambda) \in B^{-1}_{left}$ whenever $0 \neq \lambda \in \mathbf{C}$, giving $\sigma_B^{left}(Td) = \{0\}$ and hence $Td \in \mathrm{QN}(B)$ by (4.4), which is the second inclusion. This also shows that $\mathrm{Ptrb}(A^{-1}_{left}+T^{-1}(0))$ is in $T^{-1}\mathrm{QN}(B)$.

It follows that $\mathrm{Ptrb}(A^{-1}_{left} + T^{-1}(0))$ is a left ideal of $T^{-1}\mathrm{QN}(B)$. If in particular T is onto, then by Corollary 7.1 the largest of these is $T^{-1}\mathrm{Rad}(B)$, giving (9.4). Alternatively, for Fredholm elements, according to (2.2), (9.5) is (9.1) with equality. These arguments establish Theorem 9.1 for left Fredholm and Weyl elements, hence also right and therefore also two sided. □

With the aid of Ruston elements we can improve on (9.4), but only for two-sided Weyl elements:

Theorem 9.2. *If $T : A \to B$ has the strong Riesz property, then*

$$T^{-1}\mathrm{Rad}(B) \subseteq \mathrm{Ptrb}(A^{-1} + T^{-1}(0)) \ , \tag{9.6}$$

with equality if also T is onto.

Proof. If $Tr \in \mathrm{Rad}(B)$ and $a = c + d$ with $c \in A^{-1}$ and $Td = 0$, then also $T(d + r) \in \mathrm{Rad}(B)$. Thus $a + r = c + (d + r)$ is T Raubenheimer, and therefore, using the strong Riesz property, by Corollary 8.1 is Weyl. This gives (9.6), and the opposite inclusion is (9.4) for $H(A) = A^{-1}$. □

10 Commuting perturbation classes

The commuting perturbation class of the T Fredholm elements is easily derived from that of the invertibles:

Theorem 10.1. *If ω is one of $\sigma_{f,T}$, $\sigma^{left}_{f,T}$ and $\sigma^{right}_{f,T}$, then for arbitrary $a, d \in A$ there is implication*

$$d \in T^{-1}\mathrm{Rad}(B) \Longrightarrow \omega(a) = \omega(a + d) \ .$$

Also the following are equivalent:

$$d \text{ is } T \text{ Riesz} \ ;$$

$$(\forall a)\ (ad - da \in T^{-1}\mathrm{Rad}(B) \Longrightarrow \omega(a) = \omega(a + d)) \ ;$$

$$(\forall a)\ (ad - da = 0 \Longrightarrow \omega(a) = \omega(a+d))\ .$$

In particular, for each $H(B)$ of B^{-1}_{left}, B^{-1}_{right} and B^{-1} there is equality

$$\mathrm{Ptrb}_{comm} T^{-1}H(B) = T^{-1}\mathrm{QN}(B)\ .$$

Proof. Apply Theorem 4.1 in the Banach algebra B. □

The commuting perturbation class of the Weyl and of the Browder elements is also given by the Riesz elements, provided we have the strong Riesz property; the argument again goes through Ruston elements:

Theorem 10.2. *If $T : A \to B$ has the strong Riesz property, then for $r \in A$ each of the following are equivalent:*

$$r \text{ is } T \text{ Riesz}\ ; \tag{10.1}$$

$$(\forall a)\ (ar - ra \in T^{-1}\mathrm{Rad}(B) \Longrightarrow \sigma_{\omega,T}(a+r) = \sigma_{\omega,T}(a))\ ; \tag{10.2}$$

$$(\forall a)\ (ar - ra \in T^{-1}(0) \Longrightarrow \sigma_{\omega,T}(a+r) = \sigma_{\omega,T}(a))\ ; \tag{10.3}$$

$$(\forall a)\ (ar - ra = 0 \Longrightarrow \sigma_{\omega,T}(a+r) = \sigma_{\omega,T}(a))\ ; \tag{10.4}$$

$$\sigma_{\omega,T}(r) = \{0\}. \tag{10.5}$$

There is equality

$$\mathrm{Ptrb}_{comm}(A^{-1} + T^{-1}(0)) = T^{-1}\mathrm{QN}(B)\ . \tag{10.6}$$

Proof. Suppose that $r \in A$ is T Riesz, a is T Weyl and $ar - ra \in T^{-1}\mathrm{Rad}(B)$. Then $Tr \in \mathrm{QN}(B)$, $a = c+d$ with $c \in A^{-1}$, $Td = 0$ and $T(ar-ra) \in \mathrm{Rad}(B)$. Then also $c^{-1}(ar-ra)c^{-1} \in T^{-1}\mathrm{Rad}(B)$ and hence

$$rc^{-1} - c^{-1}r \in c^{-1}(ra - ar)c^{-1} + T^{-1}(0) \subseteq T^{-1}\mathrm{Rad}(B)\ .$$

Now with $S : B \to B/\mathrm{Rad}(B) = D$ the quotient map we have implication

$$STr \in \mathrm{QN}(D)_\cap \mathrm{comm}(ST(c^{-1}))\ \Longrightarrow ST(rc^{-1}) \in \mathrm{QN}(D)\ .$$

It implies $T(rc^{-1}) \in \mathrm{QN}(B)$ and $1 + r^{-1}c$ is T Ruston, by Theorem 8.3 therefore T Weyl: but now

$$a + r = (1 + rc^{-1})c + d \in A^{-1} + T^{-1}(0) + T^{-1}(0)$$

is Weyl. Therefore, if r is T Riesz and $ar - ra \in T^{-1}\mathrm{Rad}(B)$, then a is T Weyl iff $a + r$ is T Weyl. Thus if $r \in A$ is T Riesz then (10.2), and hence also (10.3) and (10.4), follow; (10.5) is obtained from (10.4) for $a = 0$. Conversely if (10.5) holds, then as $\sigma_{f,T}(r)$ is a non-empty subset of $\sigma_{\omega,T}(r)$ it follows $\sigma_{f,T}(r) = \{0\}$ and therefore $Tr \in \mathrm{QN}(B)$.

From the equivalence (10.1)$\Longleftrightarrow$(10.4) it follows that the T Riesz elements coincide with the commuting perturbation class of the T Weyl elements. $\square$

Theorem 10.3. *If $T : A \to B$ has the strong Riesz property, then for $d \in A$ each of the following are equivalent:*

$$d \text{ is } T \text{ Riesz} \ ; \tag{10.7}$$

$$(\forall a)\ (ad = da \Longrightarrow \sigma_{b,T}(a + d) = \sigma_{b,T}(a)) \ ; \tag{10.8}$$

$$\sigma_{b,T}(d) = \{0\}. \tag{10.9}$$

There is equality

$$\mathrm{Ptrb}_{comm}(A^{-1} +_{comm} T^{-1}(0)) = T^{-1}\mathrm{QN}(B)\ . \tag{10.10}$$

Proof. If $a \in A$ is T Browder and hence almost invertible Fredholm, then (5.2) applies: with $1 - q = aa^\times = a^\times a$, we have

$$aq = qa\ ;\ a + q \in A^{-1}\ ;\ aq \in \mathrm{QN}(A)\ . \tag{10.11}$$

Note that $T(q)$ is the spectral projection of $T(a)$ ([8] Proposition 2.1), and since $T(a) \in B^{-1}$ it follows $T(q) = 0$. Recall that also $q \in A$ double commutes with a: thus if $d \in T^{-1}\mathrm{QN}(B)$ commutes with $a \in A$, then it also commutes with q. Now, if $c = a + q$, then

$$a + d = c + (d - q)\ ;\ c(d - q) = (d - q)c\ ;\ c \in A^{-1}\ ;\ d - q \in T^{-1}\mathrm{QN}(B)\ ,$$

so that $a+d$ is T Ruston. By Theorem 8.3 it follows that $a+d$ is T Browder. This shows that the T Riesz elements lie in the commuting perturbation class of the T Browder elements; conversely, with no conditions on T, if $d \in A$ is in the commuting perturbation class of the T Browder elements, then

$$0 \neq \lambda \in \mathbf{C} \Longrightarrow T(\lambda^{-1} + d) \in B^{-1} \Longrightarrow T(1 + \lambda d) \in B^{-1}\ , \tag{10.12}$$

which means that (3.1) holds for $Td \in B$, i.e. $d \in T^{-1}\mathrm{QN}(B)$. Therefore, $\mathrm{Ptrb}_{comm}(A^{-1} +_{comm} T^{-1}(0)) = T^{-1}\mathrm{QN}(B)$.

This also shows the equivalence (10.7)$\Longleftrightarrow$(10.8). (10.9) follows from (10.8) for $a = 0$. If (10.9) holds, then using again (10.12) we get d is T Riesz. $\square$

We have been unable to extend Theorem 10.2 to left or right Weyl and Theorem 10.3 to left or right Browder elements. Baklouti ([4] Theorem 1.1) proved that if T is bounded and has the Riesz property, then the T null elements are included in the commuting perturbation class of the T Browder elements, and ([4] Corollary 1.1) that if also T has the strong Riesz property, then the perturbation class of the T Fredholm elements is included in the commuting perturbation class of the T Browder elements. Mouton and Raubenheimer ([19] Theorem 5.1) also show, if T is bounded with the Riesz property, that if $d \in A$ is Riesz with respect to the quotient map $A \to A/T^{-1}(0)$, then d is in the commuting perturbation class of the T Browder elements. Let us mention that the set of Riesz elements with respect to the quotient map $A \to A/T^{-1}(0)$ is contained in the set of T Riesz elements, $T^{-1}\mathrm{QN}(B)$, and the equality between these sets holds when T is onto.

11 Polynomially Riesz

Generally if $S \subseteq A$ is an arbitrary set we shall write

$$\mathrm{Poly}^{-1}(S) = \{a \in A : \exists 0 \neq p \in \mathrm{Poly} \text{ with } p(a) \in S\}\ ,$$

where Poly is the algebra of complex polynomials, $\mathrm{Poly} = \mathbf{C}[z]$. For example if $S = \{0\}$, then $\mathrm{Poly}^{-1}(S)$ consists of the *algebraic* elements of A.

We remark that, provided $S \subseteq A$ is a left or right ideal, the set

$$\mathcal{P}_a^S = \{p \in \mathrm{Poly} : p(a) \in S\}$$

of polynomials p for which $p(a) \in S$ will be an ideal of the algebra Poly. Since the natural numbers are well ordered there will be a unique monic polynomial p of minimal degree contained in $\mathcal{P}_a^S$; we shall write $p = \pi_a \equiv \pi_a^S$. Then $\mathcal{P}_a^S$ is generated by $p = \pi_a$, i.e. $\mathcal{P}_a^S = \pi_a \cdot \mathrm{Poly}$.

This remains true if more generally $S \subseteq A$ is a *commutative ideal*, in the sense that

$$S +_{comm} S \subseteq S\ ,\ A \cdot_{comm} S \subseteq S\ . \tag{11.1}$$

For example the set $\mathrm{QN}(A)$ is a commutative ideal, as well the set $T^{-1}\mathrm{QN}(B)$.

For the proof of the next theorem we need the following result [9] Theorem 2:

If $a \in A$, $f : U \to \mathbf{C}$ is holomorphic in a neighbourhood $U \subseteq \sigma_A(a)$ and $T : A \to B$ has the Riesz property, then there is equality

$$\sigma_{b,T}(f(a)) = f(\sigma_{b,T}(a)). \tag{11.2}$$

We note that for this assertion it is not necessary for T to be bounded.

With the previous notation we prove

Theorem 11.1. *If $T : A \to B$ and if $a \in \mathrm{Poly}^{-1}T^{-1}\mathrm{Rad}(B)$, then*

$$\sigma_{f,T}^{left}(a) = \sigma_{f,T}^{right}(a) = \sigma_{f,T}(a) = \pi_a^{-1}(0) \ . \tag{11.3}$$

If more generally $a \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN}(B)$, then

$$\sigma_{f,T}(a) = \pi_a^{-1}(0) \ , \tag{11.4}$$

and if in particular T has the strong Riesz property, then also

$$\sigma_{b,T}(a) = \pi_a^{-1}(0) \ . \tag{11.5}$$

Proof. With $p \in \mathrm{Poly}$ if $p(a) \in T^{-1}\mathrm{Rad}(B)$, it is clear

$$p(a) \in T^{-1}\mathrm{QN}(B) \implies p(\sigma_{f,T}(a)) = p(\sigma_B(Ta)) = \sigma_B(Tp(a)) = \{0\} \ , \tag{11.6}$$

so that certainly the Fredholm spectrum of a is a subset of the roots of the polynomial p. In the other direction, if $\lambda \in p^{-1}(0)$, then there is a polynomial $q \in \mathrm{Poly}$ for which

$$p \equiv (z - \lambda)q \in \mathrm{Poly} \implies p(a) = (a - \lambda)q(a) \in T^{-1}\mathrm{Rad}(B) \subseteq A \ .$$

If in addition $a - \lambda \in A$ is left T Fredholm, so that $T(a - \lambda) \in B_{left}^{-1}$ is left invertible, then it follows that

$$Tq(a) \in B_{right}^{-1}\mathrm{Rad}(B) \subseteq \mathrm{Rad}(B) \ ,$$

which means that the polynomial p is not minimal. Thus no root of the minimal polynomial can be outside the left, or similarly the right, T Fredholm spectrum. This gives (11.3).

If $p(a) \in T^{-1}\mathrm{QN}(B)$, from (11.6) it follows $\sigma_{f,T}(a) \subset p^{-1}(0)$. Conversely, as above if $\lambda \in p^{-1}(0)$, then there exists a polynomial q such that $p \equiv (z-\lambda)q$ and therefore

$$Tp(a) = (Ta - \lambda)Tq(a) = Tq(a)(Ta - \lambda) \in \mathrm{QN}(B). \tag{11.7}$$

If $a - \lambda$ is two-sided T Fredholm, then $Ta - \lambda$ is an invertible element which commutes with $Tp(a)$. Hence its inverse commutes with $Tp(a)$, and from (11.7) we obtain $Tq(a) \in \mathrm{QN}(B)$ which implies that the polynomial p is not minimal. Hence all roots of the minimal polynomial of a belong to the T Fredholm spectrum of a. This proves (11.4).

Inclusion one way in (11.5) follows from (11.4); conversely if T has the strong Riesz property, then by (11.2) and Theorem 10.3

$$\pi_a(a) \in T^{-1}\mathrm{QN}(B) \Longrightarrow \pi_a(\sigma_{b,T}(a)) = \sigma_{b,T}(\pi_a(a)) = \{0\} ,$$

and hence $\sigma_{b,T}(a) \subset \pi_a^{-1}(0)$. □

From (11.5) it follows that, if $T : A \to B$ has the strong Riesz property and $a \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN}(B)$, then

$$\mathrm{acc}\ \sigma_A(a) \subseteq \pi_a^{-1}(0) ,$$

and hence

$$\lambda \in \sigma_A(a) \setminus \pi_a^{-1}(0) \Longrightarrow q_\lambda \in T^{-1}(0) ,$$

where q_λ is the spectral projection corresponding to a and λ.

We remark that if $T : A \to B$ is onto, then Theorem 11.1 applies to $d \in \mathrm{Poly}^{-1}\mathrm{Ptrb}(T^{-1}B^{-1})$ (Theorem 9.1), and if it has the strong Riesz property, then also to $a \in \mathrm{Poly}^{-1}\mathrm{Ptrb}(A^{-1} + T^{-1}(0))$ (Theorem 9.2). Baklouti ([4] Theorem 1.3) derives $\sigma_{f,T}(a) = \sigma_{b,T}(a) = \pi_a^{-1}(0)$ when T is bounded and onto with the strong Riesz property and $a \in \mathrm{Poly}^{-1}\mathrm{Ptrb}(T^{-1}B^{-1})$. It is clear that if both (11.3) and (11.5) hold, then the left and right Fredholm spectrum both coincide with the Browder spectrum, and therefore also with everything in between.

Polynomially Riesz elements satisfy a curious variant of membership of a perturbation class:

Theorem 11.2. *If $a \in A$ and $d \in A$, if $H(B)$ is one of B^{-1}_{left}, B^{-1}_{right} and B^{-1}, and if $p \in \mathrm{Poly}$, then*

$$ad - da \in T^{-1}\mathrm{Rad}(B) \text{ and } p(d) \in T^{-1}\mathrm{QN}(B) , \tag{11.8}$$

implies

$$p(a) \in T^{-1}H(B) \Longrightarrow a - d \in T^{-1}H(B) . \tag{11.9}$$

Proof. Theorem 4.2, applied to $T(a)$ and $T(d)$ in B. □

12 Holomorphically Riesz

For certain subsets $S \subseteq A$ there is no increase in $\mathrm{Poly}^{-1}(S)$ if polynomials are replaced by non trivial holomorphic functions. Generally if $g : U \to \mathbf{C}$ is holomorphic on open $U \subseteq \mathbf{C}$ and $S \subseteq A$ we write

$$g(S) = \{g(a) : a \in S, \sigma(a) \subseteq U\} \ , \ g^{-1}(S) = \{a \in A : g(a) \in S, \sigma(a) \subseteq U\} \ ,$$

and

$$\mathrm{Holo}_1(K) \subseteq \mathrm{Holo}(K)$$

for those holomorphic functions $g : U \to \mathbf{C}$ which are non constant on each connected component of open $U \supseteq K$. For the "commutative ideals" of (11.1) there is a little bit of functional calculus:

Theorem 12.1. *If $S \subseteq A$ is a commutative ideal, then there is inclusion for holomorphic functions $g : U \to \mathbf{C}$,*

$$g\ \mathrm{Poly}^{-1}(S) \subseteq \mathrm{Poly}^{-1}(S) \ , \tag{12.1}$$

with implication, for $a \in \mathrm{Poly}^{-1}(S)$ and $g \in \mathrm{Holo}\ \sigma(a)$,

$$\pi_a \equiv \prod_{p(\lambda)=0} (z-\lambda)^{\nu_\lambda} \Longrightarrow \prod_{p(\lambda)=0} (z-g(\lambda))^{\nu_\lambda} \in \{h \cdot \pi_{g(a)} : h \in \mathrm{Poly}\} \ . \tag{12.2}$$

If in particular $g \in \mathrm{Holo}_1(U)$, then also

$$g^{-1}(S) \subseteq g^{-1}\mathrm{Poly}^{-1}(S) \subseteq \mathrm{Poly}^{-1}(S) \ . \tag{12.3}$$

Proof. Suppose that $a \in \mathrm{Poly}^{-1}(S)$ and $g \in \mathrm{Holo}\ \sigma(a)$. Then there are holomorphic functions φ_λ for which $(g-g(\lambda))^{\nu_\lambda} \equiv (z-\lambda)^{\nu_\lambda}\varphi_\lambda$ and hence

$$\prod_{p(\lambda)=0} (g-g(\lambda))^{\nu_\lambda} \equiv (\prod_{p(\lambda)=0} \varphi_\lambda)\pi_a \ .$$

It follows

$$\begin{aligned} \prod_{p(\lambda)=0} (g(a)-g(\lambda))^{\nu_\lambda} &= (\prod_{p(\lambda)=0} \varphi_\lambda(a))\pi_a(a) \\ &= \pi_a(a)(\prod_{p(\lambda)=0} \varphi_\lambda(a)) \in A \cdot_{comm} S \subseteq S \,, \end{aligned}$$

and therefore $g(a) \in \mathrm{Poly}^{-1}(S)$ and $\pi_{g(a)}$ divides $\prod_{p(\lambda)=0}(z-g(\lambda))^{\nu_\lambda}$.

Suppose that $g \in \mathrm{Holo}_1(U)$ and $a \in g^{-1}\mathrm{Poly}^{-1}(S)$. Then there is $0 \neq q \in \mathrm{Poly}$ for which $q(g(a)) \in S$, and by compactness there is a polynomial $p \in \mathrm{Poly}$ and a holomorphic function $\varphi \in \mathrm{Holo}\ \sigma(a)$ for which

$$q \circ g \equiv p\varphi \ ; \ \varphi^{-1}(0)_\cap \sigma(a) = \emptyset \ ,$$

so that $\varphi(a) \in A^{-1}$ and $q(g(a)) = p(a)\varphi(a) = \varphi(a)p(a)$. Hence $\varphi(a)^{-1}$ commutes with $p(a)$ and therefore also with $q(g(a))$, and

$$p(a) = \varphi(a)^{-1}q(g(a)) \in A \cdot_{comm} S \subseteq S \ .$$

This gives (12.3). □

We remark that the multiplicities ν_λ cannot be removed from the product of the $z - g(\lambda)$ in (12.2) (take $g \equiv z$), although they are not always necessary (take $g \equiv \pi_a$). With or without multiplicities, it follows from (12.2) that

$$\pi_{g(a)}^{-1}(0) \subseteq g(\pi_a^{-1}(0)) \ . \tag{12.4}$$

Theorem 11.1 has a holomorphic extension:

Theorem 12.2. *Suppose $T : A \to B$ and $a \in A$ and $g \in \mathrm{Holo}_1\sigma(a)$: if $a \in \mathrm{Poly}^{-1}T^{-1}\mathrm{Rad}(B)$, then also $g(a) \in \mathrm{Poly}^{-1}T^{-1}\mathrm{Rad}(B)$ with*

$$\sigma_{f,T}^{left}(g(a)) = \sigma_{f,T}^{right}(g(a)) = \sigma_{f,T}(g(a)) = g(\pi_a^{-1}(0)) \ . \tag{12.5}$$

If more generally $a \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN}(B)$, then also $g(a) \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN}(B)$ with

$$\sigma_{f,T}(g(a)) = g(\pi_a^{-1}(0)) \ . \tag{12.6}$$

If in particular $T : A \to B$ has the strong Riesz property and $a \in \mathrm{Poly}^{-1}T^{-1}\mathrm{QN}(B)$, then also

$$\sigma_{b,T}(g(a)) = g(\pi_a^{-1}(0)) \ . \tag{12.7}$$

Proof. By (12.1) from Theorem 12.1 with $S = T^{-1}\mathrm{Rad}(B)$ and with $S = T^{-1}\mathrm{QN}(B)$, together with Theorem 11.1 applied to $g(a)$, we get each of (12.5), (12.6) and (12.7) with $g(\pi_a^{-1}(0))$ replaced by $\pi_{g(a)}^{-1}(0)$, and hence by (12.4) inclusion at the end. Conversely if $g(\lambda) \notin \sigma_{f,T}(g(a))$, and hence also if $g(\lambda) \notin \sigma_{b,T}(g(a))$, then there is $\varphi_\lambda(a) \in A$ for which

$$T(g(a) - g(\lambda)) = T\varphi_\lambda(a)T(a - \lambda) = T(a - \lambda)T\varphi_\lambda(a) \in B^{-1} \ ,$$

giving $T(a - \lambda) \in B^{-1}$, and hence $\lambda \notin \sigma_{f,T}(a) = \pi_a^{-1}(0)$. □

Equality in (12.4) for $S = T^{-1}\mathrm{Rad}(B)$ and for $S = T^{-1}\mathrm{QN}(B)$ follows.

Theorem 11.2 also extends to holomorphic functions:

Theorem 12.3. *If $a \in A$ and $d \in A$, if $H(B)$ is one of B^{-1}_{left}, B^{-1}_{right} and B^{-1} and if $g \in \mathrm{Holo}_1(\sigma(a)_\cup\sigma(d))$, then if*

$$ad - da \in T^{-1}\mathrm{Rad}(B) \text{ and } g(d) \in T^{-1}\mathrm{QN}(B)\ , \tag{12.8}$$

then there is implication

$$g(a) \in T^{-1}H(B) \Longrightarrow a - d \in T^{-1}H(B)\ . \tag{12.9}$$

Proof. If g is holomorphic, then by compactness there is a polynomial $p \in$ Poly and a holomorphic function $\varphi \in \mathrm{Holo}_1(\sigma(a)_\cup\sigma(d))$ for which

$$g \equiv p\varphi \text{ and } \varphi^{-1}(0)_\cap(\sigma(a)_\cup\sigma(d)) = \emptyset\ ,$$

so that for each $c \in \{a, d\}$

$$g(c) = p(c)\varphi(c) = \varphi(c)p(c) \text{ with } \varphi(c) \in A^{-1}\ .$$

This means that each of (12.8) and (12.9) is equivalent to the corresponding statement of (11.8) and (11.9).

Indeed if $ad-da \in T^{-1}\mathrm{Rad}(B)$ and $g(d) \in T^{-1}QN(B)$, then $Tp(d)T\varphi(d) \in QN(B)$. Since $Tp(d)$ and $T\varphi(d)$ commute, and $T\varphi(d) \in B^{-1}$, it follows $Tp(d) \in QN(B)$.

If $g(a) \in T^{-1}H(B)$, then $Tp(a)T\varphi(a) \in H(B)$ and since $T\varphi(a) \in B^{-1}$ it follows $Tp(a) \in H(B)$.

From Theorem 11.2 we get $a - d \in T^{-1}H(B)$. □

13 Operators

The motivating example for abstract Fredholm theory is the Calkin homomorphism

$$T : B(X) = A \to B = B(X)/K(X)\ ,$$

where $B(X)$ is the bounded operators on a Banach space and $K(X)$ the closed two sided ideal of compact operators. The same Fredholm theory can also be derived from a variant,

$$T_0 : B(X) = A \to B_0 = B(X)/K_0(X)\ ,$$

where $K_0(X)$ is the, not in general closed, ideal of finite rank operators. The compact operators have the advantage of giving a Banach Calkin algebra, but the finite rank operators have the important property that every one of them has a *generalized inverse*. The T null elements of A are here the compact operators, the T almost null are the *inessential* operators and the T Riesz elements are indeed what is known as the Riesz operators. For the homomorphism T_0 the null elements are the finite rank operators and the almost null are again the inessential operators. Here $T_0^{-1}(0) \subseteq T^{-1}(0) \subseteq T_0^{-1}\mathrm{Rad}(B_0) = T^{-1}\mathrm{Rad}(B)$. By Atkinson's theorem ([6] Theorem 3.2.8; [10]; [11] Theorem 6.4.3; [21] Theorem 16.13) the T Fredholm operators and the T_0 Fredholm operators both coincide with the classical Fredholm operators, those with finite dimensional null space and closed range of finite codimension; by Schechter's theorem ([11] Theorem 6.5.3; [21] Theorem 19.7) the T Weyl operators ([21] Theorem 19.7) and the T_0 Weyl operators ([11] Theorem 6.5.3) coincide with the classical Fredholm operators of index zero, and finally the T Browder operators and the T_0 Browder operators are here ([6] Theorem 1.4.5; [21] Theorem 20.21; [1] Theorem 3.48) the Fredholm operators of finite ascent and descent. Also the left and right T Fredholm operators, and also the left and right T_0 Fredholm operators are ([5] Theorem 5.1.5; [6] Theorems 4.3.2, 4.3.3) operators with complemented null space and closed and complemented range either with the null space of finite dimension or the range of finite codimension, as are ([21] Theorem 19.7) the left and right T Weyl operators respectively, and also the left and right T_0 Weyl operators, the left and right Fredholm operators of non positive or non negative index, while the left and right T Browder operators, and also the left and right T_0 Browder operators, are respectively the left and right Fredholm operators of finite ascent and of finite descent ([22] Theorems 5, 6). Evidently the quotient homomorphism is onto, and the Riesz property is a consequence of the spectral theory of compact operators. The strong Riesz property in this case is a consequence of the punctured neighbourhood theorem ([11] Theorem 7.8.4; [21] Theorem 18.7). For operators Corollary 8.1 tells us that the Ruston operators coincide with the Browder operators, and the almost essentially Ruston and the almost Ruston coincide with the Weyl. From [21] Theorem 19.7 and [22] Theorem 8 it follows that $T \in B(X)$ is left (right) Fredholm of non positive (non negative) index iff it is the sum of a left (right) invertible and an inessential, while from [22] Theorems 5, 6, 7 it follows that $T \in B(X)$ is left (right) Fredholm of finite ascent (descent) iff it is the sum of a left (right) invertible and an inessential which commute. By Corollary 8.1 the Raubenheimer operators coincide with the Weyl operators and the commutatively

Raubenheimer operators with the Browder operators. This together with Theorem 8.2 therefore shows that also the same Fredholm theory is derived from the homomorphism $T_1 : A \to B_1$ where now $B_1 = A/J$ where J is the inessential operators $T^{-1}\mathrm{Rad}(B)$.

Theorem 11.1 is based on the Gilfeather discussion [7] of the structure of operators which are polynomially compact.

One other classical example of Fredholm theory is given by

$$T : A = A(\mathbf{D}) \to C(\partial\mathbf{D}) = B$$

embedding the disc algebra

$$A(\mathbf{D}) = C(\mathbf{D})_\cap \mathrm{Holo}(\mathrm{int}\ \mathbf{D})$$

in the continuous functions on the circle $\mathbf{S} = \partial\mathbf{D}$. Here $a \in A$ is invertible provided it does not vanish on the disc, and T Fredholm provided it does not vanish on the circle; the T Weyl elements are those T Fredholm functions $a \in A$ for which [13] the induced mapping

$$a/|a| : \mathbf{S} \to \mathbf{S}$$

is *contractible*, equivalently has zero “winding number”: in particular the complex coordinate $z \in A$ is Fredholm but not Weyl. The homomorphism T is here bounded below, and hence strong Riesz. Since A is commutative the Weyl and Browder functions coincide, and hence also all kinds of Ruston element.

Arendt [2] considers the embedding $T : A \to B$ of the “regular” operators A on a complex Banach lattice X in the bounded operators B, specifically when $X = L_p(G) \subseteq M(G)$ the measure algebra of a locally compact group; in particular T is one one, therefore Riesz, but does not have closed range. With $G = \mathbf{S}$ there exists ([2] Counterexample 3.7) a positive measure $\mu \in M(G)$ on $\mathbf{S}$, self adjoint and of norm 1, which has “disjoint powers” (relative to convolution); now $a \in A$ and $Ta \in B$ are defined as operators on X by convolution:

$$a(f) = (Ta)(f) = \mu * f \ (f \in X)$$

where

$$(\mu * f)(s) = \int_{t\in G} f(t^{-1}s)\mu(dt) \ (s \in G\ ,\ f \in X)\ .$$

Since $Ta \in B$ is compact the spectrum $\sigma_{f,T}(a)$ is a countable subset of $\mathbf{D}$, while it turns out that the whole circle $\mathbf{S} \subseteq \sigma_A(a)$: thus the strong Riesz property (7.4) fails.

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Snežana Č. Živković-Zlatanović
University of Niš
Faculty of Sciences and Mathematics
P.O. Box 224, 18000 Nis, Serbia
E-mail: `mladvlad@open.telekom.rs`

Dragan S. Djordjević
University of Niš
Faculty of Sciences and Mathematics
P.O. Box 224, 18000 Nis, Serbia
E-mail: `dragan@pmf.ni.ac.rs`

Robin E. Harte
Trinity College
Dublin, Ireland
E-mail: `rharte@maths.tcd.ie`