Mixed-type reverse order law and its equivalencies

Nebojša Č. Dinčić
Faculty of Sciences and Mathematics, University of Niš
18000 Niš, Serbia
E-mail: ndincic@hotmail.com

Dragan S. Djordjević
Faculty of Sciences and Mathematics, University of Niš
18000 Niš, Serbia
E-mail: dragandjordjevic70@gmail.com

Dijana Mosić
Faculty of Sciences and Mathematics, University of Niš
18000 Niš, Serbia
E-mail: sknme@ptt.rs

Abstract

In this paper we present new results related to various equivalencies of the mixed-type reverse order law for the Moore-Penrose inverse for operators on Hilbert spaces. Recent finite dimensional results of Tian are extend to Hilbert space operators.

1 Introduction

The reverse order law of the form $(AB)^\dagger = B^\dagger A^\dagger$ does not hold in general for the Moore-Penrose inverse. The classical equivalent condition (A^*A

2010 *Mathematics Subject Classification*: Primary 47A05; Secondary 15A09.
Key words and phrases: Moore-Penrose inverse, weighted Moore-Penrose inverse, reverse order law.
The authors are supported by the Ministry of Education and Science, Serbia, grant #174004.

commutes with $BB^\dagger$, and BB^* commutes with $AA^\dagger$) is proved in [G] for complex matrices, in [B1], [B2] and [I] for closed-range linear bounded operators on Hilbert spaces, and in [KDjC] in rings with involutions. However, various weaker conditions than the reverse order law are also investigated. A significant number of results is already published in this direction (see [Dj1], [Dj2], [DjD], [DjR], [T1], [T2], [T3], [T4], [T5], [WG], [W1], [W2]). It is also important that the reverse order law of the form $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$ is investigated in [Hw].

In this paper we present a set of equivalencies of the mixed type reverse-order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ for the ordinary and weighted Moore-Penrose inverse of linear bounded operators on Hilbert spaces. Some finite dimensional results from [T4] are extended to infinite dimensional settings. We use operator matrices, which naturally appear in the theory of closed-range linear bounded operators on Hilbert spaces. Hence, our method of proving results is essentially different than the method used in [T4].

Let X, Y, Z be Hilbert spaces, and let $\mathcal{L}(X,Y)$ be the set of all linear bounded operators from X to Y. For $A \in \mathcal{L}(X,Y)$ we use, respectively, $\mathcal{N}(A)$, $\mathcal{R}(A)$, A^*: the null space, the range space and the adjoint of A.

The Moore-Penrose inverse of $A \in L(X,Y)$ (if it exists) is the unique operator $A^\dagger \in \mathcal{L}(Y,X)$ satisfying the following:

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A.$$

It is well-known that $A^\dagger$ exists for given A if and only if $\mathcal{R}(A)$ is closed.

Let $M \in \mathcal{L}(Y)$ and $N \in \mathcal{L}(X)$ be positive and invertible operators. The weighted Moore-Penrose inverse of $A \in L(X,Y)$ with respect to the weights M and N (if it exists) is the unique operator $A^\dagger_{M,N} \in \mathcal{L}(Y,X)$ satisfying the following:

$$AA^\dagger_{M,N}A = A, \qquad A^\dagger_{M,N}AA^\dagger_{M,N} = A^\dagger_{M,N},$$

$$(MAA^\dagger_{M,N})^* = MAA^\dagger_{M,N},\ (NA^\dagger_{M,N}A)^* = NA^\dagger_{M,N}A.$$

Also, $A^\dagger_{M,N}$ exists for given A if and only if $\mathcal{R}(A)$ is closed. If $M = I_Y$ and $N = I_X$, then $A^\dagger_{I_Y,I_X}$ is the standard Moore-Penrose inverse $A^\dagger$ of A.

We assume that the reader is familiar with the generalized invertibility and the Moore-Penrose inverse (see, for example, [BIG], [C], [H]).

We continue with several auxiliary results.

Lemma 1.1. *Let $A \in \mathcal{L}(X,Y)$ have a closed range. Then A has the matrix decomposition with respect to the orthogonal decompositions of spaces $X =$*

$\mathcal{R}(A^*) \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible. Moreover,

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

The proof is straightforward.

Lemma 1.2. [DjD] *Let $A \in \mathcal{L}(X, Y)$ have a closed range. Let X_1 and X_2 be closed and mutually orthogonal subspaces of X, such that $X = X_1 \oplus X_2$. Let Y_1 and Y_2 be closed and mutually orthogonal subspaces of Y, such that $Y = Y_1 \oplus Y_2$. Then the operator A has the following matrix representations with respect to the orthogonal sums of subspaces $X = X_1 \oplus X_2 = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = Y_1 \oplus Y_2$:*

(a)

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1 A_1^* + A_2 A_2^*$ maps $\mathcal{R}(A)$ into itself and $D > 0$ (meaning $D \geq 0$ invertible). Also,

$$A^\dagger = \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

where $D = A_1^* A_1 + A_2^* A_2$ maps $\mathcal{R}(A^*)$ into itself and $D > 0$ (meaning $D \geq 0$ invertible). Also,

$$A^\dagger = \begin{bmatrix} D^{-1} A_1^* & D^{-1} A_2^* \\ 0 & 0 \end{bmatrix}.$$

Here A_i denotes different operators in any of these two cases.

The reader should notice the difference between the following notations. If $A, B \in \mathcal{L}(X)$, then $[A, B] = AB - BA$ denotes the commutator of A and B. On the other hand, if $U \in \mathcal{L}(X, Z)$ and $V \in \mathcal{L}(Y, Z)$, then $[U \;\; V] : \begin{bmatrix} X \\ Y \end{bmatrix} \to Z$ denote the matrix form of the corresponding operator. In the following lemma, a lot of well-known and important facts and properties concerning the Moore-Penrose inverse are collected, especially those which we use in the proof of the main theorem.

Lemma 1.3. [BIG], [DjR] *Let $A \in \mathcal{L}(X,Y)$ be a closed range operator, and let $M \in \mathcal{L}(Y)$ and $N \in \mathcal{L}(X)$ be positive definite and invertible operators. Then:*

(1) $A^* = A^\dagger A A^* = A^* A A^\dagger$;

(2) $A^\dagger = A^*(AA^*)^\dagger = (A^*A)^\dagger A^*$;

(3) $\mathcal{R}(A) = \mathcal{R}(AA^\dagger) = \mathcal{R}(AA^*)$;

(4) $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*) = \mathcal{R}(A^\dagger A) = \mathcal{R}(A^*A)$;

(5) $\mathcal{R}(I - A^\dagger A) = \mathcal{N}(A^\dagger A) = \mathcal{N}(A) = \mathcal{R}(A^*)^\perp$;

(6) $\mathcal{R}(I - AA^\dagger) = \mathcal{N}(AA^\dagger) = \mathcal{N}(A^\dagger) = \mathcal{N}(A^*) = \mathcal{R}(A)^\perp$;

(7) $\mathcal{R}(A^\dagger_{M,N}) = N^{-1}\mathcal{R}(A^*)$, $\mathcal{N}(A^\dagger_{M,N}) = M^{-1}\mathcal{N}(A^*)$;

(8) $A^\dagger_{M,N} = N^{-1/2}(M^{1/2}AN^{-1/2})^\dagger M^{1/2}$.

The following result is well-known, and it can be found in [C] p.127, and also [I].

Lemma 1.4. *Let $A \in \mathcal{L}(Y,Z)$ and $B \in \mathcal{L}(X,Y)$ have closed ranges. Then AB has a closed range if and only if $A^\dagger ABB^\dagger$ has a closed range.*

The following result is proved in [DjD], Lemma 2.1.

Lemma 1.5. *Let X, Y be Hilbert spaces, let $C \in \mathcal{L}(X,Y)$ has a closed range, and let $D \in \mathcal{L}(Y)$ be Hermitian and invertible. Then $\mathcal{R}(DC) = \mathcal{R}(C)$ if and only if $[D, CC^\dagger] = 0$.*

We shall also use the following result from [DW], which can easily be extended from complex matrices case to the linear bounded Hilbert space operators.

Lemma 1.6. *Let $H_i, (i = \overline{1,4})$ be Hilbert spaces, let $C \in \mathcal{L}(H_1,H_2)$, $X \in \mathcal{L}(H_2,H_3)$ and $B \in \mathcal{L}(H_3,H_4)$ be closed range operators. Then:*

$$C(BXC)^\dagger B = X^\dagger$$

if and only if:

$$\mathcal{R}(B^*BX) = \mathcal{R}(X) \ \text{ and } \ \mathcal{N}(XCC^*) = \mathcal{N}(X).$$

Let $\mathcal{A}$ be an unital C^*–algebra with the unit 1. Denote the set of all projections by $\mathcal{P}(\mathcal{A}) = \{p \in \mathcal{A} : p^2 = p = p^*\}$. In [L, Theorem 10.a] the following results are proved.

Lemma 1.7. [L] *Let $p, q \in \mathcal{P}(\mathcal{A})$. Then the following statements are equivalent:*

(a) *pq is Moore-Penrose invertible;*

(b) *qp is Moore-Penrose invertible;*

(c) *$(1-p)(1-q)$ is Moore-Penrose invertible;*

(d) *$(1-q)(1-p)$ is Moore-Penrose invertible.*

Lemma 1.8. [L] *Let $p, q \in \mathcal{P}(\mathcal{A})$. If pq is Moore-Penrose invertible, then:*

$$(qp)^\dagger = pq - p[(1-p)(1-q)]^\dagger q.$$

We shall use these results in the case of $\mathcal{A} = \mathcal{L}(X)$.

2 Main results

Many necessary and sufficient condition for $(AB)^\dagger = B^\dagger A^\dagger$ to hold were given in the literature. In the paper of Tian [T3], one can found the following important relation: $(AB)^\dagger = B^\dagger A^\dagger$ iff $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ and $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$. Therefore, it is necessary to seek various equivalent conditions for $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ to satisfy. The next theorem is our main results, and it represents the generalization of results from [T4] to infinite dimensional settings.

Theorem 2.1. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ be operators such that A, B and AB have closed ranges. The following statements are equivalent:*

(a1) $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$;

(a2) $(AB)^\dagger = B^*(A^*ABB^*)^\dagger A^*$;

(a3) $(AB)^\dagger = B^\dagger A^\dagger - B^\dagger((I - BB^\dagger)(I - A^\dagger A))^\dagger A^\dagger$;

(b1) $((A^\dagger)^*B)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^*$;

(b2) $((A^\dagger)^*B)^\dagger = B^*((A^*A)^\dagger BB^*)^\dagger A^\dagger$;

(b3) $((A^\dagger)^*B)^\dagger = B^\dagger A^* - B^\dagger((I-BB^\dagger)(I-A^\dagger A))^\dagger A^*$;

(c1) $(A(B^\dagger)^*)^\dagger = B^*(A^\dagger ABB^\dagger)^\dagger A^\dagger$;

(c2) $(A(B^\dagger)^*)^\dagger = B^\dagger(A^*A(BB^*)^\dagger)^\dagger A^*$;

(c3) $(A(B^\dagger)^*)^\dagger = B^*A^\dagger - B^*((I-BB^\dagger)(I-A^\dagger A))^\dagger A^\dagger$;

(d1) $(B^\dagger A^\dagger)^\dagger = A(BB^\dagger A^\dagger A)^\dagger B$;

(d2) $(B^\dagger A^\dagger)^\dagger = (A^\dagger)^*((BB^*)^\dagger(A^*A)^\dagger)^\dagger(B^\dagger)^*$;

(d3) $(B^\dagger A^\dagger)^\dagger = AB - A((I-A^\dagger A)(I-BB^\dagger))^\dagger B$;

(e1) $(A^\dagger AB)^\dagger A^\dagger = B^\dagger(ABB^\dagger)^\dagger$;

(e2) $(A^\dagger AB)^\dagger A^* = B^\dagger((A^\dagger)^*BB^\dagger)^\dagger$;

(e3) $(A^\dagger A(B^\dagger)^*)^\dagger A^\dagger = B^*(ABB^\dagger)^\dagger$;

(e4) $(BB^\dagger A^\dagger)^\dagger B = A(B^\dagger A^\dagger A)^\dagger$;

(e5) $(A^*AB)^\dagger A^* = B^*(ABB^*)^\dagger$;

(e6) $((A^*A)^\dagger B)^\dagger A^\dagger = B^*((A^\dagger)^*BB^*)^\dagger$;

(e7) $(A^*A(B^\dagger)^*)^\dagger A^* = B^\dagger(A(BB^*)^\dagger)^\dagger$;

(e8) $B^\dagger((A^*)^\dagger(BB^*)^\dagger)^\dagger = ((A^*A)^\dagger(B^*)^\dagger)^\dagger A^\dagger$;

(e9) $(AA^*ABB^*B)^\dagger = B^\dagger(A^*ABB^*)^\dagger A^\dagger$;

(f1) $(A^\dagger AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger$ *and* $(ABB^\dagger)^\dagger = (A^\dagger ABB^\dagger)^\dagger A^\dagger$;

(f2) $(A^\dagger AB)^\dagger = B^*(A^\dagger ABB^*)^\dagger$ *and* $(ABB^\dagger)^\dagger = (A^*ABB^\dagger)^\dagger A^*$;

(f3) $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A - B^\dagger((I-BB^\dagger)(I-A^\dagger A))^\dagger A^\dagger A$ *and* $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger - BB^\dagger((I-BB^\dagger)(I-A^\dagger A))^\dagger A^\dagger$;

(g1) $\mathcal{R}((AB)^\dagger) = \mathcal{R}(B^\dagger(A^\dagger ABB^\dagger)A^\dagger)$ *and* $\mathcal{R}(((AB)^\dagger)^*) = \mathcal{R}((B^\dagger(A^\dagger ABB^\dagger)A^\dagger)^*)$;

(g2) $\mathcal{R}((AB)^\dagger) = \mathcal{R}(B^\dagger A^\dagger)$ *and* $\mathcal{R}((B^*A^*)^\dagger) = \mathcal{R}((A^*)^\dagger(B^*)^\dagger)$;

(g3) $\mathcal{R}(AA^*AB) = \mathcal{R}(AB)$ *and* $\mathcal{R}(B^*B(AB)^*) = \mathcal{R}((AB)^*)$.

Proof. The existence of various terms appearing in the statements of the theorem follows mainly from the Lemma 1.4, and from some properties of the kernel and range of operators (see Lemma 1.3). The existence of the Moore-Penrose inverse of the products like $(I - BB^\dagger)(I - A^\dagger A)$ follows from Lemma 1.7.

Using Lemma 1.1, we conclude that the operator B has the following matrix form:

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where B_1 is invertible. Then

$$B^\dagger = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}.$$

From Lemma 1.2 also follows that the operator A has the following matrix form:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1 A_1^* + A_2 A_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A))$. Then

$$A^\dagger = \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}.$$

First we find an equivalent form for the statement (a1). We have

$$S = A^\dagger A B B^\dagger = \begin{pmatrix} A_1^* D^{-1} A_1 & 0 \\ A_2^* D^{-1} A_1 & 0 \end{pmatrix},$$

and consequently

$$S^\dagger = (S^* S)^\dagger S^* = \begin{pmatrix} (A_1^* D^{-1} A_1)^\dagger A_1^* D^{-1} A_1 & (A_1^* D^{-1} A_1)^\dagger A_1^* D^{-1} A_2 \\ 0 & 0 \end{pmatrix}.$$

It follows that

$$B^\dagger S^\dagger A^\dagger = \begin{pmatrix} B_1^{-1} (A_1^* D^{-1} A_1)^\dagger A_1^* D^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$(AB)^\dagger = B^\dagger (A^\dagger A B B^\dagger)^\dagger B^\dagger$$

is equivalent to:

$$(A_1 B_1)^\dagger = B_1^{-1} (A_1^* D^{-1} A_1)^\dagger A_1^* D^{-1} = B_1^{-1} (D^{-1/2} A_1)^\dagger D^{-1/2}.$$

By checking the Penrose equations, the last formula holds if and only if
(2.1)
$$[B_1B_1^*,(D^{-1/2}A_1)^\dagger D^{-1/2}A_1]=0 \quad \text{and} \quad [D,D^{-1/2}A_1(D^{-1/2}A_1)^\dagger]=0.$$

Hence, the statement (a1) is equivalent to (2.1).
Let us now find some more equivalent statements to the condition (a1). Using Lemma 1.5, we get that (2.1) is equivalent to:

$$\mathcal{R}(DA_1)=\mathcal{R}(A_1) \text{ and } \mathcal{R}(B_1B_1^*A_1^*)=\mathcal{R}(A_1^*).$$

or

$$\mathcal{R}(DA_1)=\mathcal{R}(A_1) \text{ and } \mathcal{N}(A_1B_1B_1^*)=\mathcal{N}(A_1),$$

If we apply Lemma 1.6, for $X=A_1B_1$, $C=B_1^{-1}$, $B=D^{-1/2}$, the equality:

$$(A_1B_1)^\dagger=B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2}$$

is equivalent to:

$$\mathcal{R}(D^{-1}A_1B_1)=\mathcal{R}(A_1B_1) \text{ and } \mathcal{N}(A_1B_1(B_1^*B_1)^{-1})=\mathcal{N}(A_1B_1),$$

or

$$\mathcal{R}(D^{-1}A_1B_1)=\mathcal{R}(A_1B_1) \text{ and } \mathcal{R}((B_1^*B_1)^{-1}(A_1B_1)^*)=\mathcal{R}((A_1B_1)^*).$$

Now, we find an equivalent statement to (g3). Conditions

$$\mathcal{R}(AA^*AB)=\mathcal{R}(AB) \quad \text{and} \quad \mathcal{R}(B^*B(AB)^*)=\mathcal{R}((AB)^*)$$

are equivalent to

$$\mathcal{R}(DA_1B_1)=\mathcal{R}(A_1B_1) \quad \text{and} \quad \mathcal{R}(B_1^*B_1(A_1B_1)^*)=\mathcal{R}((A_1B_1)^*)$$

which is equivalent to (2.1). Hence, (g3) is equivalent to (a1).

Analogously, the equivalencies: $(b1) \Leftrightarrow (g3), (c1) \Leftrightarrow (g3)$ and $(d1) \Leftrightarrow (g3)$ can be proved.

Let us now prove, for example, $(c2) \Leftrightarrow (g3)$. Using above notations, and

$$T=A^*A(BB^*)^\dagger=\begin{pmatrix} A_1^*A_1(B_1B_1^*)^{-1} & 0 \\ A_2^*A_1(B_1B_1^*)^{-1} & 0 \end{pmatrix},$$

it is easy to see that

$$\begin{aligned} T^\dagger &= (T^*T)^\dagger T^* \\ &= \begin{pmatrix} (D^{1/2}A_1(B_1B_1^*)^{-1})^\dagger D^{-1/2}A_1 & (D^{1/2}A_1(B_1B_1^*)^{-1})^\dagger D^{-1/2}A_2 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Now,

$$(A(B^\dagger)^*)^\dagger = B^\dagger(A^*A(BB^*)^\dagger)^\dagger A^*$$

if and only if

$$(A_1(B_1^*)^{-1})^\dagger = B_1^{-1}(D^{1/2}A_1(B_1B_1^*)^{-1})^\dagger D^{1/2}.$$

Applying Lemma 1.6, for $X = A_1(B_1^*)^{-1}$, $C = B_1^{-1}$, $B = D^{1/2}$, the last equality is equivalent to

$$\mathcal{R}(DA_1(B_1^*)^{-1}) = \mathcal{R}(A_1(B_1^*)^{-1}) \text{ and } \mathcal{N}(A_1(B_1^*)^{-1}B_1^{-1}(B_1^*)^{-1}) = \mathcal{N}(A_1(B_1^*)^{-1}),$$

i.e.

$$\mathcal{R}(DA_1B_1) = \mathcal{R}(A_1B_1) \text{ and } \mathcal{R}(B_1^{-1}A_1^*) = \mathcal{R}((A_1B_1)^*),$$

so we have just proved that (c2) is equivalent to (g3).

Analogously, we prove the equivalencies $(a2) \Leftrightarrow (g3), (b2) \Leftrightarrow (g3)$ and $(d2) \Leftrightarrow (g3)$.

In proving equivalencies including $e-$statements, there are no other techniques beside those we have already shown in the previous part of the proof.

The table of proper statements is given bellow as some kind od summary overview, and also for the sake of completeness:

(a1) $(A_1B_1)^\dagger = B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2}$;

(a2) $(A_1B_1)^\dagger = B_1^*(D^{1/2}A_1B_1B_1^*)^\dagger D^{1/2}$;

(b1) $(D^{-1}A_1B_1)^\dagger = B_1^{-1}(D^{-1/2}A_1)^\dagger D^{1/2}$;

(b2) $(D^{-1}A_1B_1)^\dagger = B_1^*(D^{-3/2}A_1B_1B_1^*)^\dagger D^{-1/2}$;

(c1) $(A_1(B_1^*)^{-1})^\dagger = B_1^*(D^{-1/2}A_1)^\dagger D^{-1/2}$;

(c2) $(A_1(B_1^*)^{-1})^\dagger = B_1^{-1}(D^{1/2}A_1(B_1B_1^*)^{-1})^\dagger D^{1/2}$;

(d1) $(B_1^{-1}A_1^*D^{-1})^\dagger = D^{1/2}(A_1^*D^{-1/2})^\dagger B_1$;

(d2) $(B_1^{-1}A_1^*D^{-1})^\dagger = D^{-1/2}((B_1B_1^*)^{-1}A_1^*D^{-3/2})^\dagger(B_1^*)^{-1}$;

(e1) $(D^{-1/2}A_1B_1)^\dagger D^{-1/2} = B_1^{-1}A_1^\dagger$;

(e2) $(D^{-1/2}A_1B_1)^\dagger D^{-1/2} = B_1^{-1}(D^{-1}A_1)^\dagger D^{-1}$;

(e3) $(D^{-1/2}A_1(B_1^*)^{-1})^\dagger = B_1^*A_1^\dagger D^{1/2}$;

(e4) $(B_1^{-1}A_1^*D^{-1/2})^\dagger = D^{-1/2}(A_1^*D^{-1})^\dagger B_1$;

(e5) $(D^{1/2}A_1B_1)^\dagger = B_1^*(A_1B_1B_1^*)^\dagger D^{-1/2}$;

(e6) $(D^{-1}A_1B_1B_1^*)^\dagger = (B_1^*)^{-1}(D^{-3/2}A_1B_1)^\dagger D^{-1/2}$;

(e7) $(D^{1/2}A_1(B_1^*)^{-1})^\dagger = B_1^{-1}(A_1(B_1B_1^*)^{-1})^\dagger D^{-1/2}$;

(e8) $(D^{-1}A_1(B_1B_1^*)^{-1})^\dagger = B_1(D^{-3/2}A_1(B_1^*)^{-1})^\dagger D^{-1/2}$;

(e9) $(DA_1B_1B_1^*B_1)^\dagger = B_1^{-1}(D^{1/2}A_1B_1B_1^*)^\dagger D^{-1/2}$.

Each of those statements is equivalent to:

$$\mathcal{R}(D^\alpha A_1B_1) = \mathcal{R}(A_1B_1) \text{ and } \mathcal{N}(A_1B_1(B_1^*B_1)^\beta) = \mathcal{N}(A_1B_1),$$

for some $\alpha, \beta \in \{-1, 1\}$. More precisely, we have:

α	β	statement
1	1	a2, d1, e3, e6
1	-1	b1, c2, e1, e8
-1	1	b2, c1, e4, e5
-1	-1	a1, d2, e2, e7, e9

Using Lemma 1.5, we have:

$$\begin{aligned}\mathcal{R}(D^\alpha A_1B_1) = \mathcal{R}(A_1B_1) &\Leftrightarrow [D^\alpha, A_1B_1(A_1B_1)^\dagger] = 0\\ &\Leftrightarrow [D, A_1B_1(A_1B_1)^\dagger] = 0,\end{aligned}$$

and:

$$\begin{aligned}\mathcal{N}(A_1B_1(B_1^*B_1)^\beta) = \mathcal{N}(A_1B_1) &\Leftrightarrow \mathcal{R}((B_1^*B_1)^\beta(A_1B_1)^*) = \mathcal{R}((A_1B_1)^*) \Leftrightarrow\\ &\Leftrightarrow [(B_1^*B_1)^\beta, (A_1B_1)^*((A_1B_1)^*)^\dagger] = 0 \Leftrightarrow\\ &\Leftrightarrow [(B_1^*B_1)^\beta, (A_1B_1)^\dagger A_1B_1] = 0 \Leftrightarrow\\ &\Leftrightarrow [B_1^*B_1, (A_1B_1)^\dagger A_1B_1] = 0,\end{aligned}$$

which means that each statement mentioned in the table above is equivalent to (g3). Now, we prove the equivalencies $(x3) \Leftrightarrow (x1)$, where $x \in \{a, b, c, d, f\}$.

First, we prove $(a3) \Leftrightarrow (a1)$:

$$(a3) \Leftrightarrow (AB)^\dagger = B^\dagger A^\dagger - B^\dagger[(I - BB^\dagger)(I - A^\dagger A)]^\dagger A^\dagger.$$

Using Lemma 1.8, for $P = BB^\dagger$ and $Q = A^\dagger A$, we have:

$$(2.2)\quad (A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A - BB^\dagger[(I - BB^\dagger)(I - A^\dagger A)]^\dagger A^\dagger A.$$

If we premultiply this expression by $B^\dagger$ and postmultiply it by $A^\dagger$, we obtain:

$$B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger = B^\dagger A^\dagger - B^\dagger[(I - BB^\dagger)(I - A^\dagger A)]^\dagger A^\dagger = (AB)^\dagger,$$

and we have the proof.
Analogously, way we can prove that $(b3) \Leftrightarrow (b1)$ and $(c3) \Leftrightarrow (c1)$; the part $(d3) \Leftrightarrow (d1)$ is very similar - the difference is in taking $Q = BB^\dagger$ and $P = A^\dagger A$.
Let us now prove $(f3) \Leftrightarrow (f1)$:

$$(f3.1) \Leftrightarrow (A^\dagger AB)^\dagger = B^\dagger A^\dagger A - B^\dagger((I - BB^\dagger)(I - A^\dagger A))^\dagger A^\dagger A.$$

If we premultiply (2.2) by $B^\dagger$, we have:

$$B^\dagger(A^\dagger ABB^\dagger)^\dagger = B^\dagger A^\dagger A - B^\dagger((I - BB^\dagger)(I - A^\dagger A))^\dagger A^\dagger A = (A^\dagger AB)^\dagger,$$

i.e. part $(f1.1)$. Also,

$$(f3.2) \Leftrightarrow (ABB^\dagger)^\dagger = BB^\dagger A^\dagger - BB^\dagger((I - BB^\dagger)(I - A^\dagger A))^\dagger A^\dagger.$$

If we postmultiply (2.2) by $A^\dagger$, we have:

$$(A^\dagger ABB^\dagger)^\dagger A^\dagger = BB^\dagger A^\dagger - BB^\dagger((I - BB^\dagger)(I - A^\dagger A))^\dagger A^\dagger = (ABB^\dagger)^\dagger,$$

i.e. part $(f1.2)$. We have finished this part of the proof.

Let us now see what are the equivalent of statements (f1) and (f2).
A simple computation shows that (f1) is equivalent to the following two statements:

$$(2.3)\quad (D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_i = B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2}A_i,\ i = 1, 2;$$
$$(2.4)\quad A_1^\dagger = (D^{-1/2}A_1)^\dagger D^{-1/2}.$$

Suppose that (f1) holds; if we substitute (2.4) in (2.3), then postmultiply each of modified equations (2.3) by A_i^*, and add them, we get:

$$(D^{-1/2}A_1B_1)^\dagger = B_1^{-1}A_1^\dagger D^{1/2},$$

which holds if and only if:

$$[D, A_1A_1^\dagger] = 0 \text{ and } [B_1B_1^*, A_1^\dagger A_1] = 0,$$

which is, by Lemma 1.5, equivalent to

$$\mathcal{R}(DA_1)=\mathcal{R}(A_1) \text{ and } \mathcal{R}(B_1B_1^*A_1^*)=\mathcal{R}(A_1^*),$$

i.e. we get the statement (a1). It is not difficult to see that the reverse implication also holds.
An easy computation shows that (f2) is equivalent to the following two statements:

$$(D^{-1/2}A_1B_1)^{\dagger}D^{-1/2}A_i=B_1^*(D^{-1/2}A_1B_1B_1^*)^{\dagger}D^{-1/2}A_i,\ i=1,2; \tag{2.5}$$
$$A_1^{\dagger}=(D^{-1/2}A_1)^{\dagger}D^{-1/2}. \tag{2.6}$$

Suppose that (f2) holds; if we postmultiply each equations of (2.5) by A_i^*, and add them, we obtain:

$$(D^{-1/2}A_1B_1)^{\dagger}=B_1^*(D^{-1/2}A_1B_1B_1^*)^{\dagger},$$

which holds, by Lemma 1.6, if and only if $\mathcal{N}(A_1B_1B_1^*B_1)=\mathcal{N}(A_1B_1)$. As in the previous part of the proof, (2.6) is equivalent to $\mathcal{R}(DA_1)=\mathcal{R}(A_1)$. So, we have the part $(f2)\Rightarrow(a1)$. The reverse implication can easily be obtained.

Let us now see what are the equivalent statements of (g1) and (g2). First, (g1):

$$\begin{aligned}\mathcal{R}(B^{\dagger}(A^{\dagger}ABB^{\dagger})A^{\dagger})=\mathcal{R}((AB)^{\dagger})=\mathcal{R}((AB)^*)\Leftrightarrow\\ \mathcal{R}(B_1^*A_1^*)=\mathcal{R}(B_1^{-1}(D^{-1/2}A_1)^{\dagger}D^{-1/2})=\mathcal{R}(B_1^{-1}(D^{-1/2}A_1)^{\dagger})\Leftrightarrow\\ B_1\mathcal{R}(B_1^*A_1^*)=\mathcal{R}(B_1B_1^*A_1^*)=\mathcal{R}((D^{-1/2}A_1)^{\dagger})=\mathcal{R}((D^{-1/2}A_1)^*)=\mathcal{R}(A_1^*),\end{aligned}$$

so we actually have:

$$\mathcal{R}(B_1B_1^*A_1^*)=\mathcal{R}(A_1^*).$$

The second condition: $\mathcal{R}(((AB)^{\dagger})^*)=\mathcal{R}((B^{\dagger}(A^{\dagger}ABB^{\dagger})A^{\dagger})^*)$ becomes:

$$\begin{aligned}\mathcal{N}(B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger})=\mathcal{N}((AB)^{\dagger})=\mathcal{N}((AB)^*)\Leftrightarrow\\ \mathcal{N}(A_1^*)=\mathcal{N}(B_1^*A_1^*)=\mathcal{N}(B_1^{-1}(D^{-1/2}A_1)^{\dagger}D^{-1/2})=\mathcal{N}((D^{-1/2}A_1)^{\dagger}D^{-1/2})\Leftrightarrow\\ \mathcal{R}(A_1)=\mathcal{R}(D^{-1/2}(A_1^*D^{-1/2})^{\dagger})\Leftrightarrow\\ D^{1/2}\mathcal{R}(A_1)=\mathcal{R}(D^{1/2}A_1)=\mathcal{R}((A_1^*D^{-1/2})^{\dagger})=\mathcal{R}((A_1^*D^{-1/2})^*)=\mathcal{R}(D^{-1/2}A_1),\end{aligned}$$

so we have:

$$\mathcal{R}(DA_1)=\mathcal{R}(A_1).$$

Those two things are equivalent to the (a1), so we have just proved $(g1) \Leftrightarrow (a1)$.

Now, (g2):

$$\begin{aligned}
\mathcal{R}(B^\dagger A^\dagger) = \mathcal{R}((AB)^\dagger) = \mathcal{R}((AB)^*) \Leftrightarrow \\
\mathcal{R}(B_1^* A_1^*) = \mathcal{R}(B_1^* A_1^* D^{-1}) = \mathcal{R}(B_1^{-1} A_1^*) \Leftrightarrow \\
B_1 \mathcal{R}(B_1^* A_1^*) = \mathcal{R}(B_1 B_1^* A_1^*) = \mathcal{R}(A_1^*)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}((B^* A^*)^\dagger) = \mathcal{R}((A^*)^\dagger (B^*)^\dagger) \Leftrightarrow \\
\mathcal{N}((AB)^\dagger) = \mathcal{N}(B^\dagger A^\dagger) = \mathcal{N}((AB)^*) \Leftrightarrow \\
\mathcal{N}(B_1^* A_1^*) = \mathcal{N}(B_1^{-1} A_1^* D^{-1}) \Leftrightarrow \\
\mathcal{N}(A_1^*) = \mathcal{N}(A_1^* D^{-1}) \Leftrightarrow \\
\mathcal{R}(A_1) = \mathcal{R}(D^{-1} A_1),
\end{aligned}$$

which together are equivalent to (a1), so we have just proved $(g2) \Leftrightarrow (a1)$.

□

Now we formulate analogous result for the weighted Moore-Penrose inverse.

Theorem 2.2. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ be operators such that A, B and AB have closed ranges. Suppose $M \in \mathcal{L}(Z)$ and $\mathcal{N} \in \mathcal{L}(X)$ are positive definite invertible operators. The following statements are equivalent:*

(a1) $(AB)^\dagger_{M,N} = B^\dagger_{I,N}(A^\dagger_{M,I} ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I}$;

(a2) $(AB)^\dagger_{M,N} = N^{-1}B^*(A^*MABN^{-1}B^*)^\dagger A^*M$;

(a3) $(AB)^\dagger_{M,N} = B^\dagger_{I,N}A^\dagger_{M,I} - B^\dagger_{I,N}((I - BB^\dagger_{I,N})(I - A^\dagger_{M,I}A))^\dagger A^\dagger_{M,I}$;

(b1) $((A^*)^\dagger_{I,M^{-1}}B)^\dagger_{M^{-1},N} = B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^*$;

(b2) $((A^*)^\dagger_{I,M^{-1}}B)^\dagger_{M^{-1},N} = N^{-1}B^*((A^*MA)^\dagger(BN^{-1}B^*))^\dagger A^\dagger_{M,I}M^{-1}$;

(b3) $((A^*)^\dagger_{I,M^{-1}}B)^\dagger_{M^{-1},N} = B^\dagger_{I,N}A^* - B^\dagger_{I,N}((I - BB^\dagger_{I,N})(I - A^\dagger_{M,I}A))^\dagger A^*$;

(c1) $(A(B^*)^\dagger_{N^{-1},I})^\dagger_{M,N^{-1}} = B^*(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I}$;

(c2) $(A(B^*)^\dagger_{N^{-1},I})^\dagger_{M,N^{-1}} = NB^\dagger_{I,N}((A^*MA)(BN^{-1}B^*)^\dagger)^\dagger A^*M$;

(c3) $(A(B^*)^\dagger_{N^{-1},I})^\dagger_{M,N^{-1}} = B^*A^\dagger_{M,I} - B^*((I - BB^\dagger_{I,N})(I - A^\dagger_{M,I}A))^\dagger A^\dagger_{M,I}$;

(d1) $(B^\dagger_{I,N}A^\dagger_{M,I})^\dagger_{N,M} = A(BB^\dagger_{I,N}A^\dagger_{M,I}A)^\dagger B$;

(d2) $(B^\dagger_{I,N}A^\dagger_{M,I})^\dagger_{N,M} = M^{-1}(A^*)^\dagger_{I,M^{-1}}((BN^{-1}B^*)^\dagger(A^*MA)^\dagger)^\dagger(B^*)^\dagger_{N^{-1},I}N$;

(d3) $(B^\dagger_{I,N}A^\dagger_{M,I})^\dagger_{N,M} = AB - A((I-A^\dagger_{M,I}A)(I-BB^\dagger_{I,N}))^\dagger B$;

(e1) $(A^\dagger_{M,I}AB)^\dagger_{I,N}A^\dagger_{M,I} = B^\dagger_{I,N}(ABB^\dagger_{I,N})^\dagger_{M,I}$;

(e2) $(A^\dagger_{M,I}AB)^\dagger_{I,N}A^* = B^\dagger_{I,N}((A^*)^\dagger_{I,M^{-1}}BB^\dagger_{I,N})^\dagger_{M^{-1},I}$;

(e3) $(A^\dagger_{M,I}A(B^*)^\dagger_{N^{-1},I})^\dagger_{I,N^{-1}}A^\dagger_{M,I} = B^*(ABB^\dagger_{I,N})^\dagger_{M,I}$;

(e4) $(BB^\dagger_{I,N}A^\dagger_{M,I})^\dagger_{I,M}B = A(B^\dagger_{I,N}A^\dagger_{M,I}A)^\dagger_{N,I}$;

(e5) $N(A^*MAB)^\dagger_{I,N}A^*M = B^*(ABN^{-1}B^*)^\dagger_{M,I}$;

(e6) $N((A^*MA)^\dagger B)^\dagger_{I,N}A^\dagger_{M,I} = B^*((A^*)^\dagger_{I,M^{-1}}BN^{-1}B^*)^\dagger_{M^{-1},I}M$;

(e7) $(A^*MA(B^*)^\dagger_{N^{-1},I})^\dagger_{I,N^{-1}}A^*M = NB^\dagger_{I,N}(A(BN^{-1}B^*)^\dagger)^\dagger_{M,I}$;

(e8) $NB^\dagger_{I,N}((A^*)^\dagger_{I,M^{-1}}(BN^{-1}B^*)^\dagger)^\dagger_{M^{-1},I}M = ((A^*MA)^\dagger(B^*)^\dagger_{N^{-1},I})^\dagger_{I,N^{-1}}A^\dagger_{M,I}$;

(e9) $(AA^*MABN^{-1}B^*B)^\dagger_{M,N} = B^\dagger_{I,N}(A^*MABN^{-1}B^*)^\dagger A^\dagger_{M,I}$;

(f1) $(A^\dagger_{M,I}AB)^\dagger_{I,N} = B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger$ *and*
$(ABB^\dagger_{I,N})^\dagger_{M,I} = (A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I}$;

(f2) $(A^\dagger_{M,I}AB)^\dagger_{I,N} = N^{-1}B^*(A^\dagger_{M,I}ABN^{-1}B^*)^\dagger$ *and*
$(ABB^\dagger_{I,N})^\dagger_{M,I} = (A^*MABB^\dagger_{I,N})^\dagger A^*M$;

(f3) $(A^\dagger_{M,I}AB)^\dagger_{I,N} = B^\dagger_{I,N}A^\dagger_{M,I}A - B^\dagger_{I,N}((I-BB^\dagger_{I,N})(I-A^\dagger_{M,I}A))^\dagger A^\dagger_{M,I}A$
and
$(ABB^\dagger_{I,N})^\dagger_{M,I} = BB^\dagger_{I,N}A^\dagger_{M,I} - BB^\dagger_{I,N}((I-BB^\dagger_{I,N})(I-A^\dagger_{M,I}A))^\dagger A^\dagger_{M,I}$;

(g1) $\mathcal{R}((AB)^\dagger_{M,N}) = \mathcal{R}(B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I})$ *and*
$\mathcal{R}(((AB)^\dagger_{M,N})^*) = \mathcal{R}((B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I})^*)$;

(g2) $\mathcal{R}((AB)^\dagger_{M,N}) = \mathcal{R}(B^\dagger_{I,N}A^\dagger_{M,I})$ *and*
$\mathcal{R}((B^*A^*)^\dagger_{N^{-1},M^{-1}}) = \mathcal{R}((A^*)^\dagger_{I,M^{-1}}(B^*)^\dagger_{N^{-1},I})$;

(g3) $\mathcal{R}(AA^*MAB) = \mathcal{R}(AB)$ *and* $\mathcal{R}((ABN^{-1}B^*B)^*) = \mathcal{R}((AB)^*)$.

Proof. Using the basic relation between ordinary and weighted Moore-Penrose inverse:

$$A^\dagger_{M,N} = N^{-1/2}(M^{1/2}AN^{-1/2})^\dagger M^{1/2},$$

and substitutions:

$$\tilde{A} = M^{1/2}A,\ \tilde{B} = BN^{-1/2},$$

all statements from this theorem reduces to the statements of the already-proven Theorem 2.1. For example, let we prove $(e6) \Leftrightarrow (g2)$.

$$\begin{aligned}
(e6) \quad &\Leftrightarrow \quad N((A^*MA)^\dagger B)^\dagger_{I,N} A^\dagger_{M,I} = B^*((A^*)^\dagger_{I,M^{-1}} BN^{-1}B^*)^\dagger_{M^{-1},I} M \\
&\Leftrightarrow \quad N^{1/2}((A^*MA)^\dagger BN^{-1/2})^\dagger (M^{1/2}A)^\dagger M^{1/2} = B^*((A^*M^{-1/2})^\dagger BN^{-1}B^*)^\dagger M^{1/2} \\
&\Leftrightarrow \quad ((\tilde{A}^*\tilde{A})^\dagger \tilde{B})^\dagger \tilde{A}^\dagger = \tilde{B}^*((\tilde{A}^*)^\dagger \tilde{B}\tilde{B}^*)^\dagger,
\end{aligned}$$

which is actually (e6) from Theorem 2.1.
On the other side, (g2) becomes:

$$\begin{aligned}
(g2.1) \quad &\Leftrightarrow \quad \mathcal{R}((AB)^\dagger_{M,N}) = \mathcal{R}(B^\dagger_{I,N} A^\dagger_{M,I}) \\
&\Leftrightarrow \quad \mathcal{R}(N^{-1/2}(M^{1/2}ABN^{-1/2})^\dagger M^{1/2}) = \mathcal{R}(N^{-1/2}(BN^{-1/2})^\dagger (M^{1/2}A)^\dagger M^{1/2}) \\
&\Leftrightarrow \quad \mathcal{R}(N^{-1/2}(\tilde{A}\tilde{B})^\dagger M^{1/2}) = \mathcal{R}(N^{-1/2}\tilde{B}^\dagger \tilde{A}^\dagger M^{1/2}) \\
&\Leftrightarrow \quad \mathcal{R}(N^{-1/2}(\tilde{A}\tilde{B})^\dagger) = \mathcal{R}(N^{-1/2}\tilde{B}^\dagger \tilde{A}^\dagger) \\
&\Leftrightarrow \quad \mathcal{R}((\tilde{A}\tilde{B})^\dagger) = \mathcal{R}(\tilde{B}^\dagger \tilde{A}^\dagger),
\end{aligned}$$

and

$$\begin{aligned}
(g2.2) \quad &\Leftrightarrow \quad \mathcal{R}((B^*A^*)^\dagger_{N^{-1},M^{-1}}) = \mathcal{R}((A^*)^\dagger_{I,M^{-1}} (B^*)^\dagger_{N^{-1},I}) \\
&\Leftrightarrow \quad \mathcal{R}(M^{1/2}(N^{-1/2}B^*A^*M^{1/2})^\dagger N^{-1/2}) = \mathcal{R}(M^{1/2}(A^*M^{1/2})^\dagger (N^{-1/2}B^*)^\dagger N^{-1/2}) \\
&\Leftrightarrow \quad \mathcal{R}(M^{1/2}(\tilde{B}^*\tilde{A}^*)^\dagger N^{-1/2}) = \mathcal{R}(M^{1/2}(\tilde{A}^*)^\dagger (\tilde{B}^*)^\dagger N^{-1/2}) \\
&\Leftrightarrow \quad \mathcal{R}(M^{1/2}(\tilde{B}^*\tilde{A}^*)^\dagger) = \mathcal{R}(M^{1/2}(\tilde{A}^*)^\dagger (\tilde{B}^*)^\dagger) \\
&\Leftrightarrow \quad \mathcal{R}((\tilde{B}^*\tilde{A}^*)^\dagger) = \mathcal{R}((\tilde{A}^*)^\dagger (\tilde{B}^*)^\dagger),
\end{aligned}$$

which means we have (g2) from Theorem 2.1. Since we have Theorem 2.1 already proven, the proof of this theorem follows immediately. □

3 Conclusions

In this paper we consider a number of necessary and sufficient conditions for the reverse order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)A^\dagger$ to hold for operators on Hilbert spaces. Applying this result we obtain the equivalent conditions for the reverse order rule for the weighted Moore-Penrose inverse of operators. Although these results are already known for complex matrices, we demonstrated the new technique in proving the results. In the theory of complex matrices various authors used the matrix rank to prove the equivalent conditions related to this reverse order law. In the case of linear bounded operators on Hilbert spaces, we applied the method of operator matrices. It is interesting to extend this work to the Moore–Penrose inverse and the weighted Moore-Penrose inverse of a triple product.

Acknowledgements

The authors are supported by the Ministry of Science, Republic of Serbia (grant no. 174007).

References

[BIG] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, 2nd ed., Springer, New York, 2003.

[B1] R. H. Bouldin, *The pseudo-inverse of a product*, SIAM J. Appl. Math. 25 (1973), 489–495.

[B2] R. H. Bouldin, *Generalized inverses and factorizations*, Recent applications of generalized inverses, Pitman Ser. Res. Notes in Math. No. 66 (1982), 233–248.

[C] S. R. Caradus, *Generalized inverses and operator theory*, Queen's paper in pure and applied mathematics, Queen's University, Kingston, Ontario, 1978.

[DW] T. Damm and H. K. Wimmer, *A cancellation property of the Moore-Penrose inverse of triple products*, J. Aust. Math. Soc. 86 (2009), 33–44.

[Dj1] D. S. Djordjević, *Unified approach to the reverse order rule for generalized inverses*, Acta Sci. Math. (Szeged) l67 (2001), 761–776.

[Dj2] D. S. Djordjević, *Further results on the reverse order law for generalized inverses*, SIAM J. Matrix Anal. Appl. 29 (4) (2007), 1242–1246.

[DjD] D. S. Djordjević and N. Č. Dinčić, *Reverse order law for the Moore-Penrose inverse*, J. Math. Anal. Appl. 361 (1) (2010), 252–261.

[DjR] D. S. Djordjević and V. Rakočević, *Lectures on generalized inverses*, Faculty of Sciences and Mathematics, University of Niš, 2008.

[G] T. N. E. Greville, *Note on the generalized inverse of a matrix product*, SIAM Rev. 8 (1966), 518–521.

[H] R. E. Harte, *Invertibility and Singularity for Bounded Linear Operators*, New York, Marcel Dekker, 1988.

[Hw] R. E. Hartwig, *The reverse order law revisited*, Linear Algebra Appl. 76 (1986), 241–246.

[I] S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tohoku Math. J. 34 (1982), 43–52.

[KDjC] J. J. Koliha, D. S. Djordjević and D. Cvetković Ilić, *Moore-Penrose inverse in rings with involution*, Linear Algebra Appl. 426 (2007), 371–381.

[L] Y. Li, *The Moore-Penrose inverses of products and differences of projections in a C^*-algebra*, Linear Algebra Appl. 428 (2008), 1169–1177.

[T1] Y. Tian, *Reverse order laws for the generalized inverses of multiple matrix products. Generalized inverses*, Linear Algebra Appl. 211 (1994), 85–100.

[T2] Y. Tian, *Some mixed-type reverse-order laws for the Moore-Penrose inverse of a triple matrix product*, Rocky Mountain J. Math. 37(4) (2007), 1327–1347.

[T3] Y. Tian, *The equivalence between $(AB)^\dagger = B^\dagger A^\dagger$ and other mixed-type reverse-order laws*, Internat. J. Math. Edu. Sci.Technol. 37(3) (2007), 331–339.

[T4] Y. Tian, *The reverse-order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)A^\dagger$ and its equivalent equalities*, J. Math. Kyoto Univ. (JMKYAZ), 45-4 (2005), 841–850.

[T5] Y. Tian, *Using rank formulas to characterize equalities for Moore-Penrose inverses of matrix products*, Appl. Math. Comput. 147 (2004), 581–600.

[WG] M. Wei and W. Guo, *Reverse order laws for least squares g-inverses and minimum norm g-inverses of products of two matrices*, Linear Algebra Appl. 342 (2002), 117–132.

[W1] H. J. Werner, *G-inverses of matrix products*, Data Analysis and Statistical Inference, Ed. S. Schach and G. Trenkler, Verlag Josef Eul, 1992, pp. 531–546.

[W2] H. J. Werner, *When is $B^{-}A^{-}$ is a generalized inverse of AB?*, Linear Algebra Appl. 210 (1994), 255–263.