Further results on the generalized Drazin inverse of block matrices in Banach algebras

Milica Z. Kolundžija, Dijana Mosić and Dragan S. Djordjević*

Abstract

The objective of this paper is to derive formulae for the generalized Drazin inverse of a block matrix in a Banach algebra $\mathcal{A}$ under different conditions. Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}$ relative to the idempotent $p \in \mathcal{A}$ and $a \in p\mathcal{A}p$ be generalized Drazin invertible. The formulae for the generalized Drazin inverse are obtained under the more general case that the generalized Schur complement $s = d - ca^d b$ is generalized Drazin invertible, which covers the cases that s is Drazin invertible, s is group invertible or s is equal to zero. Thus, recent results on the Drazin inverse of block matrices and block–operator matrices are extended to more general setting.

Key words and phrases: generalized Drazin inverse, Schur complement, block matrix.

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1 Introduction

Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1 and let $a \in \mathcal{A}$. Denote the spectrum, the spectral radius and the resolvent set of a by $\sigma(a)$, $r(a)$ and $\rho(a)$, respectively. The sets of all invertible, nilpotent and quasinilpotent elements ($\sigma(a) = \{0\}$) will be denoted by $\mathcal{A}^{-1}$, $\mathcal{A}^{nil}$ and $\mathcal{A}^{qnil}$, respectively. The generalized Drazin inverse of $a \in \mathcal{A}$ is the element $b \in \mathcal{A}$ which satisfies

$$bab = b, \qquad ab = ba, \qquad a - a^2 b \in \mathcal{A}^{qnil}.$$

If b exists, it is unique and will be denoted by a^d. The set $\mathcal{A}^d$ consists of all $a \in \mathcal{A}$ such that a^d exists. Koliha [16] studied the generalized Drazin inverse

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in Banach algebras. Harte gave an alternative definition of a generalized Drazin inverse in a ring [13]. The Drazin inverse is a special case of the generalized Drazin inverse for which $a - a^2b \in \mathcal{A}^{nil}$. The group inverse is a special case of the Drazin inverse for which $a - a^2b \in \mathcal{A}^{nil}$ is replaced with $a = aba$. By $a^{\#}$ will be denoted the group inverse of a.

An element $p = p^2 \in \mathcal{A}$ is a *spectral idempotent* of a if

$$ap = pa \in \mathcal{A}^{qnil}, \qquad a + p \in \mathcal{A}^{-1}.$$

Such an element is unique if it exists and will be denoted by a^π [12, 14, 16, 17]. Recall that $a^\pi = 1 - aa^d$. For the theory of generalized inverses and its applications, we refer the reader to [2, 4]. Cline's formula for the generalized Drazin inverse can be found in [18].

Let $p = p^2 \in \mathcal{A}$ be an idempotent. Then we can represent element $a \in \mathcal{A}$ as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$.

Lemma 1.1. [21][22, Theorem 1.6.15] *Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1, and let p be an idempotent of $\mathcal{A}$. If $x \in p\mathcal{A}p$, then $\sigma_{p\mathcal{A}p}(x) \cup \{0\} = \sigma_{\mathcal{A}}(x)$, where $\sigma_{\mathcal{A}}(x)$ denotes the spectrum of x in the algebra $\mathcal{A}$, and $\sigma_{p\mathcal{A}p}(x)$ denotes the spectrum of x in the algebra $p\mathcal{A}p$.*

Lemma 1.2. [5, Lemma 2.4] *Let $b, q \in \mathcal{A}^{qnil}$ such that $qb = 0$. Then $q + b \in \mathcal{A}^{qnil}$.*

Lemma 1.3. *Let $b \in \mathcal{A}^d$ and $a \in \mathcal{A}^{qnil}$.*

(i) [5, Corollary 3.4] *If $ab = 0$, then $a+b \in \mathcal{A}^d$ and $(a+b)^d = \sum_{n=0}^{\infty} (b^d)^{n+1} a^n$.*

(ii) *If $ba = 0$, then $a + b \in \mathcal{A}^d$ and $(a+b)^d = \sum_{n=0}^{\infty} a^n (b^d)^{n+1}$.*

Specializing [5, Corollary 3.4] (with multiplication reversed) to bounded linear operators Castro–González and Koliha [5] recovered [11, Theorem 2.2] which is a special case of Lemma 1.3(ii).

The Drazin inverse is very important in various applied mathematical fields like iterative methods, singular differential equations, singular difference equations, Markov chains and so on. Under specific conditions many authors have studied representations for the Drazin inverse [6, 7, 10, 15, 24, 25].

Hartwig et al. [15] gave expressions for the Drazin inverse of a 2×2 block matrix in the cases when the generalized Schur complement is nonsingular and it is equal to zero. These results are generalized in [19] under different conditions and the hypothesis the Schur complement is either nonsingular or zero.

In [6], Castro-González and Martínez-Serrano developed conditions under which the Drazin inverse of a block matrix having generalized Schur complement group invertible, can be expressed in terms of a matrix in the Banachiewicz-Schur form and its powers.

Deng and Wei [9] introduced several explicit representations for the Drazin inverse of a block–operator matrix with Drazin invertible Schur complement under different conditions.

Let

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A} \tag{1}$$

relative to the idempotent $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^d$ and let $s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$ be the generalized Schur complement of a in x.

We present explicit formulae for the generalized Drazin inverse of x in (1) in terms of the generalized Drazin inverse of a and the generalized Drazin inverse of s. Necessary and sufficient conditions for the existence and the expressions for the group inverse are obtained too. Thus, we study the more general case that s is generalized Drazin invertible, which covers the cases that s is Drazin invertible for linear bounded operators [9], s is group invertible for complex matrices [6] or s is equal to zero for complex matrices [15].

The following results will be used in the rest of the paper.

Lemma 1.4. *Let x be defined as in* (1) *and assume that $w_0 = p + a^d bs^\pi ca^d$ is invertible. Then $w_0 a^2 a^d$ is group invertible, $(w_0 a^2 a^d)^\# = a^d w_0^{-1}$ and $(wa^2a^d)^\pi = a^\pi$.*

Proof. Let us prove that $a^d w_0^{-1}$ is group inverse of $w_0^{-1}a^2a^d$. Indeed,

$$\begin{aligned}(w_0a^2a^d)(a^dw_0^{-1}) &= w_0aa^dw_0^{-1} = (p + a^dbs^\pi ca^d)aa^dw_0^{-1} \\ &= (a^da + a^daa^dbs^\pi ca^d)w_0^{-1} = a^daw_0w_0^{-1} = a^da \\ &= a^da^2a^d = (a^dw_0^{-1})(w_0a^2a^d),\end{aligned}$$

$$(w_0a^2a^d)(a^dw_0^{-1})(w_0a^2a^d) = w_0a^2a^da^da^2a^d = w_0a^2a^d,$$

$$(a^dw_0^{-1})(w_0a^2a^d)(a^dw_0^{-1}) = a^da^2a^da^dw_0^{-1} = a^dw_0^{-1}$$

implies that $(w_0a^2a^d)^\# = a^dw_0^{-1}$. Spectral idempotent of $w_0a^2a^d$ is equal to $(w_0a^2a^d)^\pi = p - (w_0a^2a^d)(a^dw_0^{-1}) = p - aa^d = a^\pi$. □

Lemma 1.5. *Let $x \in \mathcal{A}^d$ and $u \in \mathcal{A}$ be an invertible element. Then $u^{-1}xu \in \mathcal{A}^d$ and $(u^{-1}xu)^d = u^{-1}x^du$.*

2 Results

In this section, when we say that x is defined as in (1), we assume that x has a representation as in (1) relative to the idempotent $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^d$ and $s = d - ca^db \in ((1-p)\mathcal{A}(1-p))^d$.

The following lemma will extend to the generalized Drazin inverse of Banach algebra elements a well known result concerning the Drazin inverse of Hilbert space operators, see [8, Theorem 1]. Observe that condition (ii) of the next lemma which appear in [8, Theorem 1] can be replaced with equivalent condition (iii).

Lemma 2.1. *Let x be defined as in* (1)*. Then the following statements are equivalent:*

(i) *$x \in \mathcal{A}^d$ and $x^d = r$, where*

$$r = \begin{bmatrix} a^d + a^dbs^dca^d & -a^dbs^d \\ -s^dca^d & s^d \end{bmatrix}; \tag{2}$$

(ii) *$a^\pi bs^d = a^dbs^\pi$, $s^\pi ca^d = s^dca^\pi$ and $y = \begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi ca^\pi & ss^\pi \end{bmatrix} \in \mathcal{A}^{qnil}$;*

(iii) *$a^\pi b = bs^\pi$, $\quad s^\pi c = ca^\pi \quad$ and $\quad y = \begin{bmatrix} aa^\pi & bs^\pi \\ ca^\pi & ss^\pi \end{bmatrix} \in \mathcal{A}^{qnil}$.*

Proof. (i) ⇔ (ii): We can verify that $rxr = r$. Since $a^\pi bs^d = a^dbs^\pi$ and $s^\pi ca^d = s^dca^\pi$ imply $a^\pi bs^dca^d = a^dbs^dca^\pi$, by elementary computations, we observe that $xr = rx$ if and only if $a^\pi bs^d = a^dbs^\pi$ and $s^\pi ca^d = s^dca^\pi$. Now, we can obtain $x - x^2r = \begin{bmatrix} p & -a^db \\ 0 & 1-p \end{bmatrix} y \begin{bmatrix} p & a^db \\ 0 & 1-p \end{bmatrix}$ and $r(x - x^2r) = r(\begin{bmatrix} p & a^db \\ 0 & 1-p \end{bmatrix}\begin{bmatrix} p & -a^db \\ 0 & 1-p \end{bmatrix} y) = r(y)$. Hence, $x - x^2r \in \mathcal{A}^{qnil}$ is equivalent to $y \in \mathcal{A}^{qnil}$.

(ii) ⇔ (iii): First, we check that $a^\pi bs^d = a^dbs^\pi$ is equivalent to $a^\pi b = bs^\pi$. If we multiply the equality $a^\pi bs^d = a^dbs^\pi$ from the right side by s and from the left side by a, respectively, we get $a^\pi bs^d s = 0$ and $aa^d bs^\pi = 0$. So, $bs^d s = aa^d bs^d s = aa^d b$ and

$$a^\pi b = b - aa^d b = b - bs^d s = bs^\pi.$$

On the other hand, $a^\pi b = bs^\pi$ gives $(a^\pi b)s^d = bs^\pi s^d = 0$ and $a^d(bs^\pi) = a^d a^\pi b = 0$. Hence, $a^\pi bs^d = a^d bs^\pi$.

Similarly, we can prove that $s^\pi ca^d = s^d ca^\pi$ is equivalent to $s^\pi c = ca^\pi$. Thus, we deduce that (ii) ⇔ (iii). □

Remark. Using Lemma 2.1, if x is defined as in (1) and r is defined as in (2), then $x \in \mathcal{A}^\#$ and $x^\# = r$ if and only if $a \in (p\mathcal{A}p)^\#$, $s \in ((1-p)\mathcal{A}(1-p))^\#$, $a^\pi b = 0 = bs^\pi$ and $s^\pi c = 0 = ca^\pi$. This results is well-known for a complex matrix [3, Theorem 2] (see also [6, Corollary 2.3]). The expression (2) is called the generalized Banachiewicz–Schur form of x. For more details see [1, 3, 6, 8, 15, 23].

Now we present a formula for the generalized Drazin inverse of block matrix x in (1) in terms of the generalized Drazin invertible Schur complement s. We extend [9, Theorem 7] concerning the Drazin inverse of 2×2 block-operator matrix to more general setting.

Theorem 2.1. *Let x be defined as in* (1)*. If*

$$ca^\pi bss^d = 0, \qquad aa^\pi bss^d = 0, \qquad ss^\pi c = 0, \qquad a^\pi bs^\pi c = bs^\pi caa^d = 0, \tag{3}$$

then $x \in \mathcal{A}^d$ and

$$x^d = \left(\begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(1 + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix} \begin{bmatrix} aa^\pi & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix}^n\right), \tag{4}$$

where r is defined as in (2)*.*

Proof. Notice that, by $a^\pi + aa^d = p$ and $s^\pi + ss^d = 1 - p$,

$$x = \begin{bmatrix} aa^\pi & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix} + \begin{bmatrix} a^2a^d & bss^d \\ caa^d & dss^d \end{bmatrix} := y + z.$$

From $a^d a^\pi = 0 = s^\pi s^d$, $d = s + ca^d b$, $bs^\pi ca^d = (bs^\pi caa^d)a^d = 0$ and (3), we get $yz = 0$.

To prove that $y \in \mathcal{A}^{qnil}$, we observe that

$$\begin{aligned} y &= \begin{bmatrix} aa^\pi & a^\pi bs^\pi \\ 0 & ss^\pi \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ s^\pi ca^\pi & s^\pi ca^d bs^\pi \end{bmatrix} + \begin{bmatrix} 0 & aa^d bs^\pi \\ ss^d ca^\pi & ss^d ds^\pi \end{bmatrix} \\ &:= y_1 + y_2 + y_3. \end{aligned}$$

Recall that if $u = \begin{bmatrix} a_1 & 0 \\ c_1 & b_1 \end{bmatrix}$, then $\lambda 1 - u = \begin{bmatrix} \lambda p - a_1 & 0 \\ -c_1 & \lambda(1-p) - b_1 \end{bmatrix}$ and

$$\lambda \in \rho_{p\mathcal{A}p}(a_1) \cap \rho_{(1-p)\mathcal{A}(1-p)}(b_1) \Rightarrow \lambda \in \rho(u),$$

i.e.

$$\sigma(u) \subseteq \sigma_{p\mathcal{A}p}(a_1) \cup \sigma_{(1-p)\mathcal{A}(1-p)}(b_1).$$

Since $aa^\pi \in (p\mathcal{A}p)^{qnil}$ and $ss^\pi \in ((1-p)\mathcal{A}(1-p))^{qnil}$, we deduce that $y_1 \in \mathcal{A}^{qnil}$. By $r(s^\pi ca^d bs^\pi) = r(bs^\pi ca^d) = r(0) = 0$ and $\sigma_{\mathcal{A}}(s^\pi ca^d bs^\pi) = \sigma_{(1-p)\mathcal{A}(1-p)}(s^\pi ca^d bs^\pi) \cup \{0\}$(Lemma 1.1), $y_2 \in \mathcal{A}^{qnil}$. We can check that $y_1 y_2 = 0$ which gives that $y_1 + y_2 \in \mathcal{A}^{qnil}$, by Lemma 1.2. Also, by Lemma 1.2, $y_3^2 = 0$ (i.e. $y_3 \in \mathcal{A}^{nil}$) and $(y_1 + y_2)y_3 = 0$ imply $y \in \mathcal{A}^{qnil}$.

In order to show that $z \in \mathcal{A}^d$, we write

$$z = \begin{bmatrix} a^2 a^d & aa^d bss^d \\ ss^d caa^d & ss^d dss^d \end{bmatrix} + \begin{bmatrix} 0 & a^\pi bss^d \\ s^\pi caa^d & s^\pi dss^d \end{bmatrix} := z_1 + z_2.$$

We can verify that $z_1 z_2 = 0$ and $z_2^2 = 0$. If $z_1 = \begin{bmatrix} A_{z_1} & B_{z_1} \\ C_{z_1} & D_{z_1} \end{bmatrix}$, we note that $A_{z_1} \equiv a^2 a^d \in (p\mathcal{A}p)^\#$, $(a^2 a^d)^\# = a^d$, $S_{z_1} \equiv D_{z_1} - C_{z_1} A_{z_1}^\# B_{z_1} = s^2 s^d \in ((1-p)\mathcal{A}(1-p))^\#$ and $(s^2 s^d)^\# = s^d$. Using Lemma 2.1, we have $z_1 \in \mathcal{A}^d$ and $z_1^d = r$. Further, by Lemma 1.3, $z \in \mathcal{A}^d$ and $z^d = z_1^d + z_2 (z_1^d)^2$.

Applying again Lemma 1.3, we conclude that $x \in \mathcal{A}^d$ and

$$x^d = \sum_{n=0}^{\infty} (z^d)^{n+1} y^n = (1 + z_2 z_1^d) z_1^d \left(1 + \sum_{n=0}^{\infty} (z_1^d)^{n+1} y^{n+1} \right).$$

Then, observe that $z_1^d = r = r \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} = \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} r$,

$$z_2 z_1^d = \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} r = \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r$$

and

$$ry = r \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} y = r \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} \begin{bmatrix} 0 & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix} = r \begin{bmatrix} 0 & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix}$$

yield (4). □

The condition of Theorem 2.1 are cumbersome and complicated, but the theorem itself have a number of useful consequences.

By Theorem 2.1, we obtain the following corollary which recovers [6, Theorem 2.5] for the Drazin inverse of complex matrices.

Corollary 2.1. *Let x be defined as in* (1)*, $a \in (p\mathcal{A}p)^{\#}$ and let $s \in ((1-p)\mathcal{A}(1-p))^{\#}$. If $ca^{\pi} = 0$ and $bs^{\pi} = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} p - a^{\pi}bs^{\#}ca^{\#} & a^{\pi}bs^{\#} \\ s^{\pi}ca^{\#} & 1-p \end{bmatrix} \begin{bmatrix} a^{\#} + a^{\#}bs^{\#}ca^{\#} & -a^{\#}bs^{\#} \\ -s^{\#}ca^{\#} & s^{\#} \end{bmatrix}.$$

If we assume that the generalized Schur complement s is invertible in Theorem 2.1, then $s^{\pi} = 0$ and the next corollary which covers [15, Theorem 3.1] follows.

Corollary 2.2. *Let x be defined as in* (1)*, and let $s \in ((1-p)\mathcal{A}(1-p))^{-1}$. If $ca^{\pi}b = 0$ and $aa^{\pi}b = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \left(\begin{bmatrix} 0 & a^{\pi}b \\ 0 & 0 \end{bmatrix} r_1 + 1 \right) r_1 \left(1 + \sum_{n=0}^{\infty} r_1^{n+1} \begin{bmatrix} 0 & 0 \\ ca^n a^{\pi} & 0 \end{bmatrix} \right),$$

where $r_1 = \begin{bmatrix} a^d + a^d bs^{-1}ca^d & -a^d bs^{-1} \\ -s^{-1}ca^d & s^{-1} \end{bmatrix}$.

In the following result we introduce the other expression for the generalized Drazin inverse of x which include an invertible element $w_0 = p + a^d bs^{\pi}ca^d$.

Theorem 2.2. *Let x be defined as in* (1)*. If*

$$aa^{\pi} - a^{\pi}bs^d ca^{\pi} = 0, \quad s^{\pi}ca^{\pi} = 0, \quad ca^{\pi}b = 0, \quad a^{\pi}bs^{\pi} = 0, \quad ss^{\pi}c = 0 = bss^{\pi} \tag{5}$$

and $w_0 = p + a^d bs^{\pi}ca^d$ is invertible, then $x \in \mathcal{A}^d$ and

$$x^d = \left(\begin{bmatrix} 0 & a^{\pi}b \\ s^{\pi}c & s^{\pi}d \end{bmatrix} r + 1 \right) wrw \left(1 + r \begin{bmatrix} 0 & bs^{\pi} \\ ca^{\pi} & ds^{\pi} \end{bmatrix} \right), \tag{6}$$

where r is defined as in (2) *and* $w = \begin{bmatrix} w_0^{-1} & 0 \\ 0 & 1-p \end{bmatrix}$.

Proof. First, we observe that $u = \begin{bmatrix} p & a^d b \\ s^d c & (1-p) + s^d ca^d b \end{bmatrix}$ is invertible in $\mathcal{A}$ and its inverse is $u^{-1} = \begin{bmatrix} p + a^d bs^d c & -a^d b \\ -s^d c & 1-p \end{bmatrix}$.

Let us denote $X = uxu^{-1}$, so we have

$$\begin{aligned} X &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} = uxu^{-1} \\ &= \begin{bmatrix} p & a^d b \\ s^d c & (1-p) + s^d c a^d b \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p + a^d b s^d c & -a^d b \\ -s^d c & (1-p) \end{bmatrix} \\ &= \begin{bmatrix} a - a^\pi b s^d c + a^d b s^\pi c & a^\pi b + a^d b s \\ s^\pi c + s^d c(a - a^\pi b s^d c + a^d b s^\pi c) & s + s^d c(a^\pi b + a^d b s) \end{bmatrix}. \end{aligned}$$

The first and the third conditions from (5) give us equations $caa^\pi = 0$ and $aa^\pi b = 0$. The second condition implies $s^\pi c a a^d = s^\pi c$.

Applying these equations along with $a = aa^\pi + a^2 a^d$, we have

$$\begin{aligned} A &= a - a^\pi b s^d c + a^d b s^\pi c = w_0 a^2 a^d + aa^\pi - a^\pi b s^d c, \\ B &= a^\pi b + a^d b s, \\ C &= s^\pi c + s^d c(a - a^\pi b s^d c + a^d b s^\pi c) = s^\pi c + s^d c w_0 a^2 a^d, \\ D &= s + s^d c(a^\pi b + a^d b s) = s + s^d c a^d b s. \end{aligned}$$

From Lemma 1.4, we have $w_0 a^2 a^d \in (p\mathcal{A}p)^\#$, $(w_0 a^2 a^d)^\# = a^d w_0^{-1}$ and $(wa^2 a^d)^\pi = a^\pi$. Further,

$$(aa^\pi - a^\pi b s^d c)^2 = (aa^\pi - a^\pi b s^d c a^\pi)a - aa^\pi b s^d c + a^\pi b s^d c a^\pi b s^d c = 0$$

implies $(aa^\pi - a^\pi b s^d c) \in (p\mathcal{A}p)^{nil} \subseteq (p\mathcal{A}p)^{qnil}$ and it holds

$$w_0 a^2 a^d (aa^\pi - a^\pi b s^d c) = 0.$$

Applying Lemma 1.3 (ii), we conclude that $A \in (p\mathcal{A}p)^d$ and

$$\begin{aligned} A^d &= (w_0 a^2 a^d)^\# + (aa^\pi - a^\pi b s^d c)((w_0 a^2 a^d)^\#)^2 \\ &= (p - a^\pi b s^d c a^d w_0^{-1}) a^d w_0^{-1} \end{aligned}$$

Since $w_0 a a^d = a a^d w_0$ implies $(w_0 a^2 a^d)(a^d w_0^{-1}) = aa^d$ and it holds $a^d w_0^{-1} a^\pi = (w_0 a^2 a^d)^\# (w_0 a^2 a^d)^\pi = 0$, we have

$$\begin{aligned} A^\pi &= p - AA^d \\ &= p - (w_0 a^2 a^d + aa^\pi - a^\pi b s^d c)(p - a^\pi b s^d c a^d w_0^{-1}) a^d w_0^{-1} \\ &= p - w_0 a^2 a^d a^d w_0^{-1} - aa^\pi a^d w_0^{-1} + a^\pi b s^d c a^d w_0^{-1} \\ &+ w_0 a^2 a^d a^\pi b s^d c a^d w_0^{-1} a^d w_0^{-1} + aa^\pi b s^d c a^d w_0^{-1} a^d w_0^{-1} \\ &- a^\pi b s^d c a^\pi b s^d c a^d w_0^{-1} a^d w_0^{-1} \\ &= a^\pi + a^\pi b s^d c a^d w_0^{-1}. \end{aligned}$$

Notice that $AA^\pi = 0$. Therefore, $A^d = A^\#$.

Now,

$$\begin{aligned} S &= D - CA^\# B = s + s^d ca^d bs \\ &- (s^\pi c + s^d cw_0 a^2 a^d)(p - a^\pi bs^d ca^d w_0^{-1}) a^d w_0^{-1}(a^\pi b + a^d bs) \\ &= s + s^d ca^d bs - (s^\pi c + s^d cw_0 a^2 a^d) a^d w_0^{-1} a^d bs \\ &= s + s^d ca^d bs - s^\pi ca^d w_0^{-1} a^d bs - s^d cw_0 a^2 a^d a^d w_0^{-1} a^d bs \\ &= s - s^\pi ca^d w_0^{-1} a^d bs. \end{aligned}$$

Since

$$s \in ((1-p)\mathcal{A}(1-p))^d, \quad (s^\pi ca^d w_0^{-1} a^d bs)^2 = 0, \quad s(s^\pi ca^d w_0^{-1} a^d bs) = 0,$$

applying Lemma 1.3 (ii), we have that $S \in ((1-p)\mathcal{A}(1-p))^d$ and

$$S^d = s^d - s^\pi ca^d w_0^{-1} a^d bs^d.$$

Then,

$$S^\pi = (1-p) - SS^d = s^\pi + s^\pi ca^d w_0^{-1} a^d bss^d.$$

The following equations hold

$$CA^\pi = 0, \quad BS^\pi = 0, \quad AA^\pi = 0, \quad SS^\pi C = 0,$$

which implies that X satisfies the conditions (3) from Theorem 2.1. Using this Theorem, we conclude $X \in \mathcal{A}^d$ and

$$X^d = \left(\begin{bmatrix} 0 & A^\pi B \\ S^\pi C & S^\pi D \end{bmatrix} R + 1 \right) R,$$

where

$$R = \begin{bmatrix} A^\# + A^\# BS^d CA^\# & -A^\# BS^d \\ -S^d CA^\# & S^d \end{bmatrix}$$

Then, applying Lemma 1.5 on $x = u^{-1} X u$ we have

$$x^d = u^{-1} X^d u = u^{-1} \left(\begin{bmatrix} 0 & A^\pi B \\ S^\pi C & S^\pi D \end{bmatrix} R + 1 \right) Ru.$$

Observe that

$$Ru = \begin{bmatrix} A^{\#} & 0 \\ 0 & S^d \end{bmatrix} \begin{bmatrix} p + BS^dCA^{\#} & -BS^d \\ -CA^{\#} & (1-p) \end{bmatrix} \begin{bmatrix} p & a^db \\ s^dc & (1-p) + s^dca^db \end{bmatrix}.$$

Since

$$\begin{bmatrix} p + BS^dCA^{\#} & -BS^d \\ -CA^{\#} & (1-p) \end{bmatrix} \begin{bmatrix} p & a^db \\ s^dc & (1-p) + s^dca^db \end{bmatrix} =$$
$$= \begin{bmatrix} p + a^{\pi}b(s^d)^2caa^d + a^dbs^dcaa^d & -a^{\pi}bs^d - a^dbss^d \\ -s^{\pi}ca^dw_0^{-1} - s^dcaa^d & (1-p) \end{bmatrix} \begin{bmatrix} p & a^db \\ s^dc & (1-p) + s^dca^db \end{bmatrix}$$
$$= \begin{bmatrix} p - a^{\pi}b(s^d)^2ca^{\pi} - a^dbs^dca^{\pi} & a^dbs^{\pi} - a^{\pi}bs^d \\ -s^{\pi}ca^dw_0^{-1} + s^dca^{\pi} & (1-p) - s^{\pi}ca^dw_0^{-1}a^db \end{bmatrix},$$

we have

$$\begin{aligned} Ru &= \begin{bmatrix} A^{\#} & 0 \\ 0 & S^d \end{bmatrix} \begin{bmatrix} p - a^{\pi}b(s^d)^2ca^{\pi} - a^dbs^dca^{\pi} & a^dbs^{\pi} - a^{\pi}bs^d \\ -s^{\pi}ca^dw_0^{-1} + s^dca^{\pi} & (1-p) - s^{\pi}ca^dw_0^{-1}a^db \end{bmatrix} \\ &= \begin{bmatrix} A^{\#} & 0 \\ 0 & S^d \end{bmatrix} \begin{bmatrix} p - a^dbs^dca^{\pi} & a^dbs^{\pi} \\ s^dca^{\pi} & (1-p) \end{bmatrix} \\ &+ \begin{bmatrix} A^{\#} & 0 \\ 0 & S^d \end{bmatrix} \begin{bmatrix} -a^{\pi}b(s^d)^2ca^{\pi} & -a^{\pi}bs^d \\ -s^{\pi}ca^dw_0^{-1} & -s^{\pi}ca^dw_0^{-1}a^db \end{bmatrix} \\ &= \begin{bmatrix} A^{\#} & 0 \\ 0 & S^d \end{bmatrix} \left(1 + \begin{bmatrix} -a^dbs^dca^{\pi} & a^dbs^{\pi} \\ s^dca^{\pi} & 0 \end{bmatrix}\right) \\ &+ \begin{bmatrix} (p - a^{\pi}bs^dca^dw_0^{-1})a^dw_0^{-1} & 0 \\ 0 & ((1-p) - s^{\pi}ca^dw_0^{-1}a^db)s^d \end{bmatrix} \\ &\times \begin{bmatrix} -a^{\pi}b(s^d)^2ca^{\pi} & -a^{\pi}bs^d \\ -s^{\pi}ca^dw_0^{-1} & -s^{\pi}ca^dw_0^{-1}a^db \end{bmatrix} \\ &= \begin{bmatrix} A^{\#} & 0 \\ 0 & S^d \end{bmatrix} \left(1 + \begin{bmatrix} a^d + a^dbs^dca^d & -a^dbs^d \\ -s^dca^d & s^d \end{bmatrix} \begin{bmatrix} 0 & bs^{\pi} \\ ca^{\pi} & ds^{\pi} \end{bmatrix}\right) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A^{\#} & 0 \\ 0 & S^d \end{bmatrix} \left(1 + r \begin{bmatrix} 0 & bs^{\pi} \\ ca^{\pi} & ds^{\pi} \end{bmatrix}\right) \end{aligned}$$

We can write

$$\begin{aligned} \begin{bmatrix} A^{\#} & 0 \\ 0 & S^d \end{bmatrix} &= \begin{bmatrix} p - a^{\pi}bs^dca^dw_0^{-1} & 0 \\ 0 & (1-p) - s^{\pi}ca^dw_0^{-1}a^dbss^d \end{bmatrix} \\ &\times \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix} \begin{bmatrix} w_0^{-1} & 0 \\ 0 & (1-p) \end{bmatrix} \\ &= \begin{bmatrix} p - a^{\pi}bs^dca^dw_0^{-1} & 0 \\ 0 & (1-p) - s^{\pi}ca^dw_0^{-1}a^dbss^d \end{bmatrix} \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix} w. \end{aligned}$$

Therefore,

$$\begin{aligned} Ru &= \begin{bmatrix} p - a^{\pi}bs^{d}ca^{d}w_0^{-1} & 0 \\ 0 & (1-p) - s^{\pi}ca^{d}w_0^{-1}a^{d}bss^{d} \end{bmatrix} \begin{bmatrix} a^{d} & 0 \\ 0 & s^{d} \end{bmatrix} w \\ &\times \left(1 + r \begin{bmatrix} 0 & bs^{\pi} \\ ca^{\pi} & ds^{\pi} \end{bmatrix}\right) \end{aligned}$$

Denote $M = w\left(1 + r \begin{bmatrix} 0 & bs^{\pi} \\ ca^{\pi} & ds^{\pi} \end{bmatrix}\right)$. Notice $r \begin{bmatrix} a & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} a^{d} & 0 \\ 0 & s^{d} \end{bmatrix} = r$.
Using the equation $a^{d}w_0^{-1}(p + a^{d}bs^{d}ca^{d}a) = a^{d}w_0^{-1}(a^{d} + a^{d}bs^{d}ca^{d})a$, we have

$$\begin{aligned} x^{d} &= u^{-1}\left(\begin{bmatrix} 0 & A^{\pi}B \\ S^{\pi}C & S^{\pi}D \end{bmatrix} R + 1\right) \\ &\times \begin{bmatrix} p - a^{\pi}bs^{d}ca^{d}w_0^{-1} & 0 \\ 0 & (1-p) - s^{\pi}ca^{d}w_0^{-1}a^{d}bss^{d} \end{bmatrix} \begin{bmatrix} a^{d} & 0 \\ 0 & s^{d} \end{bmatrix} M \\ &= u^{-1} \begin{bmatrix} p - A^{\pi}BS^{d}CA^{\#} & A^{\pi}BS^{d} \\ S^{\pi}CA^{\#} & (1-p) \end{bmatrix} \\ &\times \begin{bmatrix} p - a^{\pi}bs^{d}ca^{d}w_0^{-1} & 0 \\ 0 & (1-p) - s^{\pi}ca^{d}w_0^{-1}a^{d}bss^{d} \end{bmatrix} \begin{bmatrix} a^{d} & 0 \\ 0 & s^{d} \end{bmatrix} M \\ &= u^{-1} \\ &\times \begin{bmatrix} p - a^{\pi}b(s^{d})^2caa^{d} - a^{\pi}bs^{d}ca^{d}w_0^{-1}a^{d}bs^{d}caa^{d} & a^{\pi}bs^{d} + a^{\pi}bs^{d}ca^{d}w_0^{-1}a^{d}bss^{d} \\ s^{\pi}ca^{d}w_0^{-1}(a^{d} + a^{d}bs^{d}ca^{d})a & (1-p) \end{bmatrix} \\ &\times \begin{bmatrix} p - a^{\pi}bs^{d}ca^{d}w_0^{-1} & 0 \\ 0 & (1-p) - s^{\pi}ca^{d}w_0^{-1}a^{d}bss^{d} \end{bmatrix} \begin{bmatrix} a^{d} & 0 \\ 0 & s^{d} \end{bmatrix} M \\ &= u^{-1} \\ &\times \begin{bmatrix} p - a^{\pi}b(s^{d})^2caa^{d} - a^{\pi}bs^{d}ca^{d}w_0^{-1}(a^{d} + a^{d}bs^{d}ca^{d})a & a^{\pi}bs^{d} + a^{\pi}bs^{d}ca^{d}w_0^{-1}a^{d}bss^{d} \\ s^{\pi}ca^{d}w_0^{-1}(a^{d} + a^{d}bs^{d}ca^{d})a & (1-p) - s^{\pi}ca^{d}w_0^{-1}a^{d}bss^{d} \end{bmatrix} \\ &\times \begin{bmatrix} a^{d} & 0 \\ 0 & s^{d} \end{bmatrix} M \\ &= u^{-1}\left(1 + \begin{bmatrix} -a^{\pi}bs^{d}ca^{d}w_0^{-1} & a^{\pi}bs^{d} \\ s^{\pi}ca^{d}w_0^{-1} & 0 \end{bmatrix} \begin{bmatrix} (a^{d} + a^{d}bs^{d}ca^{d})a & -a^{d}bs^{d}s \\ -s^{d}ca^{d}a & s^{d}s \end{bmatrix}\right) \\ &\times \begin{bmatrix} a^{d} & 0 \\ 0 & s^{d} \end{bmatrix} M \\ &= u^{-1}\left(1 + \begin{bmatrix} -a^{\pi}bs^{d}ca^{d} & a^{\pi}bs^{d} \\ s^{\pi}ca^{d} & 0 \end{bmatrix} \begin{bmatrix} w_0^{-1} & 0 \\ 0 & (1-p) \end{bmatrix} r \begin{bmatrix} a & 0 \\ 0 & s \end{bmatrix}\right) \\ &\times \begin{bmatrix} a^{d} & 0 \\ 0 & s^{d} \end{bmatrix} M \end{aligned}$$

$$
\begin{aligned}
&= \left(\begin{bmatrix} p + a^d b s^d c & -a^d b \\ -s^d c & (1-p) \end{bmatrix} \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix} \right. \\
&+ \left. u^{-1} \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r \begin{bmatrix} w_0^{-1} & 0 \\ 0 & (1-p) \end{bmatrix} r \begin{bmatrix} a & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix} \right) M \\
&= \left(r + \begin{bmatrix} p + a^d b s^d c & -a^d b \\ -s^d c & (1-p) \end{bmatrix} \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} rwr \right) M \\
&= \left(\begin{bmatrix} w_0 & 0 \\ 0 & (1-p) \end{bmatrix} + \left(\begin{bmatrix} a^d b s^d c & -a^d b \\ -s^d c & 0 \end{bmatrix} + 1 \right) \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r \right) wrM \\
&= \left(\begin{bmatrix} w_0 & 0 \\ 0 & (1-p) \end{bmatrix} + \begin{bmatrix} -a^d b s^\pi c & -a^d b s^\pi d \\ 0 & 0 \end{bmatrix} r + \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r \right) wrM \\
&= \left(\begin{bmatrix} p + a^d b s^\pi c a^d & 0 \\ 0 & (1-p) \end{bmatrix} + \begin{bmatrix} -a^d b s^\pi c a^d & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r \right) wrM \\
&= \left(1 + \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r \right) wrM.
\end{aligned}
$$

Replacing M, we get (6). $\square$

If $s \in ((1-p)\mathcal{A}(1-p))^\#$ in Theorem 2.3, then $ss^\pi = 0$ and we recover as a special case [9, Theorem 9]. If $s \in ((1-p)\mathcal{A}(1-p))^{-1}$ in Theorem 2.3, we get the following consequence.

Corollary 2.3. *Let x be defined as in* (1)*, and let $s \in ((1-p)\mathcal{A}(1-p))^{-1}$. If $aa^\pi - a^\pi b s^{-1} c a^\pi = 0$ and $ca^\pi b = 0$, then $x \in \mathcal{A}^d$ and*

$$
x^d = \left(\begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} r_1 + 1 \right) r_1 \left(1 + r_1 \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix} \right),
$$

where r_1 is defined as in Corollary 2.2.

We give a representation of x^d in the next theorem under conditions $a^\pi b = 0$ and $s^\pi c a a^d = 0$.

Theorem 2.3. *Let x be defined as in* (1)*. If $a^\pi b = 0$ and $s^\pi c a a^d = 0$, then $x \in \mathcal{A}^d$ and*

$$
x^d = \sum_{n=0}^{\infty} r^{n+1} \left(1 + \begin{bmatrix} -a^d b s^d c a^\pi & a^d b s^\pi \\ s^d c a^\pi & 0 \end{bmatrix} \right) \begin{bmatrix} aa^\pi & 0 \\ s^\pi c & s^\pi s \end{bmatrix}^n, \qquad (7)
$$

where r is defined as in (2).

Proof. By the assumption $a^\pi b = 0$ and $s^\pi caa^d = 0$, $s^\pi ca^d = 0$ and we can write

$$\begin{aligned} x &= \begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} + \begin{bmatrix} a^2a^d & aa^d b \\ ss^d c & ss^d d \end{bmatrix} = \begin{bmatrix} aa^\pi & 0 \\ s^\pi c & s^\pi s \end{bmatrix} + \begin{bmatrix} a^2a^d & aa^d b \\ ss^d c & ss^d d \end{bmatrix} \\ &:= y + z. \end{aligned}$$

Now, we obtain that $yz = 0$ and $y \in \mathcal{A}^{qnil}$, because $aa^\pi \in (p\mathcal{A}p)^{qnil}$ and $ss^\pi \in ((1-p)\mathcal{A}(1-p))^{qnil}$.

To prove that $z \in \mathcal{A}^d$, we observe that

$$z = \begin{bmatrix} a^2a^d & aa^d bss^d \\ ss^d caa^d & ss^d dss^d \end{bmatrix} + \begin{bmatrix} 0 & aa^d bs^\pi \\ ss^d ca^\pi & ss^d ds^\pi \end{bmatrix} := z_1 + z_2.$$

From Lemma 2.1, we have $z_1 \in \mathcal{A}^d$ and $z_1^d = r$. Since $z_2 z_1 = 0$ and $z_2^2 = 0$, by Lemma 1.3(i), $z \in \mathcal{A}^d$ and $z^d = z_1^d + (z_1^d)^2 z_2 = r + r^2 z_2$.

Therefore, using Lemma 1.3(i), $x \in \mathcal{A}^d$ and $x^d = \sum_{n=0}^{\infty} r^{n+1}(1 + rz_2)y^n$ which gives (7). □

Also we can obtain the following expression for the generalized Drazin inverse of block matrix x.

Theorem 2.4. *Let x be defined as in* (1)*. If $a^\pi b = 0 = bs^\pi$ and $s^\pi scaa^d = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1 \right) r \left(1 + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^n a^\pi & 0 \end{bmatrix} \right), \qquad (8)$$

where r is defined as in (2)*.*

Proof. In the similar way as in the proof of Theorem 2.1, using the following decomposition

$$x = \begin{bmatrix} aa^\pi & 0 \\ ca^\pi & ss^\pi \end{bmatrix} + \begin{bmatrix} a^2a^d & bss^d \\ caa^d & dss^d \end{bmatrix} := y + z,$$

we verify this result. □

Using Theorem 2.3, we get necessary and sufficient conditions for the existence and the expression of the group inverse of x. The following result recovers [9, Theorem 12] and [6, Theorem 2.2].

Theorem 2.5. *Let x be defined as in (1). Suppose that $a^\pi b = 0$ and $s^\pi caa^d = 0$. Then*

$$x \in \mathcal{A}^\# \text{ if and only if } a \in (p\mathcal{A}p)^\#,\ s \in ((1-p)\mathcal{A}(1-p))^\# \text{ and } s^\pi ca^\pi = 0.$$

Furthermore, if $a \in (p\mathcal{A}p)^\#$, $s \in ((1-p)\mathcal{A}(1-p))^\#$, $a^\pi b = 0$ and $s^\pi c = 0$, then

$$x^\# = \begin{bmatrix} a^\# + a^\# bs^\# ca^\# & -a^\# bs^\# \\ -s^\# ca^\# & s^\# \end{bmatrix} \begin{bmatrix} p - a^\# bs^\# ca^\pi & a^\# bs^\pi \\ s^\# ca^\pi & 1-p \end{bmatrix}. \tag{9}$$

Proof. If $x \in \mathcal{A}^\#$, by Theorem 2.3, $x^\#$ is equal to the right hand side of (7). Since $xr^2 = (xr)r = \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} r = r$, then

$$\begin{aligned} x^2 x^\# &= x^2 r \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right) \\ &+ \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right) \begin{bmatrix} aa^\pi & 0 \\ s^\pi c & s^\pi s \end{bmatrix} \\ &+ \sum_{n=2}^{\infty} r^{n-1} \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right) \begin{bmatrix} aa^\pi & 0 \\ s^\pi c & s^\pi s \end{bmatrix}^n \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

By the equality $x - x^2 x^\# = 0$, we obtain $I_3 = x - I_1 - I_2$. Now, notice that

$$\begin{aligned} x^\# &= r \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right) \\ &+ r^2 \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right) \begin{bmatrix} aa^\pi & 0 \\ s^\pi c & s^\pi s \end{bmatrix} + r^2 I_3 \\ &= r \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right) + r^2 (x - I_1) \\ &= r \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right). \end{aligned}$$

Hence,

$$\begin{aligned} x^2 x^\# &= \begin{bmatrix} a^2 a^d + a^\pi bs^d ca^\pi & aa^d bs^\pi + bss^d \\ caa^d + ss^d ca^\pi & ca^d b + s^2 s^d \end{bmatrix} \\ &= \begin{bmatrix} a^2 a^d & bs^\pi + bss^d \\ ss^d caa^d + ss^d ca^\pi & d - s + s^2 s^d \end{bmatrix} = \begin{bmatrix} a^2 a^d & b \\ ss^d c & d - s + s^2 s^d \end{bmatrix} \end{aligned}$$

and $x^2x^{\#} = x$ imply $a^2a^d = a$, $s^2s^d = s$ and $ss^dc = c$. So, $a \in (p\mathcal{A}p)^{\#}$, $s \in ((1-p)\mathcal{A}(1-p))^{\#}$ and $s^\pi ca^\pi = ca^\pi - ca^\pi = 0$.

Assume that $a \in (p\mathcal{A}p)^{\#}$, $s \in ((1-p)\mathcal{A}(1-p))^{\#}$ and $s^\pi ca^\pi = 0$. Then $s^\pi c = s^\pi ca^\pi + s^\pi caa^{\#} = 0$. Denote by u the right hand side of (9). Using Theorem 2.3, we get that $x \in \mathcal{A}^d$ and $x^d = u$. We can show that $xx^dx = xux = x$ which implies that $x \in \mathcal{A}^{\#}$ and $x^{\#} = u$. □

Applying Theorem 2.4, we prove the next result related to the group inverse $x^{\#}$ which is an extension of [9, Theorem 13].

Theorem 2.6. *Let x be defined as in* (1)*. If $a^\pi b = 0 = bs^\pi$ and $s^\pi scaa^d = 0$. Then*

$$x \in \mathcal{A}^{\#} \text{ if and only if } a \in (p\mathcal{A}p)^{\#},\ s \in ((1-p)\mathcal{A}(1-p))^{\#} \text{ and } s^\pi ca^\pi = 0.$$

Furthermore, if $a \in (p\mathcal{A}p)^{\#}$, $s \in ((1-p)\mathcal{A}(1-p))^{\#}$, $a^\pi b = 0 = bs^\pi$ and $s^\pi ca^\pi = 0$, then

$$x^{\#} = \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(1 + r \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix}\right), \tag{10}$$

where r is defined as in (2)*.*

Proof. Let $x \in \mathcal{A}^{\#}$. Using Theorem 2.4, $x^{\#}$ is equal to the right hand side of (8). From $\begin{bmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{bmatrix} x = \begin{bmatrix} 0 & 0 \\ ca^{n+1}a^\pi & 0 \end{bmatrix}$, we get

$$\begin{aligned}
x^{\#}x &= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(x + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^{n+1}a^\pi & 0 \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) \left(rx + \sum_{n=1}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) \left(rx - r\begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix} + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) \left(r\begin{bmatrix} a & b \\ caa^d & d \end{bmatrix} + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) \left(\begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{bmatrix}\right).
\end{aligned}$$

Observe that $x\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r = x\begin{bmatrix} 0 & 0 \\ s^\pi c a^d & 0 \end{bmatrix} = 0$ gives

$$x = x^2x^\# = xx^\#x = x\begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} + x\sum_{n=0}^{\infty} r^{n+1}\begin{bmatrix} 0 & 0 \\ ca^n a^\pi & 0 \end{bmatrix}.$$

So $x\sum_{n=0}^{\infty} r^{n+1}\begin{bmatrix} 0 & 0 \\ ca^n a^\pi & 0 \end{bmatrix} = x - x\begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} = x\begin{bmatrix} a^\pi & 0 \\ 0 & s^\pi \end{bmatrix}$. By this equality and the equation $rxr = r$, we obtain

$$\begin{aligned}
x^\# &= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(1 + \sum_{n=0}^{\infty} r^{n+1}\begin{bmatrix} 0 & 0 \\ ca^n a^\pi & 0 \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(1 + rx\sum_{n=0}^{\infty} r^{n+1}\begin{bmatrix} 0 & 0 \\ ca^n a^\pi & 0 \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(1 + rx\begin{bmatrix} a^\pi & 0 \\ 0 & s^\pi \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(1 + r\begin{bmatrix} aa^\pi & 0 \\ ca^\pi & ss^\pi \end{bmatrix}\right)
\end{aligned}$$

implying

$$\begin{aligned}
x^2x^\# &= x^2 r\left(1 + r\begin{bmatrix} aa^\pi & 0 \\ ca^\pi & ss^\pi \end{bmatrix}\right) \\
&= x\begin{bmatrix} aa^d & 0 \\ s^\pi ca^d & ss^d \end{bmatrix}\left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & 0 \\ s^d ca^\pi & 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} a^2a^d & bss^d \\ caa^d & dss^d \end{bmatrix}\left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & 0 \\ s^d ca^\pi & 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} a^2a^d & bss^d \\ caa^d & dss^d \end{bmatrix} + \begin{bmatrix} -aa^d bs^d ca^\pi + bs^d ca^\pi & 0 \\ -ca^d bs^d ca^\pi + ds^d ca^\pi & 0 \end{bmatrix} \\
&= \begin{bmatrix} a^2a^d & bss^d \\ caa^d & dss^d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ ss^d ca^\pi & 0 \end{bmatrix} = \begin{bmatrix} a^2a^d & b \\ c - s^\pi ca^\pi & d - ss^\pi \end{bmatrix}.
\end{aligned}$$

Because $x^2x^\# = x$, we deduce that $a^2a^d = a$, $ss^\pi = 0$ and $s^\pi ca^\pi = 0$ which yield $a \in (p\mathcal{A}p)^\#$, $s \in ((1-p)\mathcal{A}(1-p))^\#$ and $s^\pi ca^\pi = 0$.

Suppose that $a \in (p\mathcal{A}p)^\#$, $s \in ((1-p)\mathcal{A}(1-p))^\#$ and $s^\pi ca^\pi = 0$. Thus $a^n a^\pi = 0$ for all $n \geq 1$. If we denote by v the right hand side of (10), by

Theorem 2.4, $x \in \mathcal{A}^d$ and $x^d = v$. Since $xx^dx = xrx = x$, then $x \in \mathcal{A}^\#$ and $x^\# = v$. $\square$

In the recent past EP Banach algebra elements were studied, in other words, elements of an algebra such that they commute with their Moore-Penrose inverse, see [20]. It seems as a nice problem finding representations of EP block matrices.

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Address:

Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia

E-mail

M. Z. Kolundžija: `mkolundzija@pmf.ni.ac.rs`
D. Mosić: `dijana@pmf.ni.ac.rs`
D. S. Djordjević: `dragan@pmf.ni.ac.rs`