WEYL’S THEOREMS:
CONTINUITY OF THE SPECTRUM
AND QUASIHYPONORMAL OPERATORS

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Abstract. We consider various Weyl’s theorems in connection with the continuity of the reduced minimum modulus, Weyl spectrum, Browder spectrum, essential approximate point spectrum and Browder essential approximate point spectrum. If $H$ is a Hilbert space, and $T \in B(H)$ is a quasihyponormal operator, we prove the spectral mapping theorem for the essential approximate point spectrum and for arbitrary analytic function, defined on some neighbourhood of $\sigma(T)$. Also, if $T^*$ is quasihyponormal, we prove that the $\alpha$-Weyl’s theorem holds for $T$.

1. Introduction

Let $X$ be a complex infinite-dimensional Banach space and let $B(X)$ ($K(X)$) denote the Banach algebra of all bounded operators (the ideal of all compact operators) on $X$. If $T \in B(X)$, then $\sigma(T)$ denotes the spectrum of $T$ and $\rho(T)$ denotes the resolvent set of $T$. It is are well-known that the following sets form semigroups of semi-Fredholm operators on $X$: $\Phi_+(X) = \{T \in B(X) : \mathcal{R}(T) \text{ is closed and } \dim \mathcal{N}(T) < \infty\}$ and $\Phi_-(X) = \{T \in B(X) : \mathcal{R}(T) \text{ is closed and } \dim X/\mathcal{R}(T) < \infty\}$. The semigroup of Fredholm operators is $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$. If $T$ is semi-Fredholm and $\alpha(T) = \dim \mathcal{N}(T)$ and $\beta(T) = \dim X/\mathcal{R}(T)$, then we define the index by: $i(T) = \alpha(T) - \beta(T)$. We also consider the sets $\Phi_0(X) = \{T \in \Phi(X) : i(T) = 0\}$ (Weyl operators), $\Phi_+(X) = \{T \in \Phi_+(X) : i(T) \leq 0\}$ and $\Phi_+(X) = \{T \in \Phi_+(X) : i(T) \geq 0\}$. The following definitions are well-known: the Fredholm spectrum of $T$ is $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi(X)\}$, the Weyl spectrum of

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$T$ is $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_{0}(X)\}$ and the Browder spectrum of $T$ is $
abla(T) = \cap\{\sigma(T + K) : TK = KT, K \in K(X)\}$. $\sigma_a(T)$ denotes the approximate point spectrum of $T \in B(X)$. Let $\pi_{00}(T)$ be the set of all $\lambda \in \mathbb{C}$ such that $\lambda$ is an isolated point of $\sigma(T)$ and $0 < \dim N(T - \lambda) < \infty$ and let $\pi_0(T)$ be the set of all normal eigenvalues of $A$, that is, the set of all isolated points of $\sigma(T)$ for which the corresponding spectral projection has finite-dimensional range. It is well-known that, for all $T \in B(X)$ the next inclusion $\pi_0(T) \subset \pi_{00}(T)$ holds. We say that $T$ obeys Weyl’s theorem \cite{6,8,10}, if

$$\sigma_w(T) = \sigma(T) \setminus \pi_{00}(T).$$

Let $\pi_{a0}$ denote the set of all $\lambda \in \mathbb{C}$ such that $\lambda$ is isolated in $\sigma_a(T)$ and $0 < \alpha(T - \lambda) < \infty$. Also, by definition, $\sigma_{ea}(T) = \cap\{\sigma_a(T + K) : K \in K(X)\}$ is the essential approximate point spectrum \cite{11} and $\sigma_{a0}(T) = \cap\{\sigma_a(T + K) : AK = KA, K \in K(X)\}$ is the Browder essential approximate point spectrum \cite{12}. It is well-known that $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_{+}(X)\}$. We say that $T$ obeys a-Weyl’s theorem \cite{13}, if

$$\sigma_{ea}(T) = \sigma_a(T) \setminus \pi_{a0}(T).$$

It is well-known that if $T \in B(X)$ obeys a-Weyl’s theorem, then it obeys Weyl’s theorem also \cite{13}.

Let $\Gamma_{0c}(T)$ be the union of all trivial components of the set

$$(\sigma_e(T) \setminus [\rho_{-F}^{\pm}(T)]^{-}) \cup (\cup_{-\infty < n < \infty} \{[\rho_{-F}^{n}(T)]^{-} \setminus [\rho_{-F}^{n}(T)]\})$$

where $\rho_{-F}^{\pm}(T) = \{\lambda \in \mathbb{C} : T - \lambda \in \Phi_{+}(X) \cup \Phi_{-}(X), i(T - \lambda) \neq 0\}$ and $\rho_{-F}^{n}(T) = \{\lambda \in \mathbb{C} : T - \lambda \in \Phi_{+}(X) \cup \Phi_{-}(X), i(T - \lambda) = n\}$. Recall the definition of the reduced minimum modulus of $T$:

$$\gamma(T) = \inf \left\{ \frac{\|Ax\|}{\operatorname{dist}(x, N(T))} : x \notin N(T) \right\}.$$ 

It is well–known that $\gamma(T) > 0$ if and only if $R(T)$ is closed.

If $(\tau_n)$ is a sequence of compact subsets of $\mathbb{C}$, then, by the definition, its limit inferior is $\liminf \tau_n = \{\lambda \in \mathbb{C} : \text{there are } \lambda_n \in \tau_n \text{ with } \lambda_n \to \lambda\}$ and its limit superior is $\limsup \tau_n = \{\lambda \in \mathbb{C} : \text{there are } \lambda_n \in \tau_n \text{ with } \lambda_n \to \lambda\}$. If $\liminf \tau_n =$
lim sup $\tau_n$, then $\lim \tau_n$ is defined by this common limit. A mapping $p$, defined on $B(X)$, whose values are compact subsets of $C$, is said to be upper (lower) semi-continuous at $A$, provided that if $A_n \to A$ then $\lim sup p(A_n) \subset p(A)$ ($p(A) \subset \lim inf p(A_n)$). If $p$ is both upper and lower semi-continuous at $A$, then it is said to be continuous at $A$ and in this case $\lim p(A_n) = p(A)$.

Let $H$ be a Hilbert space. We say that $T \in B(H)$ is hyponormal provided that $\|T^*x\| \leq \|Tx\|$ for all $x \in H$. An operator $T \in B(H)$ is quasihyponormal, if $\|T^*Tx\| \leq \|T^2x\|$ for all $x \in H$. Note that Weyl’s theorem is proved for hyponormal and quasihyponormal operators [3,6,10]. Recall the definitions of ascent and descent of an operator in [2]. We use $a(T)$ to denote the ascent of $T$. Also, $\mathcal{F}(T)$ denotes the set of all complex–valued functions, which are defined and regular on some neighbourhood of $\sigma(T)$.

2. General results

For the sake of completeness we recall some results from [7, Theorem 2.24].

Theorem 2.1. Let the spectra $\sigma$ or $\sigma_b$ be continuous at $A \in B(X)$. Then the following conditions are equivalent:

(i) $A$ obeys Weyl’s theorem;
(ii) if $\lambda \in \pi_{00}(A)$, then $R(A - \lambda)$ is closed;
(iii) $\gamma(A - \lambda)$ is discontinuous at every $\lambda \in \pi_{00}(A)$;
(iv) $\lambda \in \pi_{00}(A)$ implies that $A - \lambda$ has finite ascent.

It is known that, if $A$ obeys Weyl’s theorem, then $\sigma_w(A) = \sigma_b(A)$ [7]. Throughout this paragraph $H$ denotes a complex infinite–dimensional separable Hilbert space, although some of the proofs are valid in Banach spaces, too.

Theorem 2.2. Let $A \in B(H)$ obey Weyl’s theorem. Then $\sigma_w$ is continuous at $A$ if and only if $\sigma$ is continuous at $A$.

Proof. Let $\sigma_w$ be continuous at $A \in B(H)$ and let $\{A_n\}$ be a sequence in $B(H)$ such that $A_n \to A$. Since $\sigma$ is upper semi-continuous, we have to show that $\sigma$ is lower semi-continuous at $A$, or $\sigma(A) \subset \lim inf \sigma(A_n)$. Let $\lambda \in \sigma(A)$. Then, if $\lambda \in \sigma_w(A) \subset \sigma(A)$, we have $\lambda \in \sigma_w(A) \subset \lim inf \sigma_w(A_n) \subset \lim inf \sigma(A_n)$. Suppose that $\lambda \in \sigma(A) \setminus \sigma_w(A)$. Since $A$ obeys Weyl’s theorem, we have that $\lambda \in \pi_{00}(A)$,
so \( \lambda \) is isolated point of \( \sigma(A) \). Now from [9, Theorem 3.26] it follows that \( \lambda \in \lim \inf \sigma(A_n) \).

Now, let \( \sigma \) be continuous at \( A \) and let \( A \) obey Weyl’s theorem. Since \( \pi_0(A) \subset \pi_{00}(A) \), we have

\[
\overline{\pi_0(A)} \cap \sigma_e(A) \subset \overline{\pi_{00}(A)} \cap \sigma_w(A) = \overline{\pi_{00}(A) \cap (\sigma(A) \setminus \pi_{00}(A))} \subset \Gamma_{oe}(A)
\]

and by [1, Theorem 14.17] \( \sigma_w \) is continuous at \( A \).

\textbf{Theorem 2.3.} Let \( A \in B(H) \) obey Weyl’s theorem. Then \( \sigma_w \) is continuous at \( A \) if and only if \( \sigma_b \) is continuous at \( A \).

\textit{Proof.} Since \( A \) obeys Weyl’s theorem, we have that \( \sigma_b(A) = \sigma_w(A) \). Now, by [1, Theorem 14.17] we have that \( \sigma_w \) is continuous at \( A \) if and only if \( \sigma_b \) is continuous at \( A \). \( \square \)

\textbf{Theorem 2.4.} Let \( \sigma_{ab} \) be continuous at \( A \in B(H) \). Then the following conditions are equivalent:

(i) \( A \) obeys \( a \)-Weyl’s theorem;
(ii) if \( \lambda \in \pi_{a0}(A) \), then \( R(A - \lambda) \) is closed.
(iii) \( \lambda \in \pi_{a0}(A) \) implies that \( \gamma \) is discontinuous at \( A - \lambda \).
(iv\(_1\)) if \( \lambda \in \pi_{00}(A) \), then descent of \( A - \lambda \) is finite, and
(iv\(_2\)) if \( \lambda \in \pi_{a0}(A) \setminus \pi_{00}(A) \), then \( R(A - \lambda) \) is closed.

\textit{Proof.} Since \( \sigma_{ab} \) is continuous at \( A \) we have that \( \sigma_{ab}(A) = \sigma_{ea}(A) \) [4, Theorem 2.2].

(i)\(\Leftrightarrow\) (ii) The implication \( \Rightarrow \) is obvious. To prove the opposite implication \( \Leftarrow \), let \( A - \lambda \in \Phi_{\pi}(H) \). Then \( \lambda \notin \sigma_{ea}(A) = \sigma_{ab}(A) \). Now, by [12, Corollary 2.4] it follows that \( \lambda \) is not a limit point of \( \sigma_a(A) \) and by [13, Theorem 1.1] \( A \) obeys \( a \)-Weyl’s theorem.

(i)\(\Leftrightarrow\) (iii) The implication \( \Rightarrow \) follows by [13, Theorem 2.4]. We prove the opposite implication. Suppose that condition (i) holds. Let \( \lambda \in \Delta^+_a(A) = \{ \mu : T - \mu \in \Phi_+(X), 0 < \alpha(A - \mu) \} \). Then \( \lambda \notin \sigma_{ea}(A) = \sigma_{ab}(A) \) and \( \lambda \) is an isolated point of \( \sigma_a(A) \). So \( \lambda \in \pi_{a0}(A) \). The rest of the proof follows again from [13, Theorem 2.4].

(i)\(\Leftrightarrow\) (iv) The implication \( \Rightarrow \) follows by [13, Theorem 2.9]. We now prove the opposite implication. We use next sets: \( \Delta^+_a(A) = \{ \lambda \in C : A - \lambda \in \Phi(X), i(A - \lambda) = \)
Then \( \sigma \) is continuous at \( A \). Suppose that \( \lambda \in \sigma_+(A) \cup \sigma_-(A) \). Then \( \lambda - A \in \Phi_+(X) \) and \( \lambda \not\in \sigma_{ea}(A) = \sigma_{ab}(A) \). Now by [12], it follows that ascent of \( A - \lambda \) is finite. Suppose that \( \lambda \in \Delta^\infty_-(A) \). Then \( A - \lambda \in \Phi_-(X) \), so \( \lambda \not\in \sigma_{ea}(A) = \sigma_{ab}(A) \). By [12] we get that \( \lambda \) is an isolated point of \( \sigma_a(A) \). There exists a neighbourhood \( B(\lambda) \) of \( \lambda \), such that for all \( \mu \in B(\lambda) \setminus \{\lambda\} \) is satisfied \( \alpha(A) = 0 \). We get that \( \lambda \) satisfies the condition \( (\lambda) \) of [13] or [8]. By [13, Theorem 2.9] it follows that \( A \) obeys the \( a \)-Weyl’s theorem. \(\square\)

**Theorem 2.5.** Let \( \sigma_a \) be continuous at \( A \in B(H) \). Then the following conditions are equivalent:

\(\begin{align*}
(\text{i}) & \quad A \text{ obeys } a \text{-Weyl’s theorem}; \\
(\text{ii}) & \quad \text{if } \lambda \in \pi_{a0}(A), \text{ then } \mathcal{R}(A - \lambda) \text{ is closed.} \\
(\text{iii}) & \quad \gamma \text{ is discontinuous at } A - \lambda, \text{ for every } \lambda \in \pi_{a0}(A). \\
(\text{iv}_1) & \quad \text{if } \lambda \in \pi_{00}(A), \text{ then descent of } A - \lambda \text{ is finite, and} \\
(\text{iv}_2) & \quad \text{if } \lambda \in \pi_{a0}(A) \setminus \pi_{00}(A), \text{ then } \mathcal{R}(A - \lambda) \text{ is closed.}
\end{align*}\)

**Proof.** The implications \( (\text{i}) \implies (\text{ii}), (\text{iii}), (\text{iv}) \) hold by [13]. Now, let \( \sigma_a \) be continuous at \( A \), then by [1, Theorem 14.19] we have that

\[
\sigma_a(A) = \pi_0(A) \cup \sigma_{t_e}(A) \cup \rho^+_a(F)(A).
\]

Then, \( \sigma_a(A) \setminus \sigma_{ea}(A) \subset \pi_0(A) \subset \pi_{00}(A) \subset \pi_{a0}(A) \), so \( \sigma_a(A) \setminus \pi_{a0}(A) \subset \sigma_{ea}(A) \).

(\text{ii}) \implies (\text{i}) Suppose that (\text{ii}) holds and let \( \lambda \in \sigma_{ea}(A) \) and \( \lambda \in \pi_{a0}(A) \). Then \( 0 < \alpha(A - \lambda) < \infty \) and by (\text{ii}) \( \mathcal{R}(A - \lambda) \) is closed. Since \( \lambda \in \sigma_{ea}(A) \) if and only if \( A - \lambda \not\in \Phi^+_a(H) \) [13], we have that \( i(A - \lambda) > 0 \). By the continuity of the index, we have that \( \lambda \) is an interior point of \( \sigma_a(A) \) and we get the contradiction, since \( \lambda \in \pi_{a0}(A) \).

(\text{iii}) \implies (\text{i}) Suppose that (\text{iii}) is valued and let \( \lambda_0 \in \pi_{a0}(A) \). Since \( \lambda_0 \) is isolated in \( \sigma_a(A) \), there is some \( \epsilon > 0 \) and a ball \( B(\lambda_0, \epsilon) \) centered in \( \lambda_0 \), such that \( B(\lambda_0, \epsilon) \cap \sigma_a(A) = \{\lambda_0\} \). For every \( \mu \in B(\lambda_0, \epsilon) \setminus \{\lambda_0\} \) we have
\[
\gamma(A - \mu) = \inf_{x \neq 0} \frac{\|(A - \mu)x\|}{\|x\|} \leq \inf_{x \in N(A - \mu)} \frac{\|(A - \lambda_0) - (\mu - \lambda_0))x\|}{\|x\|} = \\
= \inf_{x \in N(A - \mu)} \frac{\|(\mu - \lambda_0))x\|}{\|x\|} = |\mu - \lambda_0|.
\]

Since \(\gamma(\cdot)\) is discontinuous at \(A - \lambda_0\) and \(\gamma(A - \mu) \to 0\), as \(A - \mu \to A - \lambda_0\), we have that \(\gamma(A - \lambda_0) > 0\). Now, by (ii) we have that \(A\) obeys a-Weyl's theorem.

(iv) \(\implies\) (i) Suppose that (iv) is valid and let \(\lambda \in \Delta^*_4(A) \cup \Delta^*_\pi(A)\). Then \(\lambda - A \in \Phi^+_\pi(H)\) and \(\lambda \notin \sigma_e(a(A))\). Since \(\sigma_a\) is continuous at \(A\), then \(\lambda \in \pi_0(A)\). \(\lambda\) is an isolated point of \(\sigma_a(A)\), so, by [12, Corollary 2.3], \(a(A - \lambda) = \infty\) implies \(\lambda \in \sigma_e(a(A))\). This is a contradiction, so \(a(A - \lambda) < \infty\).

Suppose that \(\lambda \in \Delta^*_\infty(A)\). Since \(\lambda \notin \sigma_e(a(A))\) we get that \(\lambda\) is an isolated point of \(\sigma_a(A)\). From Theorem 2.4. (iv2) we have that \(\lambda\) satisfies condition (\(\lambda\)) of [13]. By [13, Theorem 2.9] we have that \(A\) obeys the a-Weyl's theorem. \(\square\)

**Lemma 2.6.** If \(A \in B(H)\) obeys a-Weyl’s theorem, then \(\sigma_e(a(A)) = \sigma_{ab}(A)\).

**Proof.** Since \(\sigma_e(a(A)) \subset \sigma_{ab}(A)\) for every \(A \in B(H)\), we have to show only the opposite inclusion.

It is known that \(\lambda \in \sigma_{ab}(A)\) if and only if \(A - \lambda \notin \Phi^+_{\pi}(H)\), or \(a(A - \lambda) = \infty\) [12]. If \(A - \lambda \notin \Phi^+_{\pi}(H)\), then \(\lambda \in \sigma_e(a(A))\). Suppose that \(A - \lambda \in \Phi^+_{\pi}(H)\) and \(a(A - \lambda) = \infty\). Then, by [6, Theorem 2.9 (ii)], we have that \(\lambda \notin \Delta^*_4(A) \cup \Delta^*_\pi(A)\), so

\[
i(A - \lambda) \neq 0 \text{ and } \alpha(A - \lambda) \geq \beta(A - \lambda) \text{ implies that } A - \lambda \notin \Phi^+_{\pi}(H).
\]

This contradiction completes the proof. \(\square\)

**Corollary 2.7.** Let \(A \in B(H)\) obey a-Weyl’s theorem. Then \(\sigma_e(a)\) is continuous at \(A\) if and only if \(\sigma_{ab}\) is continuous at \(A\).

**Proof.** By Lemma 2.6 and [4, Theorem 2.2]. \(\square\)

We shall improve Prasanna’s result, concerning Weyl’s theorem [12]. See also a paper of Gustafson [8]. Let \(\Delta^-(T)\) denote the set of all \(\lambda \in \sigma_a(T)\), such that \(T - \lambda \in \Phi^-\pi(X)\).
Theorem 2.8. Suppose that $T \in B(X)$ such that $\pi_{a0}(T) = \pi_0(T)$ and $\Delta_+(T) \subseteq \partial\sigma_a(T)$. Then $a$-Weyl’s theorem holds for $T$.

Proof. Suppose that $\lambda \in \pi_{a0}(T) = \pi_0(T)$. Then $\lambda$ has the finite algebraic multiplicity, so $X = \mathcal{N}((T - \lambda)^p) \oplus \mathcal{R}((T - \lambda)^p)$ for some non-negative integer $p$ [3]. Now, $0 < \dim\mathcal{N}(T - \lambda) < \infty$, so $\dim\mathcal{N}((T - \lambda)^p) < \infty$. We get that $(T - \lambda)^p \in \Phi_0(X)$. Since $\mathcal{R}((T - \lambda)^p) \subseteq \mathcal{R}(T - \lambda)$, we obtain $T - \lambda \in \Phi(X)$ and $\iota(T - \lambda) = \frac{1}{p} \iota((T - \lambda)^p) = 0$, so $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$.

To prove the opposite inclusion, suppose that $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. We know that $T - \lambda \in \Phi_+(X)$ and $0 < \alpha(T - \lambda) < \infty$. There exists some $\epsilon > 0$, such that for all $\mu$ satisfying $0 < |\mu - \lambda| < \epsilon$, we have that $\alpha(T - \mu)$ is a constant, not greater than $\alpha(T - \lambda)$ and also $T - \mu \in \Phi_+(X)$. A ball $B(\lambda, \epsilon)$ centered at $\lambda$, intersects the set $C \setminus \sigma_a(T)$, since $\Delta_+(T) \subseteq \partial\sigma_a(T)$, so we get that $\alpha(T - \mu) = 0$ for all such $\mu$. Now, it is obvious that $\lambda$ must be an isolated point of $\sigma_a(T)$, so $\lambda \in \pi_{a0}(T)$. □

Notice that if $\sigma_a(T)$ is nowhere dense, then the inclusion $\Delta_+(T) \subseteq \partial\sigma_a(T)$ is valid.

Corollary 2.9. Let $T \in B(X)$. If $\pi_{a0}(T) = \pi_0(T)$ and $\sigma_a(T)$ is nowhere dense in $C$, then $a$-Weyl’s theorem holds for $T$. If $\pi_{00}(T) = \pi_0(T)$ and $\sigma(T)$ is nowhere dense in $C$, then Weyl’s theorem holds for $T$.

Proof. We shall prove the second statement. Since $\sigma(T)$ is nowhere dense, we get that $\sigma(T) = \partial\sigma(T) = \sigma_a(T)$, so the conditions of Theorem 2.5 are valued. We get that the $a$-Weyl’s theorem holds for $T$, so the Weyl’s theorem holds for $T$ [13]. □

3. Quasihyponormal operators

Through this paragraph $H$ denotes a complex infinite–dimensional complex Hilbert space. The next theorem is proved by Heuser [2].

Theorem 3.1. Let $T$ be a bounded operator on a Banach space $X$ and let $a(T) < \infty$. If $\alpha(T) < \infty$, or $\beta(T) < \infty$, then $\alpha(T) \leq \beta(T)$.

The following lemma is proved in the Erovenko’s paper [5]. For the sake of completeness, we give details of the proof.
Lemma 3.2. Let $T$ be a quasihyponormal operator on $H$. If $\lambda \in \mathbb{C}\setminus\{0\}$, then $\alpha(T - \lambda) \leq \alpha(T - \lambda^*)$. If $\alpha(T) < \infty$, or $\beta(T) < \infty$, then $\alpha(T) < \alpha(T^*)$.

Proof. Suppose that $\lambda \neq 0$. If $x \in \mathcal{N}(T - \lambda)$, then $Tx = \lambda x$ and we get $\|T^*x\| \leq |\lambda| \|x\|$. Now $((T - \lambda)^*x, (T - \lambda^*)x) \leq 0$, so $x \in \mathcal{N}((T - \lambda)^*)$. To prove the second statement, let $T^2x = 0$. Now $(Tx, Tx) = 0$, so $x \in \mathcal{N}(T)$. We get that $a(T) = 1$ and the rest of the proof follows by Theorem 3.1. □

The following theorem is an improvement of Erovenko’s result [5]. Using this method, Erovenko proved the next result for the Weyl spectrum and an arbitrary polynomial.

Theorem 3.3. Let $T \in B(H)$ be quasihyponormal and $f \in \mathcal{F}(T)$. Then

$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T)) \quad \text{and} \quad \sigma_w(f(T)) = f(\sigma_w(T)).$$

Proof. We prove the first statement. Note that it is enough to prove the inclusion $\supset$. Suppose that $\lambda \notin \sigma_{ea}(f(T))$. Then $f(T) - \lambda \in \Phi^-(H)$ and

$$f(T) - \lambda = c(T - \mu_1) \cdots (T - \mu_n)g(T),$$

where $c \in \mathbb{C}$, $g(T)$ is invertible and the operators on the right side of (1) mutually commute. Now, $T - \mu_i \in \Phi^+(H)$. By Lemma 3.2 we get that $i(T) = \alpha(T) - \alpha(T^*) \leq 0$, so $T - \mu_i \in \Phi^+(H)$ for all $i = 1, \ldots, n$. So $\lambda \notin f(\sigma_{ea}(T))$. The proof of the second statement is analogous. □

Now, we give a generalisation of Rakočević’s result [11]. Notice that Rakočević proved Theorem 3.4 assuming that $T^*$ is hyponormal.

Theorem 3.4. Let $T \in B(H)$, such that $T^*$ is quasihyponormal. Then $a$-Weyl’s theorem holds for $T$.

Proof. Suppose that $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then $T - \lambda \in \Phi^-(H)$ and $0 < \alpha(T - \lambda) < \infty$. If $\lambda \neq 0$, since $T^*$ is quasihyponormal, by Lemma 3.2 we get that $\alpha((T - \lambda)^*) \leq \alpha(T - \lambda) < \infty$. If $\lambda = 0$, then $T \in \Phi^-(H)$ and $T^* \in \Phi^+(H)$, so we get again $\alpha(T^*) \leq \alpha(T) = \beta(T^*) < \infty$. Anyway, we get $\alpha((T - \lambda)^*) \leq \alpha(T - \lambda) < \infty$. Obviously, $i(T - \lambda) = \alpha(T - \lambda) - \alpha((T - \lambda)^*) \geq 0$. Since $T - \lambda \in \Phi^+(H)$, we get that $0 = i(T - \lambda) = i((T - \lambda)^*)$, so $\lambda \notin \sigma_{w}(T^*)$. It is well-known that quasihyponormal
operators obey Weyl’s theorem [6,10], so \( \lambda \in \pi_{00}(T^*) \) and \( \lambda \) is an isolated point of \( \sigma(T) \). Now, \( \lambda \) is isolated in \( \sigma_a(T) \) and we get that \( \lambda \in \pi a_0(T) \).

To prove the other inclusion, suppose that \( \lambda_0 \in \pi a_0(T) \). Then \( 0 < \alpha(T - \lambda_0) < \infty \) and there is some \( \epsilon > 0 \), such that for all \( \lambda \in \mathbb{C} \), if \( 0 < |\lambda - \lambda_0| < \epsilon \), then \( \lambda \notin \sigma_a(T) \). For all such \( \lambda \), using Lemma 3.2, we get \( \alpha((T - \lambda)^*) \leq \alpha(T - \lambda) = 0. \) Now \( \alpha(T - \lambda) = 0 \) and \( \lambda_0 \) must be an isolated point of \( \sigma a(T) \), so \( 0 \) must be an isolated point of \( \sigma((T - \lambda_0)^*) \). We see that \( \beta((T - \lambda_0)^*) = \alpha(T - \lambda_0) < \infty \), so \( (T - \lambda_0)^* \in \Phi(H) \). Since \( 0 \) is an isolated point of \( \sigma((T - \lambda_0)^*) \), we get \( \beta((T - \lambda)^*) = 0 \) and \( \lambda_0 \notin \sigma_w(T) \supset \sigma ea(T) \). □

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