WEYL'S THEOREMS: CONTINUITY OF THE SPECTRUM AND QUASIHYPONORMAL OPERATORS

Slaviša V. Djordjević and Dragan S. Djordjević

Abstract. We consider various Weyl's theorems in connection with the continuity of the reduced minimum modulus, Weyl spectrum, Browder spectrum, essential approximate point spectrum and Browder essential approximate point spectrum. If H is a Hilbert space, and $T \in B(H)$ is a quasihyponormal operator, we prove the spectral mapping theorem for the essential approximate point spectrum and for arbitrary analytic function, defined on some neighbourhood of $\sigma(T)$. Also, if T^* is quasihyponormal, we prove that the a-Weyl's theorem holds for T.

1. Introduction

Let X be a complex infinite–dimensional Banach space and let $B(X)$ $(K(X))$ denote the Banach algebra of all bounded operators (the ideal of all compact operators) on X. If $T \in B(X)$, then $\sigma(T)$ denotes the spectrum of T and $\rho(T)$ denotes the resolvent set of T. It is are well-known that the following sets form semigroups of semi–Fredholm operators on X: $\Phi_+(X) = \{T \in B(X) : \mathcal{R}(T) \text{ is closed and } \dim\mathcal{N}(T) < \infty\}$ and $\Phi_-(X) = \{T \in B(X) : \mathcal{R}(T) \text{ is closed and } \dim X/\mathcal{R}(T) < \infty\}$. The semigroup of Fredholm operators is $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$. If T is semi–Fredholm and $\alpha(T) = \dim\mathcal{N}(T)$ and $\beta(T) = \dim X/\mathcal{R}(T)$, then we define the index by: $i(T) = \alpha(T) - \beta(T)$. We also consider the sets $\Phi_0(X) = \{T \in \Phi(X) : i(T) = 0\}$ (Weyl operators), $\Phi_+^-(X) = \{T \in \Phi_+(X) : i(T) \leq 0\}$ and $\Phi_-^+(X) = \{T \in \Phi_-(X) : i(T) \geq 0\}$. The following definitions are well-known: the Fredholm spectrum of T is $\sigma_e(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin \Phi(X)\}$, the Weyl spectrum of

1991 *Mathematics Subject Classification.* 47A53, 47B20.

Key words and phrases. Semi–Fredholm operators, essential spectra, Weyl's theorems, spectral continuity, quasihyponormal operators.

T is $\sigma_w(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin \Phi_0(X)\}$ and the Browder spectrum of T is $\sigma_b(T) = \cap\{\sigma(T+K) : TK = KT, K \in K(X)\}$. $\sigma_a(T)$ denotes the approximate point spectrum of $T \in B(X)$. Let $\pi_{00}(T)$ be the set of all $\lambda \in \mathbf{C}$ such that λ is an isolated point of $\sigma(T)$ and $0 < \dim\mathcal{N}(T-\lambda) < \infty$ and let $\pi_0(T)$ be the set of all normal eigenvalues of A, that is the set of all isolated points of $\sigma(T)$ for which the corresponding spectral projection has finite-dimensional range. It is well-known that, for all $T \in B(X)$ the next inclusion $\pi_0(T) \subset \pi_{00}(T)$ holds . We say that T obeys Weyl's theorem [**6,8,10**], if

$$\sigma_w(T) = \sigma(T) \backslash \pi_{00}(T).$$

Let π_{a0} denote the set of all $\lambda \in \mathbf{C}$ such that λ is isolated in $\sigma_a(T)$ and $0 < \alpha(T-\lambda) < \infty$. Also, by definition, $\sigma_{ea}(T) = \cap\{\sigma_a(T+K) : K \in K(X)\}$ is the essential approximate point spectrum [**11**] and $\sigma_{ab}(T) = \cap\{\sigma_a(T+K) : AK = KA, K \in K(X)\}$ is the Browder essential approximate point spectrum [**12**]. It is well-known that $\sigma_{ea}(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin \Phi_+^-(X)\}$. We say that T obeys a-Weyl's theorem [**13**], if

$$\sigma_{ea}(T) = \sigma_a(T) \backslash \pi_{a0}(T).$$

It is well-known that if $T \in B(X)$ obeys a-Weyl's theorem, then it obeys Weyl's theorem also [**13**].

Let $\Gamma_{0e}(T)$ be the union of all trivial components of the set

$$(\sigma_e(T) \backslash [\rho_{s-F}^{\pm}(T)]^-) \cup (\cup_{-\infty<n<\infty}\{[\rho_{s-F}^n(T)]^- \backslash \rho_{s-F}^n(T)\}),$$

where $\rho_{s-F}^{\pm}(T) = \{\lambda \in \mathbf{C} : T - \lambda \in \Phi_+(X) \cup \Phi_-(X), i(T-\lambda) \neq 0\}$ and $\rho_{s-F}^n(T) = \{\lambda \in \mathbf{C} : T - \lambda \in \Phi_+(X) \cup \Phi_-(X), i(T-\lambda) = n\}$. Recall the definition of the reduced minimum modulus of T:

$$\gamma(T) = \inf\left\{\frac{\|Ax\|}{\mathrm{dist}(x, \mathcal{N}(T))} : x \notin \mathcal{N}(T)\right\}.$$

It is well–known that $\gamma(T) > 0$ if and only if $\mathcal{R}(T)$ is closed.

If (τ_n) is a sequence of compact subsets of $\mathbf{C}$, then, by the definition, its limit inferior is $\liminf \tau_n = \{\lambda \in \mathbf{C} :$ there are $\lambda_n \in \tau_n$ with $\lambda_n \to \lambda\}$ and its limit superior is $\limsup \tau_n = \{\lambda \in \mathbf{C} :$ there are $\lambda_{n_k} \in \tau_{n_k}$ with $\lambda_{n_k} \to \lambda\}$. If $\liminf \tau_n =$

$\limsup \tau_n$, then $\lim \tau_n$ is defined by this common limit. A mapping p, defined on $B(X)$, whose values are compact subsets of $\mathbf{C}$, is said to be upper (lower) semi-continuous at A, provided that if $A_n \to A$ then $\limsup p(A_n) \subset p(A)$ ($p(A) \subset \liminf p(A_n)$). If p is both upper and lower semi-continuous at A, then it is said to be continuous at A and in this case $\lim p(A_n) = p(A)$.

Let H be a Hilbert space. We say that $T \in B(H)$ is hyponormal provided that $\|T^*x\| \le \|Tx\|$ for all $x \in H$. An operator $T \in B(H)$ is quasihyponormal, if $\|T^*Tx\| \le \|T^2x\|$ for all $x \in H$. Note that Weyl's theorem is proved for hyponormal and quasihyponormal operators [**3,6,10**]. Recall the definitions of ascent and descent of an operator in [**2**]. We use $a(T)$ to denote the ascent of T. Also, $\mathcal{F}(T)$ denotes the set of all complex–valued functions, which are defined and regular on some neighbourhood of $\sigma(T)$.

2. General results

For the sake of completeness we recall some results from [**7**, Theorem 2.24].

Theorem 2.1. *Let the spectra σ or σ_b be continuous at $A \in B(X)$. Then the following conditions are equivalent:*

(i) *A obeys Weyl's theorem;*
(ii) *if $\lambda \in \pi_{00}(A)$, then $R(A-\lambda)$ is closed;*
(iii) *$\gamma(A-\lambda)$ is discontinuous at every $\lambda \in \pi_{00}(A)$;*
(iv) *$\lambda \in \pi_{00}(A)$ implies that $A-\lambda$ has finite ascent.*

It is known that, if A obeys Weyl's theorem, then $\sigma_w(A) = \sigma_b(A)$ [**7**]. Throughout this paragraph H denotes a complex infinite–dimensional separable Hilbert space, although some of the proofs are valid in Banach spaces, too.

Theorem 2.2. *Let $A \in B(H)$ obey Weyl's theorem. Then σ_w is continuous at A if and only if σ is continuous at A.*

Proof. Let σ_w be continuous at $A \in B(H)$ and let $\{A_n\}$ be a sequence in $B(H)$ such that $A_n \to A$. Since σ is upper semi-continuous, we have to show that σ is lower semi-continuous at A, or $\sigma(A) \subset \liminf \sigma(A_n)$. Let $\lambda \in \sigma(A)$. Then, if $\lambda \in \sigma_w(A) \subset \sigma(A)$, we have $\lambda \in \sigma_w(A) \subset \liminf \sigma_w(A_n) \subset \liminf \sigma(A_n)$. Suppose that $\lambda \in \sigma(A) \backslash \sigma_w(A)$. Since A obeys Weyl's theorem, we have that $\lambda \in \pi_{00}(A)$, so λ is isolated point of $\sigma(A)$. Now from [**9**, Theorem 3.26] it follows that $\lambda \in \liminf \sigma(A_n)$.

Now, let σ be continuous at A and let A obey Weyl's theorem. Since $\pi_0(A) \subset \pi_{00}(A)$, we have

$$\overline{\pi_0(A)} \cap \sigma_e(A) \subset \overline{\pi_{00}(A)} \cap \sigma_w(A) = \overline{\pi_{00}(A)} \cap (\sigma(A) \backslash \pi_{00}(A)) \subset \overline{\Gamma_{oe}(A)}$$

and by [**1**, Theorem 14.17] σ_w is continuous at A. $\square$

Theorem 2.3. *Let $A \in B(H)$ obey Weyl's theorem. Then σ_w is continuous at A if and only if σ_b is continuous at A.*

Proof. Since A obeys Weyl's theorem, we have that $\sigma_b(A) = \sigma_w(A)$. Now, by [**1**, Theorem 14.17] we have that σ_w is continuous at A if and only if σ_b is continuous at A. $\square$

Theorem 2.4. *Let σ_{ab} be continuous at $A \in B(H)$. Then the following conditions are equivalent:*

(i) *A obeys a-Weyl's theorem;*
(ii) *if $\lambda \in \pi_{a0}(A)$, then $\mathcal{R}(A-\lambda)$ is closed.*
(iii) *$\lambda \in \pi_{a0}(A)$ implies that γ is discontinuous at $A-\lambda$.*

(iv$_1$) *if $\lambda \in \pi_{00}(A)$, then descent of $A-\lambda$ is finite, and*
(iv$_2$) *if $\lambda \in \pi_{a0}(A) \backslash \pi_{00}(A)$, then $\mathcal{R}(A-\lambda)$ is closed.*

Proof. Since σ_{ab} is continuous at A we have that $\sigma_{ab}(A) = \sigma_{ea}(A)$ [**4**, Theorem 2.2].

(i)$\Leftrightarrow$(ii) The implication $\Longrightarrow$ is obvious. To prove the opposite implication $\Leftarrow$, let $A-\lambda \in \Phi_+^-(H)$. Then $\lambda \notin \sigma_{ea}(A) = \sigma_{ab}(A)$. Now, by [**12**, Corollary 2.4] it follows that λ is not a limit point of $\sigma_a(A)$ and by [**13**, Theorem 1.1] A obeys a-Weyl's theorem.

(i)$\Leftrightarrow$(iii) The implication $\Longrightarrow$ follows by [**13**, Theorem 2.4]. We prove the opposite implication. Suppose that condition (i) holds. Let $\lambda \in \Delta_a^s(A) = \{\mu : T-\mu \in \Phi_+^-(X), 0 < \alpha(A-\mu)\}$. Then $\lambda \notin \sigma_{ea}(A) = \sigma_{ab}(A)$ and λ is an isolated point of $\sigma_a(A)$. So $\lambda \in \pi_{a0}(A)$. The rest of the proof follows again from [**13**, Theorem 2.4].

(i)$\Leftrightarrow$(iv) The implication $\Longrightarrow$ follows by [**13**, Theorem 2.9]. We now prove the opposite implication. We use next sets: $\Delta_4^s(A) = \{\lambda \in \mathbf{C} : A-\lambda \in \Phi(X), i(A-\lambda) =$

$0\}$, $\Delta_-^s(A) = \{\lambda \in \Delta_a^s(A) : \alpha(A-\lambda) < \beta(A-\lambda) < \infty\}$ and $\Delta_{-\infty}^s(A) = \{\lambda \in \Delta_a^s(A) : \beta(A-\lambda) = \infty\}$. Suppose that $\lambda \in \Delta_4^s(A) \cup \Delta_-^s(A)$. Then $\lambda - A \in \Phi_-^+(X)$ and $\lambda \notin \sigma_{ea}(A) = \sigma_{ab}(A)$. Now by [**12**], it follows that ascent of $A-\lambda$ is finite. Suppose that $\lambda \in \Delta_{-\infty}^s(A)$. Then $A - \lambda \in \Phi_+^-(X)$, so $\lambda \notin \sigma_{ea}(A) = \sigma_{ab}(A)$. By [**12**] we get that λ is an isolated point of $\sigma_a(A)$. There exists a neighbourhood $B(\lambda)$ of λ, such that for all $\mu \in B(\lambda)\backslash\{\lambda\}$ is satisfied $\alpha(A) = 0$. We get that λ satisfies the condition (λ) of [**13**] or [**8**]. By [**13**, Theorem 2.9] it follows that A obeys the a-Weyl's theorem. $\square$

Theorem 2.5. *Let σ_a be continuous at $A \in B(H)$. Then the following conditions are equivalent:*

(i) *A obeys a-Weyl's theorem;*
(ii) *if $\lambda \in \pi_{a0}(A)$, then $\mathcal{R}(A-\lambda)$ is closed.*
(iii) *γ is discontinuous at $A-\lambda$, for every $\lambda \in \pi_{a0}(A)$.*

(iv$_1$) *if $\lambda \in \pi_{00}(A)$, then descent of $A-\lambda$ is finite, and*
(iv$_2$) *if $\lambda \in \pi_{a0}(A)\backslash\pi_{00}(A)$, then $\mathcal{R}(A-\lambda)$ is closed.*

Proof. The implications (i) $\Longrightarrow$ (ii), (iii), (iv) hold by [**13**]. Now, let σ_a be continuous at A, then by [**1**, Theorem 14.19] we have that

$$\sigma_a(A) = \pi_0(A) \cup \sigma_{le}(A) \cup \rho_{s-F}^+(A)\,.$$

Then, $\sigma_a(A)\backslash\sigma_{ea}(A) \subset \pi_0(A) \subset \pi_{00}(A) \subset \pi_{a0}(A)$, so $\sigma_a(A)\backslash\pi_{a0}(A) \subset \sigma_{ea}(A)$.

(ii) $\Longrightarrow$ (i) Suppose that (ii) holds and let $\lambda \in \sigma_{ea}(A)$ and $\lambda \in \pi_{a0}(A)$. Then $0 < \alpha(A-\lambda) < \infty$ and by (ii) $\mathcal{R}(A-\lambda)$ is closed. Since $\lambda \in \sigma_{ea}(A)$ if and only if $A - \lambda \notin \Phi_+^-(H)$ [**13**], we have that $i(A-\lambda) > 0$. By the continuity of the index, we have that λ is an interior point of $\sigma_a(A)$ and we get the contradiction, since $\lambda \in \pi_{a0}(A)$.

(iii) $\Longrightarrow$ (i) Suppose that (iii) is valued and let $\lambda_0 \in \pi_{a0}(A)$. Since λ_0 is isolated in $\sigma_a(A)$, there is some $\epsilon > 0$ and a ball $B(\lambda_0, \epsilon)$ centered in λ_0, such that $B(\lambda_0,\epsilon) \cap \sigma_a(A) = \{\lambda_0\}$. For every $\mu \in B(\lambda_0,\epsilon)\backslash\{\lambda_0\}$ we have

$$\gamma(A-\mu)=\inf_{x\neq 0}\frac{\|(A-\mu)x\|}{\|x\|}\leq \inf_{\substack{x\in\mathcal{N}(A-\mu)\\ x\neq 0}}\frac{\|((A-\lambda_0)-(\mu-\lambda_0))x\|}{\|x\|}=$$
$$=\inf_{\substack{x\in\mathcal{N}(A-\mu)\\ x\neq 0}}\frac{\|(\mu-\lambda_0))x\|}{\|x\|}=|\mu-\lambda_0|\,.$$

Since $\gamma(\cdot)$ is discontinuous at $A-\lambda_0$ and $\gamma(A-\mu)\to 0$, as $A-\mu\to A-\lambda_0$, we have that $\gamma(A-\lambda_0)>0$. Now, by (ii) we have that A obeys a-Weyl's theorem.

(iv) $\Longrightarrow$ (i) Suppose that (iv) is valid and let $\lambda\in\Delta_4^s(A)\cup\Delta_-^s(A)$. Then $\lambda-A\in\Phi_+^-(H)$ and $\lambda\notin\sigma_{ea}(A)$. Since σ_a is continuous at A, then $\lambda\in\pi_0(A)$. λ is an isolated point of $\sigma_a(A)$, so, by [**12**, Corollary 2.3], $a(A-\lambda)=\infty$ implies $\lambda\in\sigma_{ea}(A)$. This is a contradiction, so $a(A-\lambda)<\infty$.

Suppose that $\lambda\in\Delta_{-\infty}^s(A)$. Since $\lambda\notin\sigma_{ea}(A)$ we get that λ is an isolated point of $\sigma_a(A)$. From Theorem 2.4. (iv$_2$) we have that λ satisfies condition (λ) of [**13**]. By [**13**, Theorem 2.9] we have that A obeys the a-Weyl's theorem. $\square$

Lemma 2.6. *If $A\in B(H)$ obeys a-Weyl's theorem, then $\sigma_{ea}(A)=\sigma_{ab}(A)$.*

Proof. Since $\sigma_{ea}(A)\subset\sigma_{ab}(A)$ for every $A\in B(H)$, we have to show only the opposite inclusion.

It is known that $\lambda\in\sigma_{ab}(A)$ if and only if $A-\lambda\notin\Phi_+^-(H)$, or $a(A-\lambda)=\infty$ [**12**]. If $A-\lambda\notin\Phi_+^-(H)$, then $\lambda\in\sigma_{ea}(A)$. Suppose that $A-\lambda\in\Phi_+^-(H)$ and $a(A-\lambda)=\infty$. Then, by [**6**,Theorem 2.9 (ii)], we have that $\lambda\notin\Delta_4^s(A)\cup\Delta_-^s(A)$, so

$$i(A-\lambda)\neq 0\ \text{ and }\ \alpha(A-\lambda)\geq\beta(A-\lambda)\quad\text{implies that}\quad A-\lambda\notin\Phi_+^-(H)\,.$$

This contradiction completes the proof. $\square$

Corollary 2.7. *Let $A\in B(H)$ obey a-Weyl's theorem. Then σ_{ea} is continuous at A if and only if σ_{ab} is continuous at A.*

Proof. By Lemma 2.6 and [**4**, Theorem 2.2]. $\square$

We shall improve Prasanna's result, concerning Weyl's theorem [**12**]. See also a paper of Gustafson [**8**]. Let $\Delta_+^-(T)$ denote the set of all $\lambda\in\sigma_a(T)$, such that $T-\lambda\in\Phi_+^-(X)$.

Theorem 2.8. *Suppose that $T \in B(X)$ such that $\pi_{a0}(T) = \pi_0(T)$ and $\Delta_+^-(T) \subseteq \partial\sigma_a(T)$. Then a-Weyl's theorem holds for T.*

Proof. Suppose that $\lambda \in \pi_{a0}(T) = \pi_0(T)$. Then λ has the finite algebraic multiplicity, so $X = \mathcal{N}((T-\lambda)^p) \oplus \mathcal{R}((T-\lambda)^p)$ for some non-negative integer p [**3**]. Now, $0 < \dim\mathcal{N}(T-\lambda) < \infty$, so $\dim\mathcal{N}((T-\lambda)^p) < \infty$. We get that $(T-\lambda)^p \in \Phi_0(X)$. Since $\mathcal{R}((T-\lambda)^p) \subseteq \mathcal{R}(T-\lambda)$, we obtain $T-\lambda \in \Phi(X)$ and $i(T-\lambda) = \frac{1}{p}\, i((T-\lambda)^p) = 0$, so $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$.

To prove the opposite inclusion, suppose that $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$. We know that $T-\lambda \in \Phi_+^-(X)$ and $0 < \alpha(T-\lambda) < \infty$. There exists some $\epsilon > 0$, such that for all μ satisfying $0 < |\mu - \lambda| < \epsilon$, we have that $\alpha(T-\mu)$ is a constant, not greater than $\alpha(T-\lambda)$ and also $T-\mu \in \Phi_+^-(X)$. A ball $B(\lambda,\epsilon)$ centered at λ, intersects the set $\mathbf{C}\backslash\sigma_a(T)$, since $\Delta_+^-(T) \subseteq \partial\sigma_a(T)$, so we get that $\alpha(T-\mu) = 0$ for all such μ. Now, it is obvious that λ must be an isolated point of $\sigma_a(T)$, so $\lambda \in \pi_{a0}(T)$. $\square$

Notice that if $\sigma_a(T)$ is nowhere dense, then the inclusion $\Delta_+^-(T) \subseteq \partial\sigma_a(T)$ is valid.

Corollary 2.9. *Let $T \in B(X)$. If $\pi_{a0}(T) = \pi_0(T)$ and $\sigma_a(T)$ is nowhere dense in $\mathbf{C}$, then a-Weyl's theorem holds for T. If $\pi_{00}(T) = \pi_0(T)$ and $\sigma(T)$ is nowhere dense in $\mathbf{C}$, then Weyl's theorem holds for T.*

Proof. We shall prove the second statement. Since $\sigma(T)$ is nowhere dense, we get that $\sigma(T) = \partial\sigma(T) = \sigma_a(T)$, so the conditions of Theorem 2.5 are valued. We get that the a-Weyl's theorem holds for T, so the Weyl's theorem holds for T [**13**]. $\square$

3. Quasihyponormal operators

Through this paragraph H denotes a complex infinite–dimensional complex Hilbert space. The next theorem is proved by Heuser [**2**].

Theorem 3.1. *Let T be a bounded operator on a Banach space X and let $a(T) < \infty$. If $\alpha(T) < \infty$, or $\beta(T) < \infty$, then $\alpha(T) \leq \beta(T)$.*

The following lemma is proved in the Erovenko's paper [**5**]. For the sake of completeness, we give details of the proof.

Lemma 3.2. *Let T be a quasihyponormal operator on H. If $\lambda \in \mathbf{C}\backslash\{0\}$, then $\alpha(T-\lambda) \leq \alpha(T-\lambda)^*$. If $\alpha(T)<\infty$, or $\beta(T)<\infty$, then $\alpha(T)<\alpha(T^*)$.*

Proof. Suppose that $\lambda \neq 0$. If $x \in \mathcal{N}(T-\lambda)$, then $Tx=\lambda x$ and we get $\|T^*x\| \leq |\lambda|\,\|x\|$. Now $((T-\lambda)^*x,(T-\lambda)^*x) \leq 0$, so $x \in \mathcal{N}((T-\lambda)^*)$. To prove the second statement, let $T^2x=0$. Now $(Tx,Tx)=0$, so $x \in \mathcal{N}(T)$. We get that $a(T)=1$ and the rest of the proof follows by Theorem 3.1. $\square$

The following theorem is an improvement of Erovenko's result [**5**]. Using this method, Erovenko proved the next result for the Weyl spectrum and an arbitrary polynomial.

Theorem 3.3. *Let $T \in B(H)$ be quasihyponormal and $f \in \mathcal{F}(T)$. Then*

$$\sigma_{ea}(f(T))=f(\sigma_{ea}(T)) \quad \textit{and} \quad \sigma_w(f(T))=f(\sigma_w(T)).$$

Proof. We prove the first statement. Note that it is enough to prove the inclusion $\supset$. Suppose that $\lambda \notin \sigma_{ea}(f(T))$. Then $f(T)-\lambda \in \Phi_+^-(H)$ and

$$f(T)-\lambda=c(T-\mu_1)\cdots(T-\mu_n)g(T), \tag{1}$$

where $c \in \mathbf{C}$, $g(T)$ is invertible and the operators on the right side of (1) mutually commute. Now, $T-\mu_i \in \Phi_+(H)$. By Lemma 3.2 we get that $i(T)=\alpha(T)-\alpha(T^*) \leq 0$, so $T-\mu_i \in \Phi_+^-(H)$ for all $i=1,\dots,n$. So $\lambda \notin f(\sigma_{ea}(T))$. The proof of the second statement is analogous. $\square$

Now, we give a generalisation of Rakočević's result [**11**]. Notice that Rakočević proved Theorem 3.4 assuming that T^* is hyponormal.

Theorem 3.4. *Let $T \in B(H)$, such that T^* is quasihyponormal. Then a-Weyl's theorem holds for T.*

Proof. Suppose that $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$. Then $T-\lambda \in \Phi_+^-(H)$ and $0<\alpha(T-\lambda)<\infty$. If $\lambda \neq 0$, since T^* is quasihyponormal, by Lemma 3.2 we get that $\alpha((T-\lambda)^*) \leq \alpha(T-\lambda)<\infty$. If $\lambda=0$, then $T \in \Phi_+^-(H)$ and $T^* \in \Phi_-^+(H)$, so we get again $\alpha(T^*) \leq \alpha(T)=\beta(T^*)<\infty$. Anyway, we get $\alpha((T-\lambda)^*) \leq \alpha(T-\lambda)<\infty$. Obviously, $i(T-\lambda)=\alpha(T-\lambda)-\alpha((T-\lambda)^*) \geq 0$. Since $T-\lambda \in \Phi_+^-(H)$, we get that $0=i(T-\lambda)=i((T-\lambda)^*)$, so $\overline{\lambda} \notin \sigma_w(T^*)$. It is well-known that quasihyponormal operators obey Weyl's theorem [**6,10**], so $\overline{\lambda} \in \pi_{00}(T^*)$ and λ is an isolated point of $\sigma(T)$. Now, λ is isolated in $\sigma_a(T)$ and we get that $\lambda \in \pi_{a0}(T)$.

To prove the other inclusion, suppose that $\lambda_0 \in \pi_{a0}(T)$. Then $0 < \alpha(T-\lambda_0) < \infty$ and there is some $\epsilon > 0$, such that for all $\lambda \in \mathbf{C}$, if $0 < |\lambda - \lambda_0| < \epsilon$, then $\lambda \notin \sigma_a(T)$. For all such λ, using Lemma 3.2, we get $\alpha((T-\lambda)^*) \leq \alpha(T-\lambda) = 0$. Now $i(T-\lambda) = 0$ and λ_0 must be an isolated point of $\sigma(T)$, so 0 must be an isolated point of $\sigma((T-\lambda_0)^*)$. We see that $\beta((T-\lambda_0)^*) = \alpha(T-\lambda_0) < \infty$, so $(T-\lambda_0)^* \in \Phi(H)$. Since 0 is an isolated point of $\sigma((T-\lambda_0)^*)$, we get $i((T-\lambda)^*) = 0$ and $\lambda_0 \notin \sigma_w(T) \supset \sigma_{ea}(T)$. $\square$

Acknowledgement. We are grateful to Professor Vladimir Rakočević for helpful suggestions and conversations concerning the paper.

References

[1] C.Apostol, L.A.Fialkow, D.A.Herrero and D.Voiculescu, *Aproximation of Hilbert space operators, Vol. II*, Research Notes in Mathematics 102, Pitman, Boston, 1984.

[2] S. R. Caradus, W. E. Pfaffenberger, B. Yood, *Calkin algebras and algebras of operators on Banach spaces*, Marcel Dekker, New York, 1974.

[3] L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288.

[4] S. Djordjević, *On the continuity of the Browder essential approximate point spectrum*, Comment. Math. (to appear) **36** (1996).

[5] V. A. Erovenko, *K voprosu Oberai o suestvennom spektre*, Dokladi Akademii navuk BSSR **28, 12** (1984), 1068–1071.

[6] V. A. Erovenko, *Teorema Vel o suestvennom spektre dl k-paranormalnih operatorov*, Vesci Akademii navuk BSSR, Ser. Fiz – Mat. **5** (1986), 30–35.

[7] V. A. Erovenko, *O suestvennom spektre funkci ot linenih operatorov v Banahovom prostranstve*, Vestnik Belorusskogo Gosudarstvennogo Universiteta **2, 1** (1987), 50–54.

[8] K. Gustafson, *Weyl's theorems*, in **Linear Operators and Approximation**, Edited by P. L. Butzer, J. -P. Kahane and B. Sz. -Nagy, Proceedings of the Conference in Oberwolfach, Birkhäuser Verlag, Basel und Stuttgart, 1971.

[9] T. Kato, *Perturbation theory of linear operators*, Springer-Verlag, Berlin-Heideberg-New York, 1976.

[10] S. Prasanna, *Weyl's theorem and thin spectra*, Proc. Indian Acad. Sci. Math. Sci. **91, 1** (1982), 59–63.

[11] V. Rakočević, *On the essential approximate point spectrum II*, Mat. Vesnik **36** (1984), 89–97.

[12] V. Rakočević, *Approximate point spectrum and commuting compact perturbations*, Glasgow Math. J. **28** (1986), 193–198.

[13] V. Rakočević, *Operators obeying a-Weyl's theorem*, Rev. Roumaine Math. Pures Appl. **34, 10** (1989), 915–919.

University of Niš
Faculty of Philosophy
Department of Mathematics

Ćirila i Metodija 2
18000 Niš, Yugoslavia

Acta Sci. Math. (Szeged) **64** (1998), 259–269.