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RIGHT AND LEFT FREDHOLM OPERATOR MATRICES

Dragan S. Djordjević and Milica Z. Kolundžija

Abstract. We consider right and left Fredholm operator matrices of the form $\left[\begin{smallmatrix} A & C \\ T & S \end{smallmatrix}\right]$, which are linear and bounded on the Banach space $Z = X \oplus Y$.

1. Introduction

Let Z be an infinite dimensional Banach space, such that $Z = X \oplus Y$ for some closed subspaces X and Y. This sum will be also denoted by $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$. If W is a finite dimensional subspace of X, then $\dim W$ denotes its dimension. If W is infinite dimensional, then we simply write $\dim W = \infty$. However, if U is a closed subspace of a Hilbert space, then $\dim_H(U)$ denotes the orthogonal dimension of U.

Let $\mathcal{L}(X,Y)$ denote the set of all linear bounded operators from X to Y. We abbreviate $\mathcal{L}(X) = \mathcal{L}(X,X)$. The set of all finite rank operators from X to Y is denoted by $\mathcal{F}(X,Y)$. For $A \in \mathcal{L}(X,Y)$ we use $\mathcal{R}(A)$ and $\mathcal{N}(A)$ to denote the range and the null-space of A, respectively.

If $Z = X \oplus Y$, then any $M \in \mathcal{L}(Z)$ can be decomposed as the following operator matrix

$$M = \begin{bmatrix} A & C \\ T & S \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix}$$

for some $A \in \mathcal{L}(X)$, $C \in \mathcal{L}(Y,X)$, $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y)$. On the other hand, any choice of A, C, T, S (linear and bounded operators on the corresponding subspaces), produces a linear and bounded operator M on the space Z. Moreover, M is finite rank if and only if all A, C, T, S are finite rank operators.

If A and C are fixed, then we use the notation $M_{(T,S)}$ to show that M depends on T and S. For given A and C, we are interested to find T and S, such that $M_{(T,S)}$ is right or left Fredholm operator.

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For this purpose we need to review some properties of right and left Fredholm operators [9]. An operator $A \in \mathcal{L}(X,Y)$ is right Fredholm, if $\operatorname{def}(A) = \dim Y/\mathcal{R}(A) < \infty$, and $\mathcal{N}(A)$ is complemented in X. Notice that if A is right Fredholm, then it follows that $\mathcal{R}(A)$ has to be a closed and complemented subspace of Y. The set of all right Fredholm operators from X to Y is denoted by $\Phi_r(X,Y)$. It is well-known that $A \in \Phi_r(X,Y)$ if and only if there exist $B \in \mathcal{L}(Y,X)$ and $F \in \mathcal{F}(Y)$ such that $AB = I_Y + F$ holds.

An operator $A \in \mathcal{L}(X,Y)$ is left Fredholm, if $\operatorname{nul}(A) = \dim \mathcal{N}(A) < \infty$, and $\mathcal{R}(A)$ is closed and complemented in Y. The set of all left Fredholm operators from X to Y is denoted by $\Phi_l(X,Y)$. It is well-known that $A \in \Phi_l(X,Y)$ if and only if there exist $B \in \mathcal{L}(Y,X)$ and $F \in \mathcal{F}(X)$ such that $BA = I_X + F$ holds.

If $A \in \Phi_r(X,Y)$ and $B \in \Phi_r(Y,Z)$, then $BA \in \Phi_r(X,Z)$. The similar result holds for the class Φ_l. The set of Fredholm operators is defined as $\Phi(X,Y) = \Phi_r(X,Y) \cap \Phi_l(X,Y)$.

We formulate the following well-known results.

Lemma 1.1. *Let X, Y, Z be Banach spaces and let $A \in \mathcal{L}(X,Y)$, $B \in \mathcal{L}(Y,Z)$. If $BA \in \Phi(X,Z)$, then the following holds: $A \in \Phi(X,Y)$ if and only if $B \in \Phi(Y,Z)$.*

Lemma 1.2. *Let X, Y be Banach spaces, and let $A \in \Phi_r(X,Y)$, $P \in \mathcal{F}(X,Y)$. Then $A + P \in \Phi_r(X,Y)$. The analogous result holds for classes Φ_l and Φ.*

Lemma 1.3. *Let M_1, M_2 and N be the vector subspaces of the vector space X. If $M_1 \subseteq M_2$, then $\dim M_1/(M_1 \cap N) \leq \dim M_2/(M_2 \cap N)$.*

Properties of right (left) Fredholm and related operators can be found in [6] and [9]. For the importance and applications of operator matrices we refer to [1], [2], [3], [4], [5], [7], [8] and [10]. Particularly, this paper is related to the research in [4] and [7], where the left and right invertibility of $M_{(T,S)}$ is considered.

2. Right Fredholm operators

We consider right Fredholm properties of $M_{(T,S)}$.

Theorem 2.1. *Let $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(Y,X)$ be given. The following statements are equivalent:*

(a) *$[A \quad C] \in \Phi_r(X \oplus Y, X) \setminus \Phi(X \oplus Y, X)$, and there exists an operator $J \in \Phi_l(Y, \mathcal{N}([A \quad C]) \setminus \Phi(Y, \mathcal{N}([A \quad C]))$.*

(b) *$M_{(T,S)} \in \Phi_r(X \oplus Y) \setminus \Phi(X \oplus Y)$ for some $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y)$.*

Proof. (a) $\Longrightarrow$ (b): Suppose that $[A \quad C] \in \Phi_r(X \oplus Y, X) \setminus \Phi(X \oplus Y, X)$. It follows that $\mathcal{N}([A \quad C])$ is infinite dimensional. By the assumption, there exists an operator $J \in \Phi_l(Y, \mathcal{N}([A \quad C]) \setminus \Phi(Y, \mathcal{N}([A \quad C]))$, so $\mathcal{N}(J)$ is finite dimensional and $\mathcal{N}([A \quad C])/R(J)$ is infinite dimensional. The operator J has the form

$$J = \begin{bmatrix} E \\ G \end{bmatrix} : Y \to \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Since $\mathcal{R}(J)$ is closed and complemented in $\mathcal{N}([A \quad C])$, and $\mathcal{N}([A \quad C])$ is closed and complemented in $X \oplus Y$, we obtain that there exist closed subspaces V and W such that $\mathcal{N}[A \quad C]) = R(J) \oplus V$ and $X \oplus Y = \mathcal{N}([A \quad C]) \oplus W = R(J) \oplus V \oplus W$. Notice that V is infinite dimensional.

There exists a closed subspace Y_1 such that $Y = \mathcal{N}(J) \oplus Y_1$. Now, the reduction operator $J : Y_1 \to \mathcal{R}(J)$ is invertible, so let $K_1 : \mathcal{R}(J) \to Y_1$ denote its inverse. Define the operator $K \in \mathcal{L}(X \oplus Y, Y)$ in the following way:

$$Kx = \begin{cases} K_1 x, & x \in \mathcal{R}(J), \\ 0, & x \in V \oplus W. \end{cases}$$

Then $K \in \mathcal{L}(X \oplus Y, Y)$ is a right Fredholm operator, such that $\mathcal{N}(K) = V \oplus W$. The operator K has the matrix form

$$K = [T \quad S] : \begin{bmatrix} X \\ Y \end{bmatrix} \to Y.$$

We also have

$$KJ = [T \quad S] \begin{bmatrix} E \\ G \end{bmatrix} = I_Y - P_1, \tag{1}$$

where P_1 is the projection from Y onto the finite dimensional subspace $\mathcal{N}(J)$, parallel to Y_1.

From $\mathcal{R}(J) \subset \mathcal{N}([A \quad C])$ we get that

$$[A \quad C] \begin{bmatrix} E \\ G \end{bmatrix} = 0. \tag{2}$$

Since $[A \quad C] \in \Phi_r(X \oplus Y, X)$, we have the following decompositions of spaces: $X \oplus Y = \mathcal{N}([A \quad C]) \oplus W$ and $X = \mathcal{R}([A \quad C]) \oplus U$, where U is finite dimensional. Since the reduction $[A \quad C] : W \to \mathcal{R}([A \quad C])$ is invertible, define $L_1 : \mathcal{R}([A \quad C]) \to W$ to be its inverse. Then consider the operator $L \in \mathcal{L}(X, X \oplus Y)$, which is defined as follows:

$$Lx = \begin{cases} L_1 x, & x \in \mathcal{R}([A \quad C]) \\ 0, & x \in U. \end{cases}$$

The operator L has the matrix form

$$L = \begin{bmatrix} D \\ F \end{bmatrix} : X \to \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Then $L \in \Phi_l(X, X \oplus Y)$, $\mathcal{R}(L) = W$, and

$$[A \quad C]L = [A \quad C] \begin{bmatrix} D \\ F \end{bmatrix} = I_X - P_2, \tag{3}$$

where P_2 is the projection from X onto the finite dimensional subspace U, parallel to $\mathcal{R}([A \quad C])$. Since $\mathcal{N}([T \quad S]) = V \oplus W$, we conclude that

$$[T \quad S] \begin{bmatrix} D \\ F \end{bmatrix} = 0. \tag{4}$$

Finally, from (1), (2), (3) and (4), we get that for $M = \left[\begin{smallmatrix} A & C \\ T & S \end{smallmatrix}\right]$, $N = \left[\begin{smallmatrix} D & E \\ F & G \end{smallmatrix}\right]$ the following holds:

$$MN = \begin{bmatrix} A & C \\ T & S \end{bmatrix} \begin{bmatrix} D & E \\ F & G \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix} + \begin{bmatrix} -P_2 & 0 \\ 0 & -P_1 \end{bmatrix}. \tag{5}$$

Since $\left[\begin{smallmatrix} -P_2 & 0 \\ 0 & -P_1 \end{smallmatrix}\right]$ is finite rank, we conclude that M is right Fredholm. Moreover, we notice that

$$\mathcal{N}(M) = \mathcal{N}([A \quad C]) \cap \mathcal{N}([T \quad S]) = V,$$
$$\mathcal{R}(N) = \mathcal{R}\left(\begin{bmatrix} D \\ F \end{bmatrix}\right) + \mathcal{R}\left(\begin{bmatrix} E \\ G \end{bmatrix}\right) = W \oplus \mathcal{R}(J),$$
$$X \oplus Y = \mathcal{R}(J) \oplus V \oplus W.$$

Since V is infinite dimensional, we obtain that both M and N are not Fredholm operators.

(b) $\Longrightarrow$ (a): Suppose that there exist some $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y)$ such that $M_{(T,S)} \in \Phi_r(X \oplus Y) \setminus \Phi(X, Y)$. Then there exist operators $N \in \mathcal{L}(X \oplus Y)$ and $P \in \mathcal{F}(X \oplus Y)$ such that $MN = I + P$. The last equality holds in the matrix form as follows:

$$\begin{bmatrix} A & C \\ T & S \end{bmatrix} \begin{bmatrix} D & E \\ F & G \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

where all P_{ij} are finite rank operators. It also follows that $N = \left[\begin{smallmatrix} D & E \\ F & G \end{smallmatrix}\right] \in \Phi_l(X \oplus Y)$.

In particular, we obtain

$$[A \quad C] \begin{bmatrix} D \\ F \end{bmatrix} = I_X + P_{11},$$

so $[A \quad C]$ is right Fredholm. The operator $I_X + P_{11}$ is Fredholm. If we suppose that $[A \quad C]$ is Fredholm, by Lemma 1.1 it follows that $\left[\begin{smallmatrix} D \\ F \end{smallmatrix}\right]$ is also Fredholm. Since

$$\mathcal{R}\left(\begin{bmatrix} D & E \\ F & G \end{bmatrix}\right) = \mathcal{R}\left(\begin{bmatrix} D \\ F \end{bmatrix}\right) + \mathcal{R}\left(\begin{bmatrix} E \\ G \end{bmatrix}\right) \supset \mathcal{R}\left(\begin{bmatrix} D \\ F \end{bmatrix}\right),$$

it follows that $\left[\begin{smallmatrix} D & E \\ F & G \end{smallmatrix}\right]$ belongs to $\Phi_r(X \oplus Y)$, so $\left[\begin{smallmatrix} D & E \\ F & G \end{smallmatrix}\right]$ is Fredholm. By Lemma 1.1 again, we obtain that $\left[\begin{smallmatrix} A & C \\ T & S \end{smallmatrix}\right]$ is Fredholm (since $I + P$ is Fredholm from Lemma 1.2). The last statement is not possible, so we obtain that $[A \quad C] \in \Phi_r(X \oplus Y, X) \setminus \Phi(X \oplus Y, X)$.

Denote with $L = \left[\begin{smallmatrix} E \\ G \end{smallmatrix}\right] \in \mathcal{L}(Y, X \oplus Y)$. We have $[T \quad S]L = I_Y + P_{22}$, so $L \in \Phi_l(Y, X \oplus Y) \setminus \Phi(Y, X \oplus Y)$. Otherwise, if L is Fredholm, then also $\left[\begin{smallmatrix} D & E \\ F & G \end{smallmatrix}\right]$ is Fredholm, so $\left[\begin{smallmatrix} A & C \\ T & S \end{smallmatrix}\right]$ is Fredholm.

Since we have the following decomposition of space $X \oplus Y = \mathcal{N}([A \quad C]) \oplus W$, the operator L has the matrix form

$$L = \begin{bmatrix} J \\ K \end{bmatrix} : Y \to \begin{bmatrix} \mathcal{N}([A \quad C]) \\ W \end{bmatrix}.$$

From the fact that

$$\mathcal{R}(P_{12}) = \mathcal{R}([A \quad C]L) = \mathcal{R}\left([A \quad C] \begin{bmatrix} J \\ K \end{bmatrix}\right) = [A \quad C](\mathcal{R}(K))$$

is a finite space and the reduction $[A \quad C] : W \to \mathcal{R}([A \quad C])$ is a bijection, we obtain that $\mathcal{R}(K)$ is a finite dimensional subspace of W.

Since $L \in \Phi_l(Y, X \oplus Y) \setminus \Phi(Y, X \oplus Y)$, we have the following decompositions of spaces $Y = \mathcal{N}(L) \oplus U$ and $X \oplus Y = \mathcal{R}(L) \oplus U_1$, where $\dim \mathcal{N}(L) < \infty$ and $\dim U_1 = \infty$. The reduction operator $L : U \to \mathcal{R}(L)$ is invertible, so let $L_1 : \mathcal{R}(L) \to U$ be its inverse.

As it was shown, $\mathcal{R}(K)$ is a finite dimensional subspace, so $Y_1 = L_1(\mathcal{R}(K))$ have to be a finite dimensional subspace of U and there exists a closed subspace Y_2 such that $U = Y_1 \oplus Y_2$.

Now, the operator L has the following matrix form

$$L = \begin{bmatrix} J & 0 & 0 \\ 0 & K & 0 \end{bmatrix} : \begin{bmatrix} Y_2 \\ Y_1 \\ \mathcal{N}(L) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}([A \quad C]) \\ W \end{bmatrix},$$

where Y_1 is finite dimensional. We obtain that $\mathcal{N}(J) = Y_1 \oplus \mathcal{N}(L)$, so $\dim \mathcal{N}(J) < \infty$.

From the fact that $[T \quad S]L = I_Y + P_{22}$ follows that

$$L_1(\mathcal{N}([T \quad S]) \cap \mathcal{R}(L)) \subseteq \mathcal{N}(I_Y + P_{22}).$$

Since $I_Y + P_{22}$ is a Fredholm operator, we have that $L_1(\mathcal{N}([T \quad S]) \cap \mathcal{R}(L))$ is finite dimensional, so $\mathcal{N}([T \quad S]) \cap \mathcal{R}(L)$ is also a finite dimensional subspace.

Denote with $V = \mathcal{N}([A \quad C]) \cap \mathcal{N}([T \quad S]) \cap \mathcal{R}(J)$. Further,

$$V \subseteq \mathcal{N}([T \quad S]) \cap \mathcal{R}(J) \subseteq \mathcal{N}([T \quad S]) \cap \mathcal{R}(L),$$

so it follows that $\dim V < \infty$. Then, there exists a closed subspace V_1 such that $\mathcal{N}(M_{(T,S)}) = \mathcal{N}([A \quad C]) \cap \mathcal{N}([T \quad S]) = V \oplus V_1$. Since $\mathcal{N}(M_{(T,S)})$ is infinite dimensional, then V_1 is also an infinite dimensional subspace.

Now, applying Lemma 1.3 on the spaces $\mathcal{N}([A \quad C]) \cap \mathcal{N}([T \quad S])$, $\mathcal{N}([A \quad C])$ and $\mathcal{R}(J)$, we obtain

$$\dim V_1 = \dim(\mathcal{N}([A \quad C]) \cap \mathcal{N}([T \quad S]))/V \leq \dim \mathcal{N}([A \quad C])/\mathcal{R}(J).$$

We conclude that $\dim \mathcal{N}([A \quad C])/\mathcal{R}(J) = \infty$.

Lastly, we proved for the operator $J : Y \to \mathcal{N}([A \quad C])$ that $\dim \mathcal{N}(J) < \infty$ and $\dim \mathcal{N}([A \quad C])/\mathcal{R}(J) = \infty$.

So, there exists the operator $J \in \Phi_l(Y, \mathcal{N}([A \quad C]) \setminus \Phi(Y, \mathcal{N}([A \quad C]))$. $\square$

3. Left Fredholm operators

Now we investigate the left Fredholm properties of $M_{(T,S)}$. We consider two separate cases according to the dimension of Y.

Theorem 3.1. *Let X be infinite dimensional, and let Y be finite dimensional. For given $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(Y,X)$, the following statements are equivalent*:

(a) $M_{(T,S)} \in \Phi_l(X \oplus Y) \setminus \Phi(X \oplus Y)$ *for every* $T \in \mathcal{L}(X,Y)$ *and every operator* $S \in \mathcal{L}(Y)$;

(b) $A \in \Phi_l(X) \setminus \Phi(X)$.

Proof. Before the proof of the equivalence, note that

$$\mathcal{N}\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = \mathcal{N}(A) \oplus Y, \quad \mathcal{R}\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = \mathcal{R}(A) \oplus \{0\}.$$

Since Y is finite dimensional, we have that $A \in \Phi_l(X) \setminus \Phi(X)$ if and only if $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \Phi_l(X \oplus Y) \setminus \Phi(X \oplus Y)$.

(a) $\Longrightarrow$ (b): Suppose that $M_{(T,S)}$ is left Fredholm but not Fredholm, for every $T \in \mathcal{L}(X,Y)$ and every $S \in \mathcal{L}(Y)$. We have that $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} A & C \\ T & S \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & -C \\ -T & -S \end{smallmatrix}\right]$ where $\left[\begin{smallmatrix} 0 & -C \\ -T & -S \end{smallmatrix}\right]$ is a finite rank operator. Applying Lemma 1.2, we obtain that $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is a left Fredholm operator.

Suppose that $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is Fredholm. Applying Lemma 1.2 to $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ we conclude that $M_{(T,S)}$ has to be Fredholm, which does not hold. Hence, $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is left Fredholm but not Fredholm, so we have that $A \in \Phi_l(X) \setminus \Phi(X)$.

(b) $\Longrightarrow$ (a): Suppose that A is left Fredholm but not Fredholm, so the operator $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is also left Fredholm but not Fredholm.

Let $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y)$ be arbitrary operators. Then the operator $M_{(T,S)}$ is a finite-rank perturbation of $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Indeed, $\left[\begin{smallmatrix} A & C \\ T & S \end{smallmatrix}\right] = \left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & C \\ T & S \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} 0 & C \\ T & S \end{smallmatrix}\right]$ is a finite rank operator because Y is a finite dimensional space. Applying Lemma 1.2 to $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ we get that $M_{(T,S)}$ is a left Fredholm operator. If we suppose that $M_{(T,S)}$ is Fredholm, from Lemma 1.2, we conclude that $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ have to be Fredholm, which does not hold. We obtain that $M_{(T,S)}$ is left Fredholm but not Fredholm operator. □

Theorem 3.2. *Let X and Y be infinite dimensional, such that Y is isomorphic to $Z = X \oplus Y$. Let $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(Y,X)$ be arbitrary. Then $M_{(T,S)} \in \Phi_l(X \oplus Y) \setminus \Phi(X \oplus Y)$ for some $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y)$.*

Proof. Since Y is isomorphic with Z, then $Y = Y_1 \oplus Y_2$, where X is isomorphic to Y_1, and Y is isomorphic to Y_2. Let $T \in \mathcal{L}(X, Y_1)$ and $S \in \mathcal{L}(Y, Y_2)$ be those isomorphisms. Then $T \in \mathcal{L}(X,Y)$ is left invertible with a left inverse $K \in \mathcal{L}(Y,X)$ and $\mathcal{N}(K) = Y_2$. Also, $S \in \mathcal{L}(Y, Y_2)$ is left invertible with a left inverse L and $\mathcal{N}(L) = Y_1$. Then

$$\begin{bmatrix} 0 & K \\ 0 & L \end{bmatrix} \begin{bmatrix} A & C \\ T & S \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix},$$

so $M_{(T,S)}$ is left invertible. It follows that $M_{(T,S)}$ is left Fredholm for chosen operators T and S. Suppose that $M_{(T,S)}$ is Fredholm. Since $\left[\begin{smallmatrix} I_X & 0 \\ 0 & I_Y \end{smallmatrix}\right]$ is Fredholm, from Lemma 1.1 it follows that N is also Fredholm. However, we notice $\mathcal{N}(N) = X$, which is infinite dimensional. Hence, N is not Fredholm. Then $M_{(T,S)}$ is not Fredholm also, i.e., $M_{(T,S)} \in \Phi_l(X \oplus Y) \setminus \Phi(X \oplus Y)$. □

We formulate a corollary for Hilbert space operators.

Corollary 3.1. *Let X and Y be infinite dimensional and mutually orthogonal subspaces of a Hilbert space $Z = X \oplus Y$. Suppose that $\dim_H Y = \dim_H Z$. Let $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(Y, X)$ be arbitrary. Then $M_{(T,S)} \in \Phi_l(X \oplus Y) \setminus \Phi(X \oplus Y)$ for some $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y)$.*

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Dragan S. Djordjević
Faculty of Sciences and Mathematics
University of Niš
Višegradska 33, P.O. Box 224, 18000 Niš, Serbia
E-mail address: dragan@pmf.ni.ac.rs

Milica Z. Kolundžija
Faculty of Sciences and Mathematics
University of Niš
Višegradska 33, P.O. Box 224, 18000 Niš, Serbia
E-mail address: mkolundzija@pmf.ni.ac.rs