WEIGHTED MOORE-PENROSE INVERTIBLE AND WEIGHTED EP BANACH ALGEBRA ELEMENTS

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Abstract

The weighted Moore-Penrose inverse will be introduced and studied in the context of Banach algebras. In addition, weighted EP Banach algebra elements will be characterized. The Banach space operator case will be also considered.

KEYWORDS: *(weighted) Moore-Penrose inverse, (weighted) EP element, group inverse, Banach algebra, Banach space operator.*

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1. Introduction

The Moore-Penrose inverse in Banach algebras was introduced by V. Rakočević in [22]. Some of its basic properties were studied in this article, and latter in [23], [16] and [1]. In the recent past a particular class of Moore-Penrose invertible objects were studied, namely EP Banach space operators and EP Banach algebra elements, in other words, Moore-Penrose invertible operators or elements of an algebra such that they commute with their Moore-Penrose inverse, see [1, 2, 3, 17]. It is worth noticing that EP objects have been intensively studied in different contexts such as matrices, Hilbert space bounded and linear maps, C^*-algebras, rings with involution and naturally in the above mentioned two ones; moreover they consist in a generalization of the notion normal and hermitian objects, see [3, Theorems 3.1 and 3.3].

In the recent work by Y. Tian and H. Wang [24] the concept of weighted EP matrices (matrices that commute with their weighted Moore-Penrose inverse) was introduced. What is more, the second and third authors studied weighted EP C^*-algebra elements in [18, 19]. The main objective of this article is to introduce and study similar objects in the contexts of Banach space operators and Banach algebra elements.

In fact, in section 2, after having recalled some basic definitions and results, the notion of weighted Moore-Penrose invertible Banach space operators and Banach algebra elements is introduced and studied. Furthermore, in section 3 weighted EP Banach space operators and Banach algebra elements are characterized using the concept of group inverse. It is worth noticing that the results presented in this article apply to arbitrary Banach algebras, operators on Banach spaces and matrices with any norm.

2. The weighted Moore-Penrose inverse

From now on X will denote a Banach space and $L(X)$ the Banach algebra of all bounded and linear maps defined on and with values in X. If $T \in L(X)$, then $N(T)$ and $R(T)$ will stand for the null space and the range of T respectively. In addition, X^* will denote the dual space of X and if $T \in L(X)$, then $T^* \in L(X^*)$ will stand for the adjoint map of $T \in L(X)$.

On the other hand, A will denote a complex unital Banach algebra with unit 1. In addition, the set of all invertible elements of A will be denoted by A^{-1}. If $a \in A$, then $L_a \colon A \to A$ and $R_a \colon A \to A$ will denote the map defined by left and right multiplication, respectively:

$$L_a(x) = ax, \qquad\qquad R_a(x) = xa,$$

where $x \in A$. Moreover, the following notation will be used: $N(L_a) = a^{-1}(0)$ and $R(L_a) = aA$.

Recall that an element $a \in A$ is called *regular,* if it has a *generalized inverse,* namely if there exists $b \in A$ such that

$$a = aba.$$

Furthermore, a generalized inverse b of a regular element $a \in A$ will be called *normalized,* if b is regular and a is a generalized inverse of b, equivalently,

$$a = aba, \qquad\qquad b = bab.$$

Next follows the key notion in the definition of the (weighted) Moore-Penrose inverse in the context of Banach algebras.

Definition 2.1. *Given a unital Banach algebra A, an element $a \in A$ is said to be hermitian, if $\| exp(ita) \| = 1$, for all $t \in \mathbb{R}$.*

As regard equivalent definitions and the main properties of hermitian Banach algebra elements and Banach space operators, see for example [25, 15, 20, 4, 7]. Recall that if A is a C^*-algebra, then $a \in A$ is hermitian if and only if a is self-adjoint, see [4, Proposition 20, Chapter I, Section 12]. Given A a unital Banach algebra, the set of all Hermitian elements of A will be denoted by $H(A)$.

Now the notion of Moore-Penrose invertible Banach algebra element will be recalled.

Definition 2.2. *Let A be a unital Banach algebra and $a \in A$. If there exists $x \in A$ such that x is a normalized generalized inverse of a and ax and xa are hermitian, then x will be said to be the Moore-Penrose inverse of a, and it will be denoted by $a^\dagger$.*

Recall that according to [22, Lemma 2.1], there is at most one Moore-Penrose inverse. Concerning the Moore-Penrose inverse in Banach algebras, see [22, 23, 16, 1, 3, 2, 17]. In the context of C^*-algebras, see [9, 10, 13]. For the original definition of the Moore-Penrose inverse for matrices, see [21].

Next the weighted Moore-Penrose inverse will be considered. Recall that given a C^*-algebra A and $a \in A$, $b \in A$ is said to be the *weighted Moore-Penrose inverse* of a with weights e and f, if the following identities hold:

$$aba = a, \quad bab = b, \quad (eab)^* = eab, \quad (fba)^* = fba,$$

where e and f are positive and invertible elements in A, see [14, 18, 19].

According to [14], the conditions defining the weighted Moore-Penrose inverse can be rewritten as

$$aba = a, \quad bab = b, \quad (ab)^{*e} = ab, \quad (ba)^{*f} = ba,$$

where $A^{*e} = (A^{*e}, \|\cdot\|_e)$ (respectively $A^{*f} = (A^{*f}, \|\cdot\|_f)$) is the C^*-algebra with underlying space A, involution $x \to x^{*e} = e^{-1}x^*e$ (respectively $x \to x^{*f} = f^{-1}x^*f$) and norm $\| x \|_e = \| e^{1/2}xe^{-1/2} \|$ (respectively $\| x \|_f = \| f^{1/2}xf^{-1/2} \|$), $x \in A$. Next weighted Moore-Penrose invertible Banach algebra elements will be introduced. To this end, however, some preparation in needed.

Let A be a complex unital Banach algebra and consider $a \in A$. The element a will be said to be *positive*, if $V(a) \subset \mathbb{R}_+$, where $V(a) = \{f(a) : f \in A^*, \| f \| \leq 1, f(1) = 1\}$ ([4, Definition 5, Chapter V, Section 38]). Denote by A_+ the set of all positive elements of A. Note that necessary and sufficient for $a \in A$ to be positive is that a is hermitian and $\sigma(a) \subset \mathbb{R}_+$ ([4, Definition 5, Chapter V, Section 38]). Recall that according to [4, Lemma 7, Chapter V, Section 38], if $c \in A_+$, then there exists $d \in A_+$ such that $d^2 = c$. Moreover, according to [8, Theorem], the square root is unique. In particular, the square root of c will be denoted by $c^{1/2}$. For the definition and equivalent conditions of positive C^*-algebra elements, see [5, Definition 3.1 and Theorem 3.6, Chapter VIII, Section 3].

Given a complex unital Banach algebra A and $u \in A^{-1} \cap A_+$, denote by $A^u = (A^u, \|\cdot\|_u)$ the complex unital Banach algebra with underlying space A and norm $\| x \|_u = \| u^{1/2}xu^{-1/2} \|$. When A is a C^*-algebra, according to [4, Proposition 20, Chapter I, Section 12], a is self-adjoint in $(A^{*u}, \|\cdot\|_u)$ if and only if a is hermitian in $(A^u, \|\cdot\|_u)$, where if $x \in A$, then as before the involution in A^{*u} is defined as follows: $x \to x^{*u} = u^{-1}x^*u$. These facts lead to the following definition.

Definition 2.3. *Let A be a complex unital Banach algebra and consider e and f two positive and invertible elements in A. The element $a \in A$ will be said to be weighted Moore-Penrose invertible with weights e and f, if there exists $b \in A$ such that b is a normalized generalized inverse of a and ab (respectively ba) is a hermitian element of A^e (respectively of A^f).*

Clearly, the conditions in Definition 2.3 extend the notion of weighted Moore-Penrose invertible C^*-algebra element to Banach algebras ([14]). Furthermore, note that if $e = f$, then necessary and sufficient for $a \in A$ to be weighted Moore-Penrose invertible with weight e is that $a \in A^e$ (respectively A^{*e}, when A is a C^*-algebra) is Moore-Penrose invertible. In particular, when both weights coincide, the weighted Moore-Penrose inverse reduces to the Moore-Penrose inverse, naturally changing the the structure of the Banach or C^*-algebra. In what follows, some basic properties of weighted Moore-Penrose invertible Banach algebra elements will be studied.

In first place, the unicity of the weighted Moore-Penrose inverse will be proved. Moreover, ideas similar to the ones in [22, Lemma 2.1] will be used. However, before considering the mentioned property, some preparation is needed.

Lemma 2.4. *Let A be a unital Banach algebra and consider $u \in A^{-1} \cap A_+$. Then, $L_u \in L(A)$ is invertible and positive.*

Proof. It is clear that $L_u \in L(A)$ is invertible. In addition, note that since A is unital, $\| L_u \| = \| u \|$. In particular, since

$$\| exp(itL_u) \| = \| L_{exp(itu)} \| = \| exp(itu) \| = 1,$$

$L_u \in H(L(A))$. Moreover, since according to [4, Proposition 4, Chapter I, Section 5] $\sigma(L_u) = \sigma(u)$, $L_u \in L(A)_+$ ([4, Definition 5, Chapter V, Section 38]). □

Proposition 2.5. *Let A be a complex unital Banach algebra and consider e, $f \in A^{-1} \cap A_+$. Then, if $a \in A$, there is at most one weighted Moore-Penrose inverse of a with weights e and f.*

Proof. Suppose that b and c are two weighted Moore-Penrose inverses of a with weights e and f and consider L_{ab}, $L_{ac} \in L(A)^{L_e}$. A straightforward calculation proves that

$$\| exp(itL_{ab}) \|_{L_e} = \| exp(itab) \|_e = 1, \quad \| exp(itL_{ac}) \|_{L_e} = \| exp(itac) \|_e = 1,$$

equivalently, L_{ab} and L_{ac} are two hermitian idempotents of $L(A)^{L_e}$. Moreover, since b and c are two normalized generalized inverse of a, it is not difficult to prove that $R(L_{ab}) = R(L_a) = R(L_{ac})$. Therefore, according to [20, Theorem 2.2], $L_{ab} = L_{ac}$. In particular, $ab = ac$.

A similar argument, using in particular R_{ba} and R_{ca} instead of L_{ab} and L_{ac} respectively, proves that $ba = ca$.

Then,

$$b = bab = cab = cac = c.$$

□

According to Proposition 2.5, given a complex unital Banach algebra and $a \in A$, if the weighted Moore-Penrose inverse of a exists, then it will be denoted by $a^{\dagger}_{e,f}$. In the following remarks some elementary facts that will be used in this article will be presented.

Remark 2.6. (a) Let A be a complex unital Banach algebra and consider e, $f \in A^{-1} \cap A_+$. Suppose that $a^{\dagger}_{e,f}$ exists, $a \in A$. Then, the following statements can be easily derived from the conditions characterizing the weighted Moore-Penrose inverse of a.
(i) $(a^{\dagger}_{e,f})^{\dagger}_{f,e} = a$.
(ii) $a^{\dagger}_{e,f}A = a^{\dagger}_{e,f}aA$, $aa^{\dagger}_{e,f}A = aA$.
(iii) $(aa^{\dagger}_{e,f})^{-1}(0) = (a^{\dagger}_{e,f})^{-1}(0)$, $(a^{\dagger}_{e,f}a)^{-1}(0) = a^{-1}(0)$.
(iv) $A = a^{\dagger}_{e,f}A \oplus a^{-1}(0) = aA \oplus (a^{\dagger}_{e,f})^{-1}(0)$.
(b) Suppose that $A = L(X)$, X a Banach space. Let E, $F \in L(X)$ be two invertible and positive operators and consider $T \in L(X)$ such that $T^{\dagger}_{E,F}$ exists. Then, it is not difficult to prove the following facts.
(v) $R(T^{\dagger}_{E,F}T) = R(T^{\dagger}_{E,F})$, $R(TT^{\dagger}_{E,F}) = R(T)$.
(vi) $N(T^{\dagger}_{E,F}T) = N(T)$, $N(TT^{\dagger}_{E,F}) = N(T^{\dagger}_{E,F})$.
(vii) $X = R(T^{\dagger}_{E,F}) \oplus N(T) = R(T) \oplus N(T^{\dagger}_{E,F})$.

Next conditions equivalent to the existence of the weighted Moore-Penrose inverse will be given. Firstly, the case $A = L(X)$, X a Banach space, will be considered.

Theorem 2.7. *Let X be a Banach space and consider E, $F \in L(X)$ two invertible and positive operators. Then, if $T \in L(X)$, the following statements are equivalent:*
(i) $T^{\dagger}_{E,F}$ exists;
(ii) there exist two idempotents P,$Q \in L(X)$ such that $P \in H(L(X)^E)$, $R(P) = R(T)$, and $Q \in H(L(X)^F)$, $N(Q) = N(T)$.
Furthermore, if such P and Q exist, then they are unique.

Proof. If $T^{\dagger}_{E,F}$ exists, then consider $P = TT^{\dagger}_{E,F}$ and $Q = T^{\dagger}_{E,F}T$.

On the other hand, suppose that statement (ii) holds. Consider the invertible operator $T' \in L(R(Q), R(T))$,

$$T' = T \mid^{R(T)}_{R(Q)} : R(Q) \to R(T),$$

and define $S \in L(X)$ as follows:

$$S \mid_{N(P)} \equiv 0, \qquad S \mid_{R(T)}^{R(Q)} = (T')^{-1} \in L(R(T), R(Q)).$$

Since $R(P) = R(T)$, an easily calculation proves that S is a normalized generalized inverse of T. Moreover, since TS and P are idempotents of $L(X)$ such that

$$R(TS) = R(T) = R(P), \qquad N(TS) = N(S) = N(P),$$

it is clear that $TS = P$. In particular, $TS \in H(L(X)^E)$. A similar argument proves that $ST \in H(L(X)^F)$. Consequently, $S = T^{\dagger}_{E,F}$.

Now suppose that P' and Q' are two idempotents that satisfy statement (ii). Then, $R(P) = R(P')$ and $R(I - Q) = R(I - Q')$. Then, according to [25, Hilfssatz 2(a)-(b)] and [20, Theorem 2.2], $P = P'$ and $Q = Q'$. □

In the following theorem weighted Moore-Penrose invertible Banach algebra elements will be characterized.

Theorem 2.8. *Let A be a complex unital Banach algebra and consider e, $f \in A^{-1} \cap A_+$. If a and $b \in A$ are such that b is a normalized generalized inverse of a, then the following statements are equivalent.*
(i) $b = a^{\dagger}_{e,f}$;
(ii) $L_b = (L_a)^{\dagger}_{L_e,L_f} \in L(A)$.
In particular, if statements (i)-(ii) hold, then $(L_a)^{\dagger}_{L_e,L_f} = L_{a^{\dagger}_{e,f}}$.

Proof. First of all recall that L_e and $L_f \in L(A)$ are invertible and positive (Lemma 2.4).

Clearly, b is a normalized generalized inverse of a in A if and only if L_b is a normalized generalized inverse of L_a in $L(A)$. In addition, according to [8, Theorem], $(L_e)^{1/2} = L_{e^{1/2}}$ and $(L_f)^{1/2} = L_{f^{1/2}}$ (Lemma 2.4). Furthermore, since

$$\parallel exp(itL_aL_b) \parallel_{L_e} = \parallel exp(itL_{ab}) \parallel_{L_e} = \parallel L_{exp(itab)} \parallel_{L_e} = \parallel exp(itab) \parallel_e,$$

and

$$\parallel exp(itL_bL_a) \parallel_{L_f} = \parallel exp(itL_{ba}) \parallel_{L_f} = \parallel L_{exp(itba)} \parallel_{L_f} = \parallel exp(itba) \parallel_f,$$

$ab \in H(A^e)$ and $ba \in H(A^f)$ if and only if $L_aL_b \in H(L(A)^{L_e})$ and $L_bL_a \in H(L(A)^{L_f})$. Therefore, statements (i) and (ii) are equivalent. □

In the following proposition the weighted Moore-Penrose inverse will be described in a particular case. Recall that according to [9, Thorem 6], the condition of being regular is equivalent to the one of being Moore-Penrose invertible, for C^*-algebra elements. However, according to [1, Remark 4], in a general Banach algebra these two notions are not in general equivalent. Compare the next proposition with [14, Theorem 5].

Proposition 2.9. *Let A be a complex unital Banach algebra and consider e, $f \in A^{-1} \cap A_+$. Then, if $a \in A$ is such that $e^{1/2}af^{-1/2} \in A$ is Moore-Penrose invertible,*

$$a^{\dagger}_{e,f} = f^{-1/2}(e^{1/2}af^{-1/2})^{\dagger}e^{1/2}.$$

Proof. Let $c = e^{1/2}af^{-1/2}$ and $b = f^{-1/2}c^{\dagger}e^{1/2}$. Then, using in particular that $a = e^{-1/2}cf^{1/2}$, it is not difficult to prove that $aba = a$, $bab = b$, $ab = e^{-1/2}cc^{\dagger}e^{1/2}$ and $ba = f^{-1/2}c^{\dagger}cf^{1/2}$. However, since $cc^{\dagger}$ and $c^{\dagger}c$ are hermitian elements of A, $ab \in H(A^e)$ and $ba \in H(A^f)$. □

Next weighted Moore-Penrose inverses in quotient algebras will be considered. First of all, given a complex unital Banach algebra A and J a proper and closed two sided ideal of A, the quotient map will be denoted by $\Pi\colon A \to A/J$. In addition, if $a \in A$, then the quotient class of a will be denoted by $\tilde{a} = \Pi(a)$.

Lemma 2.10. *Let A be a complex unital Banach algebra and consider $J \subset A$ a proper and closed two sided ideal. Then, if $u \in A^{-1} \cap A_+$, $\tilde{u}$ is invertible and positive in A/J.*

Proof. Clearly, $\tilde{u} \in A/J$ is invertible. In addition, if $\| \cdot \|'$ denotes the norm in A/J, then

$$\| exp(it\tilde{u}) \|' = \| \Pi(exp(itu)) \|' \leq \| exp(itu) \| = 1.$$

Consequently, according to [1, Remark 2], $\tilde{u} \in H(A/J)$. Furthermore, since according to [4, Lemma 7, Chapter V, Section 38] there exists $v \in A_+$ such that $v^2 = u$, $\tilde{v}^2 = \tilde{u}$ and $\sigma(\tilde{u}) \subset \mathbb{R}_+$. Therefore, according to [4, Definition 5, Chapter V, Section 38], $\tilde{u} \in (A/J)_+$. □

Theorem 2.11. *Let A be a complex unital Banach algebra and consider e, $f \in A^{-1} \cap A_+$. If $a^{\dagger}_{e,f}$ exists, then $\tilde{a} \in A/J$ is weighted Moore-Penrose invertible with weights $\tilde{e}$ and $\tilde{f}$. Furthermore, $\tilde{a}^{\dagger}_{\tilde{e},\tilde{f}} = \Pi(a^{\dagger}_{e,f})$.*

Proof. It is clear that $\Pi(a^{\dagger}_{e,f})$ is a normalized generalized inverse of $\tilde{a}$. On the other hand, note that according to [8, Theorem], $\Pi(e^{1/2}) = \tilde{e}^{1/2}$, $\Pi(e^{-1/2}) = \tilde{e}^{-1/2}$, $\Pi(f^{1/2}) = \tilde{f}^{1/2}$ and $\Pi(f^{-1/2}) = \tilde{f}^{-1/2}$. As a result,

$$\begin{aligned}
\| exp(it\tilde{a}\Pi(a^{\dagger}_{e,f})) \|'_{\tilde{e}} &= \| exp(it\Pi(aa^{\dagger}_{e,f})) \|'_{\tilde{e}} = \| \Pi(e^{1/2} exp(itaa^{\dagger}_{e,f}) e^{-1/2}) \|' \\
&\leq \| e^{1/2} exp(itaa^{\dagger}_{e,f}) e^{-1/2} \| = \| exp(itaa^{\dagger}_{e,f}) \|_e = 1; \\
\| exp(it\Pi(a^{\dagger}_{e,f})\tilde{a}) \|'_{\tilde{f}} &= \| exp(it\Pi(a^{\dagger}_{e,f}a)) \|'_{\tilde{f}} = \| \Pi(f^{1/2} exp(ita^{\dagger}_{e,f}a) f^{-1/2}) \|' \\
&\leq \| f^{1/2} exp(ita^{\dagger}_{e,f}a) f^{-1/2} \| = \| exp(ita^{\dagger}_{e,f}a) \|_f = 1.
\end{aligned}$$

However, according to [1, Remark 2], the proof is concluded. □

The weighted Moore-Penrose inverse in closed invariant subspaces will be now studied. To this end, some preparation is needed. Note that if X is a Banach space and $Y \subseteq X$ is a closed and invariant subspace for $T \in L(X)$, then $T' = T \mid_Y \in L(Y)$ will stand for the restriction map of T to Y.

Proposition 2.12. *Let X be a Banach space and consider $U \in L(X)$ an invertible and positive operator. Let $Y \subseteq X$ be a closed subspace such that $U(Y) = Y$. Then $U' \in L(Y)$ is invertible and positive. What is more, $U^{1/2}(Y) \subseteq Y$, $(U')^{1/2} = (U^{1/2})' \in L(Y)$, $U^{-1/2}(Y) \subseteq Y$ and $(U')^{-1/2} = (U^{-1/2})' \in L(Y)$.*

Proof. Clearly, $U^{-1}(Y) = Y$ and $U' \in L(Y)$ is invertible. On the other hand, according to [4, Lemma 5, Chapter I, Section 10], if $S \in A = L(W)$, W a Banach space, then $V(S) = \cap_{z\in\mathbb{C}} B[z, \| z - S \|]$, where if $z \in \mathbb{C}$ and $r \in \mathbb{R}_+$, then $B[z, r] = \{z' \colon \mid z - z' \mid \leq r\}$. Since

$$V(U') = \cap_{z\in\mathbb{C}} B[z, \| z - U' \|] \subseteq \cap_{z\in\mathbb{C}} B[z, \| z - U \|] = V(U) \subset \mathbb{R}_+,$$

$U' \in L(Y)$ is positive.

To prove that $(U')^{1/2} = (U^{1/2})' \in L(Y)$, consider $K = \sigma(U) \cup \sigma(U')$. Since $U \in L(X)$ and $U' \in L(Y)$ are invertible and positive, $K \subset G = \mathbb{C} \setminus \{x \in \mathbb{R} \colon x \leq 0\}$. Let $f \colon G \to \mathbb{C}$ be the principal branch of the square root function in G, i.e., $f(z) = z^{1/2}$ $(z \in G)$. Now well, if $\lambda \in G \setminus K$, then according to [7, Lemma 1.28], $(U - \lambda)^{-1}(Y) \subseteq Y$. As a result, according to the formula of the Riesz Functional Calculus for $f \colon G \to \mathbb{C}$, using in particular an appropriate system of curves $\Gamma \subseteq G \setminus K$, it is not difficult to prove that $f(U)(Y) \subseteq Y$. However, since $f(U)^2 = U$ and $\sigma(f(U)) = f(\sigma(U)) \subseteq \mathbb{R}_+ \setminus \{0\}$, according to [8, Theorem], $f(U) = U^{1/2}$; in particular, $U^{1/2}(Y) \subseteq Y$. Now well, since $f(U') = f(U) \mid_Y = (U^{1/2})'$, $\sigma((U^{1/2})') = \sigma(f(U')) = f(\sigma(U')) \subseteq R_+$. Therefore, since $((U^{1/2})')^2 = U'$, according again to [8, Theorem], $(U')^{1/2} = (U^{1/2})'$.

Concerning the last two facts, note that according to what has been proved and [7, Lemma 1.28], $U^{-1/2}(Y) \subseteq Y$. However, a direct calculation, using in particular that $(U')^{1/2} = (U^{1/2})'$, proves that $(U')^{-1/2} = (U^{-1/2})'$. □

Theorem 2.13. *Let X be a Banach space and consider E, F, $T \in L(X)$ such that E and F are invertible and positive and $T^{\dagger}_{E,F}$ exists. Suppose in addition that there is $Y \subseteq X$ a closed and invariant subspace for T and $T^{\dagger}_{E,F}$ such that $E(Y) = Y = F(Y)$. Then, the weighted Moore-Penrose inverse of $T' \in L(Y)$ with respect to the weights E', $F' \in L(Y)$ exists. What is more, $(T')^{\dagger}_{E',F'} = (T^{\dagger}_{E,F})'$.*

Proof. It is clear that $(T^{\dagger}_{E,F})'$ is a normalized generalized inverse of T'. On the other hand, according to Proposition 2.12,

$$\begin{aligned}
\parallel exp(itT'(T^{\dagger}_{E,F})') \parallel_{E'} &= \parallel (E')^{1/2} exp(itT'(T^{\dagger}_{E,F})')(E')^{-1/2} \parallel \leq \parallel E^{1/2} exp(itTT^{\dagger}_{E,F}) E^{-1/2} \parallel \\
&= \parallel exp(itTT^{\dagger}_{E,F}) \parallel_E = 1, \\
\parallel exp(it(T^{\dagger}_{E,F})'T') \parallel_{F'} &= \parallel (F')^{1/2} exp(it(T^{\dagger}_{E,F})'T')(F')^{-1/2} \parallel \leq \parallel F^{1/2} exp(itT^{\dagger}_{E,F}T) F^{-1/2} \parallel \\
&= \parallel exp(itT^{\dagger}_{E,F}T) \parallel_F = 1.
\end{aligned}$$

Therefore, according to [1, Remark 2], the proof is concluded. □

The weighted Moore-Penrose inverse in quotient spaces will be studied now. However, first some preliminary facts must be recalled.

Remark 2.14. Let X be a Banach space and consider $U \in L(X)$ an invertible and positive operator. Suppose that $Y \subseteq X$ is a closed invariant subspace for U and denote by $\Pi \colon X \to X/Y$ the quotient map. Then, if $\tilde{U} \in L(X/Y)$ is the quotient operator induced by U, $\tilde{U}$ is invertible and positive. In fact, it is clear that $\tilde{U} \in L(X/Y)$ is invertible. In addition, according to [7, Theorem 4.12(ii)], $\tilde{U}$ is hermitian, and since $\sigma(\tilde{U}) \subseteq \sigma(U) \subset \mathbb{R}_+$, $\tilde{U}$ is positive ([4, Definition 5, Chapter V, Section 38]).

Moreover, according to [8, Theorem], $(\tilde{U})^{1/2} = \widetilde{U^{1/2}}$ and $(\tilde{U})^{-1/2} = \widetilde{U^{-1/2}}$.

Theorem 2.15. *Let X be a Banach space and consider E, $F \in L(X)$ two invertible and positive operators. Let $T \in L(X)$ such that $T^{\dagger}_{E,F}$ exists. Suppose in addition that $Y \subseteq X$ is a closed and invariant subspace for T, $T^{\dagger}_{E,F}$, E and F. Then $\tilde{T} \in L(X/Y)$ is weighted Moore-Penrose invertible with weights $\tilde{E}$ and $\tilde{F}$. Furthermore, $(\tilde{T})^{\dagger}_{\tilde{E},\tilde{F}} = \widetilde{T^{\dagger}_{E,F}}$.*

Proof. It is clear that $\widetilde{T^\dagger_{E,F}}$ is a normalized generalized inverse of $\tilde{T}$. On the other hand,

$$\begin{aligned}
\| exp(it\tilde{T}\widetilde{T^\dagger_{E,F}}) \|_{\tilde{E}} &= \| \tilde{E}^{1/2} exp(it\tilde{T}\widetilde{T^\dagger_{E,F}})\tilde{E}^{-1/2} \| = \| \Pi(E^{1/2}exp(itTT^\dagger_{E,F})E^{-1/2}) \| \\
&\leq \| E^{1/2}exp(itTT^\dagger_{E,F})E^{-1/2} \| = \| exp(itTT^\dagger_{E,F}) \|_E = 1, \\
\| exp(it\widetilde{T^\dagger_{E,F}}\tilde{T}) \|_{\tilde{F}} &= \| \tilde{F}^{1/2} exp(it\widetilde{T^\dagger_{E,F}}\tilde{T})\tilde{F}^{-1/2} \| = \| \Pi(F^{1/2}exp(itT^\dagger_{E,F}T)F^{-1/2}) \| \\
&\leq \| F^{1/2}exp(itT^\dagger_{E,F}T)F^{-1/2} \| = \| exp(itT^\dagger_{E,F}T) \|_F = 1.
\end{aligned}$$

Consequently, according to [1, Remark 2], $(\tilde{T})^\dagger_{\tilde{E},\tilde{F}} = \widetilde{T^\dagger_{E,F}}$. □

3. Weighted EP elements

In this section, weighted EP Banach space operators and Banach algebra elements will be considered. In particular, these elements will be characterized extending results for matrices ([24]) and elements of C^*-algebras ([18]). In first place, the main notion of this section will be introduced.

Definition 3.1. *Given a unital Banach algebra A and e, $f \in A$ two invertible and positive elements, $a \in A$ is said to be weighted EP with weights e and f, if $a^\dagger_{e,f}$ exists and commutes with a.*

Under the same conditions of Definition 3.1 and as it was pointed out in the paragraph that follows Definition 2.3, note that if $e = f$, then necessary and sufficient for $a \in A$ to be weighted EP with weight e is that $a \in A^e$ is EP; EP Banach algebra elements were studied in [1, 2, 3, 17]. In the context of C^*-algebras, see [13].

To study the objects introduced in Definition 3.1, the notion of group inverse need to be recalled. In fact, weighted EP Banach algebra elements consists in a particular class of group invertible elements.

Definition 3.2. *Given a unital Banach algebra A and $a \in A$, an element $b \in A$ will be said to be the group inverse of a, if the following set of equations is satisfied:*

$$a = aba, \qquad b = bab, \qquad ab = ba.$$

In the conditions of Definition 3.2, note that according to [9, Theorem 9], if the group inverse of $a \in A$ exists, then it is unique. In this case, the group inverse of $a \in A$ will be denoted by $a^\sharp$. In the following remark some of the most relevant properties of the group inverse will be recalled.

Remark 3.3. (i) Let A be a unital Banach algebra and consider $a \in A$. Suppose that $b \in A$ is a normalized generalized inverse of a. Then, necessary and sufficient for b to be the group inverse of a is that $L_b \in L(A)$ (respectively $R_b \in L(A)$) is the group inverse of $L_a \in L(A)$ (respectively $R_a \in L(A)$). In fact, since L_b is a normalized generalized inverse of L_a, according to Definition 3.2 the statement under consideration is equivalent to prove that a and b commute if and only if L_a and L_b commute, which is clear. A similar argument proves the statement for the right multiplication operator on $L(A)$. In addition, note that in this case, according to [9, Theorem 9], $(L_a)^\sharp = L_{a^\sharp}$ (respectively $(R_a)^\sharp = R_{a^\sharp}$).

(ii) Suppose that e, $f \in A$ are invertible and positive (A as in (i)). Note that if $a \in A$ is group invertible, then necessary and sufficient for a to be weighted EP with weights e and f is that $aa^\sharp \in A^e$ and $a^\sharp a \in A^f$ are hermitian. In fact, if $a \in A$ is weighted EP with weights e and f, then according to [9, Theorem 9], $a^\sharp$ exists, actually $a^\sharp = a^\dagger_{e,f}$. In particular, $aa^\sharp \in A^e$ and $a^\sharp a \in A^f$ are hermitian. On the other hand, if $aa^\sharp \in A^e$ and $a^\sharp a \in A^f$ are hermitian, then according to Definition 2.3, $a^\dagger_{e,f}$ exists. What is more, according to Proposition 2.5, $a^\sharp = a^\dagger_{e,f}$. Since $aa^\dagger_{e,f} = aa^\sharp = a^\sharp a = a^\dagger_{e,f}a$, a is weighted EP with weights e and f. Consequently, weighted EP elements are group invertible elements for which the weighted Moore-Penrose inverse exists and coincides with the group inverse.
(iii) Let $A = L(X)$, X a Banach space, and consider $T \in L(X)$. Then, according to [11, Lemma 1], necessary and sufficient for $T^\sharp$ to exists is that $X = N(T) \oplus R(T)$.
(iv) In the conditions of (iii), note that if $T \in L(X)$ is group invertible, then, as in Remark 2.6(b), it is not difficult to prove that $N(T) = N(T^\sharp T) = N(TT^\sharp) = N(T^\sharp)$ and $R(T) = R(TT^\sharp) = R(T^\sharp T) = R(T^\sharp)$.
(v) Observe that according to items (iii) and (iv), $R(T^k) = R((T^\sharp)^k) = R(T)$ and $N(T^k) = N((T^\sharp)^k) = N(T)$.

In the following theorem the first characterization of weighted EP elements will be given.

Theorem 3.4. *(a) Let X be a Banach space and consider E, $F \in L(X)$ two invertible and positive operators. Then, if $T \in L(X)$, the following statements are equivalent.*
(i) T is weighted EP with weights E and F;
(ii) there exists an idempotent $P \in L(X)$ such that $P \in H(L(X)^E) \cap H(L(X)^F)$, $R(P) = R(T)$ and $N(P) = N(T)$.
(b) Let A be a unital Banach algebra and consider e, $f \in A$ two invertible and positive elements. Then, if $a \in A$ is such that $a^\dagger_{e,f}$ exists, the following statements are equivalent.
(iii) a is weighted EP with weights e and f;
(iv) there exists an idempotent $P \in L(A)$ such that $P \in H(L(A)^{L_e}) \cap H(L(A)^{L_f})$, $R(P) = aA$ and $N(P) = a^{-1}(0)$,
(v) $L_a \in L(A)$ is weighted EP with weights L_e and L_f.
Furthermore, if in (a) or (b) such P exists, then it is unique.

Proof. (a) If T is weighted EP with weights E and F, then $P = TT^\dagger_{E,F} = T^\dagger_{E,F}T$ satisfies the required property (Remark 2.6(b)).

On the other hand, if statement (ii) holds, then according to Theorem 2.7, $T^\dagger_{E,F}$ exists. Further, observe that $R(TT^\dagger_{E,F}) = R(T) = R(P)$ and $R(I - T^\dagger_{E,F}T) = N(T^\dagger_{E,F}T) = N(T) = N(P) = R(I - P)$. Therefore, according to [25, Hilfssatz 2(a)-(b)] and [20, Theorem 2.2], $T^\dagger_{E,F}T = P = TT^\dagger_{E,F}$.
(b) According to what has been proved, statement (iv) and (v) are equivalent. In addition, since according to Theorem 2.8, $(L_a)^\dagger_{L_e,L_f} = L_{a^\dagger_{e,f}}$, statements (iii) and (v) are equivalent.

The last statement is a consequence of Theorem 2.7. □

Next some basic characterizations of weighted EP Banach space operators will be given.

Theorem 3.5. *Let X be a Banach space and consider E, $F \in L(X)$ two invertible and positive operators. Suppose in addition that $T \in L(X)$ is such that $T^\dagger_{E,F}$ and $T^\sharp$ exist. Then, the following statements are equivalent.*

(i) *T is weighted EP with weights E and F;*

(ii) $R(T^{\dagger}_{E,F}) = R(T)$ *and* $N(T^{\dagger}_{E,F}) = N(T)$*;*

(iii) $R(T^{\dagger}_{E,F}) \subset R(T)$ *and* $N(T) \subset N(T^{\dagger}_{E,F})$*;*

(iv) $R(T) \subset R(T^{\dagger}_{E,F})$ *and* $N(T) \subset N(T^{\dagger}_{E,F})$*;*

(v) $R(T^{\dagger}_{E,F}) \subset R(T)$ *and* $N(T^{\dagger}_{E,F}) \subset N(T)$*;*

(vi) $R(T) \subset R(T^{\dagger}_{E,F})$ *and* $N(T^{\dagger}_{E,F}) \subset N(T)$*;*

(vii) $T^{\dagger}_{E,F} = T^{\sharp}$*;*

(viii) $T^{\dagger}_{E,F} = T(T^{\dagger}_{E,F})^2 = (T^{\dagger}_{E,F})^2 T$*;*

(ix) $T = T^{\dagger}_{E,F} T^2 = T^2 T^{\dagger}_{E,F}$*;*

(x) $T^{\dagger}_{E,F}$ *is weighted EP with weights* F *and* E*, moreover* $(T^{\dagger}_{E,F})^{\dagger}_{F,E} = T$*;*

(xi) T *is weighted EP with weights* F *and* E*;*

(xii) T *is both weighted EP with weights* E *and* E *and with weights* F *and* F*;*

(xiii) T^k *is weighted EP with weights* E *and* F*, for some integer* $k \geq 1$*;*

(xiv) $T^{\sharp}$ *is weighted EP with weights* E *and* F*;*

(xv) $TT^{\sharp} = TT^{\dagger}_{E,E} = TT^{\dagger}_{F,F}$ *(or* $TT^{\sharp} = T^{\dagger}_{E,E}T = T^{\dagger}_{F,F}T$*);*

(xvi) $TT^{\sharp} = TT^{\dagger}_{E,F} = TT^{\dagger}_{F,E}$ *(or* $TT^{\sharp} = T^{\dagger}_{F,E}T = T^{\dagger}_{E,F}T$*).*

Proof. If statement (i) holds, then according to Remark 2.6(b), $R(T^{\dagger}_{E,F}) = R(T^{\dagger}_{E,F}T) = R(TT^{\dagger}_{E,F}) = R(T)$ and $N(T^{\dagger}_{E,F}) = N(TT^{\dagger}_{E,F}) = N(T^{\dagger}_{E,F}T) = N(T)$. Consequently, statement (ii) holds, which in turn clearly implies statements (iii)-(vi).

On the other hand, if one of the statements (iii)-(vi) holds, say statement (iii) for example, then, using the decompositions of X

$$X = R(T^{\dagger}_{E,F}) \oplus N(T) = R(T) \oplus N(T^{\dagger}_{E,F}) = R(T) \oplus N(T)$$

(Remark 2.6(b) and Remark 3.3(iii)), it is not difficult to prove that $R(T^{\dagger}_{E,F}) = R(T)$ and $N(T^{\dagger}_{E,F}) = N(T)$, equivalently statement (ii) holds. However, in this case $R(T^{\dagger}_{E,F}T) = R(TT^{\dagger}_{E,F})$ and $N(T^{\dagger}_{E,F}T) = N(TT^{\dagger}_{E,F})$. Since $T^{\dagger}_{E,F}T$ and $TT^{\dagger}_{E,F} \in L(X)$ are idempotents, $T^{\dagger}_{E,F}T = TT^{\dagger}_{E,F}$, i.e., T is weighted EP with weights E and F.

An argument similar to the one considered in Remark 3.3(ii) proves the equivalence between statements (i) and (vii).

Statement (i) implies statements (viii)-(ix). On the other hand, statement (viii) implies statement (iii) and statement (ix) implies statement (vi).

The equivalence between statements (i) and (x) is a consequence of Definition 2.3.

Since according to Remark 3.3(ii), T is weighted EP with weights F and E if and only if $TT^{\sharp} \in H(L(X)^F) \cap H(L(X)^E)$, statements (i) and (xi) are equivalent. Similarly, it is possible to prove that statement (i) is equivalent to statements (xii)-(xiv) using that $T^k(T^k)^\sharp = T^k(T^\sharp)^k = TT^\sharp$ (for some integer $k \geq 1$) and $(T^\sharp)^\sharp = T^\sharp$.

If statement (xii) holds, then $T^\sharp = T^\dagger_{E,E} = T^\dagger_{F,F}$ and statement (xv) is satisfied, which in turn implies statement (i) (Remark 3.3(ii)). In a similar way it is possible to prove that statement (xvi) implies statement (i) and statement (xi) implies statement (xvi). □

In the following theorem more characterizations concerning weighted EP operators will be proved.

Theorem 3.6. *Let X be a Banach space and consider $E, F \in L(X)$ two invertible and positive operators. Suppose in addition that $T \in L(X)$ is such that $T^\dagger_{E,F}$ and $T^\sharp$ exist. Then, necessary and sufficient for T to be weighted EP with weights E and F is that one of the following statements holds.*

(i) $TT^\sharp T^\dagger_{E,F} = T^\dagger_{E,F} T^\sharp T$;

(ii) $TT^\dagger_{E,F} T^\dagger_{E,F} T = T^\dagger_{E,F} TTT^\dagger_{E,F}$;

(iii) $T^\dagger_{E,F} T^\sharp T + TT^\sharp T^\dagger_{E,F} = 2T^\dagger_{E,F}$;

(iv) $(T^\dagger_{E,F})^2 T^\sharp = T^\dagger_{E,F} T^\sharp T^\dagger_{E,F} = T^\sharp (T^\dagger_{E,F})^2$;

(v) $T(T^\dagger_{E,F})^2 = T^\sharp = (T^\dagger_{E,F})^2 T$;

(vi) $T^k = T^\dagger_{E,F} TT^k = T^k TT^\dagger_{E,F}$;

(vii) $(T^\dagger_{E,F})^k = (T^\sharp)^k$;

(viii) $(T^\sharp)^k T^\dagger_{E,F} = T^\dagger_{E,F}(T^\sharp)^k$;

(ix) $T^k T^\dagger_{E,F} = T^\dagger_{E,F} T^k$;

(x) $T^{2k-1} = T^\dagger_{E,F} T^{2k+1} T^\dagger_{E,F}$;

(xi) $(T^\sharp)^k T^\dagger_{E,F} T = (T^\dagger_{E,F})^k$;

(xii) $T(T^\dagger_{E,F})^{k+1} = (T^\sharp)^k = (T^\dagger_{E,F})^{k+1} T$;

(xiii) $T^k TT^\dagger_{E,F} + T^\dagger_{E,F} TT^k = 2T^k$;

(xiv) $(T^\sharp)^{k+l-1} = (T^\dagger_{E,F})^l (T^\sharp)^{k-1} = (T^\sharp)^{k-1} (T^\dagger_{E,F})^l$;

(xv) $TT^\dagger_{E,F}(T^\sharp)^{k+l-1} = (T^\dagger_{E,F})^k (T^\sharp)^l T = (T^\sharp)^l T (T^\dagger_{E,F})^k$;

(xvi) $TT^\dagger_{E,F}(T^k + \lambda T^\dagger_{E,F}) = (T^k + \lambda T^\dagger_{E,F}) TT^\dagger_{E,F}$;

(xvii) $T^\dagger_{E,F} T(T^k + \lambda T^\dagger_{E,F}) = (T^k + \lambda T^\dagger_{E,F}) T^\dagger_{E,F} T$;

(xviii) $R(T + \lambda T^\dagger_{E,E}) = R(T + \lambda T^\dagger_{F,F}) = R(\lambda T + T^3)$ *and*
$N(T + \lambda T^\dagger_{E,E}) = N(T + \lambda T^\dagger_{F,F}) = N(\lambda T + T^3)$;

(xix) $R(T + \lambda T^\dagger_{E,F}) = R(\lambda T + T^3)$ *and* $N(T + \lambda T^\dagger_{E,F}) = N(\lambda T + T^3)$,

where k, $l \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{0\}$.

Proof. It is not difficult to prove that the condition of being weighted EP with weights E and F implies statements (i)–(xvii). On the other hand, if statement (i) holds, then

$$R(TT^\sharp T^\dagger_{E,F}) = R(T^\sharp TT^\dagger_{E,F}) = T^\sharp R(TT^\dagger_{E,F}) = R(T).$$

As a result, $R(T) \subseteq R(T^\dagger_{E,F})$. In addition, if $x \in N(T^\dagger_{E,F})$, then $T^\sharp T(x) \in R(T) \cap N(T^\dagger_{E,F}) = \{0\}$, Thus, $x \in N(T^\sharp T) = N(T)$ and $N(T^\dagger_{E,F}) \subseteq N(T)$. Consequently, according to Theorem 3.5(vi), the condition of being weighted EP is satisfied.

Suppose that statement (ii) holds. If $x \in N(T)$, then $TT^\dagger_{E,F}(x) \in R(T) \cap N(T^\dagger_{E,F}T) = R(T) \cap N(T) = \{0\}$. Thus, $x \in N(TT^\dagger_{E,F}) = N(T^\dagger_{E,F})$ and $N(T) \subseteq N(T^\dagger_{E,F})$. On the other hand, since

$$R(T^\dagger_{E,F}TTT^\dagger_{E,F}) = T^\dagger_{E,F}T(R(TT^\dagger_{E,F})) = T^\dagger_{E,F}T(R(T)) = R(T^\dagger_{E,F}T) = R(T^\dagger_{E,F}),$$

$R(T^\dagger_{E,F}) \subseteq R(T)$. Therefore, according to Theorem 3.5(iii), T is weighted EP with weights E and F.

If statement (iii) holds, then $R(T) = R(TT^\dagger_{E,F}) = R(T^\sharp TT^\dagger_{E,F}) \subseteq R(T^\dagger_{E,F})$. In addition, if $x \in N(T^\dagger_{E,F})$, then $T^\sharp T(x) \in R(T) \cap N(T^\dagger_{E,F}) = \{0\}$. Thus, $x \in N(T^\sharp T) = N(T)$ and $N(T^\dagger_{E,F}) \subseteq N(T)$. In particular, according to Theorem 3.5(vi), the condition of being weighted EP is satisfied.

Suppose that statement (iv) holds. According to the equality $(T^\dagger_{E,F})^2 T^\sharp = T^\dagger_{E,F}T^\sharp T^\dagger_{E,F}$,

$$\begin{aligned}(T^\dagger_{E,F})^2 T^\sharp &= ((T^\dagger_{E,F})^2 T^\sharp)TT^\sharp = T^\dagger_{E,F}T^\sharp T^\dagger_{E,F}TT^\sharp \\ &= T^\dagger_{E,F}(T^\sharp)^2 TT^\dagger_{E,F}TT^\sharp = T^\dagger_{E,F}(T^\sharp)^2.\end{aligned}$$

As a result,

$$\begin{aligned}TT^\dagger_{E,F} &= T^3(T^\sharp)^2 T^\dagger_{E,F} = T^3 T^\dagger_{E,F}T(T^\sharp)^2 T^\dagger_{E,F} = T^3(T^\dagger_{E,F}T^\sharp T^\dagger_{E,F}) \\ &= T^3((T^\dagger_{E,F})^2 T^\sharp) = T^3 T^\dagger_{E,F}(T^\sharp)^2 = T^3 T^\dagger_{E,F}T(T^\sharp)^3 = TT^\sharp.\end{aligned}$$

A similar argument, using in particular the equality $T^\dagger_{E,F}T^\sharp T^\dagger_{E,F} = T^\sharp(T^\dagger_{E,F})^2$, proves that $T^\dagger_{E,F}T = TT^\sharp$. Therefore, T is weighted EP with weights E and F.

Next consider statement (v). The condition $T(T^\dagger_{E,F})^2 = T^\sharp = (T^\dagger_{E,F})^2 T$ implies that

$$TT^\dagger_{E,F} = TT^\sharp TT^\dagger_{E,F} = TT(T^\dagger_{E,F})^2 TT^\dagger_{E,F} = T(T(T^\dagger_{E,F})^2) = TT^\sharp$$

and

$$T^\dagger_{E,F}T = T^\dagger_{E,F}TT^\sharp T = T^\dagger_{E,F}T(T^\dagger_{E,F})^2 TT = ((T^\dagger_{E,F})^2 T)T = T^\sharp T.$$

Consequently, T is weighted EP with weights E and F.

If statement (vi) holds, then $R(T) = R(T^k) \subseteq R(T^\dagger_{E,F})$ and $N(T^\dagger_{E,F}) \subseteq N(T^k) = N(T)$. Therefore, according to Theorem 3.5(vi), T is weighted EP with weights E and F. A similar argument, using in particular that the ranges and null spaces of T and $T^\sharp$ coincide, proves that statement (vii) is equivalent to the condition of being weighted EP.

Suppose that statement (viii) holds. Then,

$$R(T^\dagger_{E,F}) = R(T^\dagger_{E,F}T) = T^\dagger_{E,F}(R(T)) = T^\dagger_{E,F}(R((T^\sharp)^k)) = R(T^\dagger_{E,F}(T^\sharp)^k) \subseteq R((T^\sharp)^k) = R(T).$$

In addition, if $x \in N(T) = N((T^\sharp)^k)$, then $T^\dagger_{E,F}(x) \in R(T^\dagger_{E,F}) \cap N((T^\sharp)^k) = R(T^\dagger_{E,F}) \cap N(T) = \{0\}$. As a result, $N(T) \subseteq N(T^\dagger_{E,F})$. Consequently, according to Theorem 3.5(iii), T is weighted EP with weights E and F. A similar argument can be applied to statement (ix).

Statement (x) implies that $N(T^\dagger_{E,F}) \subseteq N(T^{2k-1}) = N(T)$ and $R(T) = R(T^{2k-1}) \subseteq R(T^\dagger_{E,F})$. As a result, according to Theorem 3.5(vi), the condition of being weighted EP is satisfied.

Suppose that statement (xi) holds. Now well, since $X = R(T^\dagger_{E,F}T) \oplus N((T^\sharp)^k)$, $R(T) = R((T^\sharp)^k) \subseteq R(T^\dagger_{E,F})$. In addition, if $x \in N(T^\dagger_{E,F})$, then $T^\dagger_{E,F}T(x) \in R(T^\dagger_{E,F}) \cap N((T^\sharp)^k) = R(T^\dagger_{E,F}) \cap N(T) = \{0\}$. Hence, $x \in N(T^\dagger_{E,F}T) = N(T)$ and $N(T^\dagger_{E,F}) \subseteq N(T)$. Consequently, according to Theorem 3.5(vi), T is weighted EP with weights E and F.

Suppose that statement (xii) holds. Then, it is not difficult to prove that $R(T) \subseteq R(T^\dagger_{E,F})$ and $N(T^\dagger_{E,F}) \subseteq N(T)$. In particular, according to Theorem 3.5(vi), T is weighted EP with weights E and F.

If statement (xiii) holds, then $R(T^\dagger_{E,F}) = R(T^\dagger_{E,F}T) = R(T^\dagger_{E,F}TT^k) \subseteq R(T^k) = R(T)$. In addition, if $x \in N(T) = N(T^k)$, then $T^\dagger_{E,F}(x) \in N(T^{k+1}) = N(T)$, which implies that $x \in N(T^\dagger_{E,F})$. Thus, $N(T) \subseteq N(T^\dagger_{E,F})$. However, according to Theorem 3.5(iii), the condition of being weighted EP holds.

Statement (xiv) implies that $R(T) \subseteq R(T^\dagger_{E,F})$ and $N(T^\dagger_{E,F})) \subseteq N(T)$. Therefore, according to Theorem 3.5(vi), T is weighted EP with weights E and F.

Suppose that statement (xv) holds. Then,

$$R(T) = R(TT^\dagger_{E,F}T) = TT^\dagger_{E,F}R(T) = TT^\dagger_{E,F}R((T^\sharp)^{k+l-1}) = R(TT^\dagger_{E,F}(T^\sharp)^{k+l-1}) \subseteq R(T^\dagger_{E,F}).$$

In addition, if $x \in N(T^\dagger_{E,F})$, then $(T^\sharp)^{k+l-1}(x) \in N(TT^\dagger_{E,F}) \cap R(T) = N(T^\dagger_{E,F}) \cap R(T) = \{0\}$. Then $N(T^\dagger_{E,F}) \subseteq N(T)$. Therefore, according to Theorem 3.5(vi), the condition of being weighted EP is satisfied.

Statement (xvi) can be rewritten as

$$T^k + \lambda TT^\dagger_{E,F}T^\dagger_{E,F} = T^kTT^\dagger_{E,F} + \lambda T^\dagger_{E,F}. \tag{1}$$

Then, multiplying (1) from the left side by $TT^\dagger_{E,F}$,

$$T^k + \lambda TT^\dagger_{E,F}T^\dagger_{E,F} = T^kTT^\dagger_{E,F} + \lambda TT^\dagger_{E,F}T^\dagger_{E,F}.$$

Hence, $T^k = T^kTT^\dagger_{E,F}$, which implies that $N(T^\dagger_{E,F}) \subseteq N(T^k) = N(T)$. Similarly, multiplying (1) from the right side by $TT^\dagger_{E,F}$,

$$T^kTT^\dagger_{E,F} + \lambda TT^\dagger_{E,F}T^\dagger_{E,F} = T^kTT^\dagger_{E,F} + \lambda T^\dagger_{E,F}.$$

As a result, $TT^{\dagger}_{E,F}T^{\dagger}_{E,F} = T^{\dagger}_{E,F}$, which implies that $R(T^{\dagger}_{E,F}) \subseteq R(T)$. Consequently, according to Theorem 3.5(v), T is weighted EP with weights E and F.

A similar argument proves that statement (xvii) implies statement (i).

Suppose that statements (xviii) holds. The equality $R(T + \lambda T^{\dagger}_{E,E}) = R(\lambda T + T^3)$ implies that for $x \in X$ there exists $y \in X$ such that $(T + \lambda T^{\dagger}_{E,E})x = (\lambda T + T^3)y$. Then

$$\begin{aligned}(T + \lambda TT^{\dagger}_{E,E}T^{\dagger}_{E,E})x &= TT^{\dagger}_{E,E}(T + \lambda T^{\dagger}_{E,E})x = TT^{\dagger}_{E,E}(\lambda T + T^3)y\\ &= (\lambda T + T^3)y = (T + \lambda T^{\dagger}_{E,E})x,\end{aligned}$$

which implies that $TT^{\dagger}_{E,E}T^{\dagger}_{E,E}x = T^{\dagger}_{E,E}x$ $(x \in X)$. Thus, $R(T^{\dagger}_{E,E}) \subseteq R(T)$. Let $x \in N(T)$. Now well, $(\lambda T + T^3)x = 0$ and, since $N(T + \lambda T^{\dagger}_{E,E}) = N(\lambda T + T^3)$, $(T + \lambda T^{\dagger}_{E,E})x = 0$, i.e., $T^{\dagger}_{E,E}x = 0$. Therefore, $N(T) \subseteq N(T^{\dagger}_{E,E})$, and according to Theorem 3.5(iii), T is weighted EP with weights E and E. Similarly, using that $R(T + \lambda T^{\dagger}_{F,F}) = R(\lambda T + T^3)$ and $N(T + \lambda T^{\dagger}_{F,F}) = N(\lambda T + T^3)$, it is possible to prove that that T is weighted EP with weights F and F. According then to Theorem 3.5(xii), T is weighted EP with weights E and F.

On the other hand, if T is weighted EP with weights E and F, then $T^{\sharp} = T^{\dagger}_{E,E} = T^{\dagger}_{F,F}$ (Theorem 3.5(xii)). Since

$$T + \lambda T^{\sharp} = (T^3 + \lambda T)(T^{\sharp})^2 = (T^{\sharp})^2(T^3 + \lambda T)$$

and

$$T^3 + \lambda T = (T + \lambda T^{\sharp})T^2 = T^2(T + \lambda T^{\sharp}),$$

statement (xviii) holds. A similar argument proves that statement (xix) is equivalent to the condition of being weighted EP. □

Next weighted EP Banach algebra elements will be characterized.

Theorem 3.7. *Let A be a unital Banach algebra and consider e, $f \in A$ two invertible and positive elements. Suppose in addition that $a \in A$ is such that $a^{\dagger}_{e,f}$ and $a^{\sharp}$ exist. Then, the following statements are equivalent.*

(i) *a is weighted EP with weights e and f;*

(ii) *$a^{\dagger}_{e,f}A = aA$ and $(a^{\dagger}_{e,f})^{-1}(0) = a^{-1}(0)$;*

(iii) *$a^{\dagger}_{e,f}A \subset aA$ and $a^{-1}(0) \subset (a^{\dagger}_{e,f})^{-1}(0)$;*

(iv) *$aA \subset a^{\dagger}_{e,f}A$ and $a^{-1}(0) \subset (a^{\dagger}_{e,f})^{-1}(0)$;*

(v) *$a^{\dagger}_{e,f}A \subseteq aA$ and $(a^{\dagger}_{e,f})^{-1}(0) \subset a^{-1}(0)$;*

(vi) *$aA \subset a^{\dagger}_{e,f}A$ and $(a^{\dagger}_{e,f})^{-1}(0) \subset a^{-1}(0)$;*

(vii) *$a^{\dagger}_{e,f} = a(a^{\dagger}_{e,f})^2 = (a^{\dagger}_{e,f})^2a$;*

(viii) *$a^{\dagger}_{e,f}$ is weighted EP with weights f and e;*

(ix) $a^{\dagger}_{e,f}a^{\sharp}a + aa^{\sharp}a^{\dagger}_{e,f} = 2a^{\dagger}_{e,f}$;

(x) $(a^{\dagger}_{e,f})^2a^{\sharp} = a^{\dagger}_{e,f}a^{\sharp}a^{\dagger}_{e,f} = a^{\sharp}(a^{\dagger}_{e,f})^2$;

(xi) $a(a^{\dagger}_{e,f})^2 = a^{\sharp} = (a^{\dagger}_{e,f})^2a$;

(xii) $aa^{\dagger}_{e,f}a^{\dagger}_{e,f}a = a^{\dagger}_{e,f}aaa^{\dagger}_{e,f}$;

(xiii) $aa^{\sharp}a^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^{\sharp}a$;

(xiv) $a \in a^{\dagger}_{e,f}A^{-1} \cap A^{-1}a^{\dagger}_{e,f}$;

(xv) *there exist* u, $v \in A$ *such that* $a = a^{\dagger}_{e,f}u = va^{\dagger}_{e,f}$ *and* $uA = A$ *and* $v^{-1}(0) = \{0\}$;

(xvi) $a \in a^{\dagger}_{e,f}A \cap Aa^{\dagger}_{e,f}$;

(xvii) *if* $b \in A$ *is such that* $ab = ba$, *then* $a^{\dagger}_{e,f}b = ba^{\dagger}_{e,f}$;

(xviii) *there exists some holomorphic function* $f\colon U \to \mathbb{C}$, $\sigma(a) \subseteq U$, *such that* $a^{\dagger}_{e,f} = f(a)$;

(xix) $(a^{\sharp})^ka^{\dagger}_{e,f}a = (a^{\dagger}_{e,f})^k$;

(xx) $a^k = a^{\dagger}_{e,f}aa^k = a^kaa^{\dagger}_{e,f}$;

(xxi) $(a^{\dagger}_{e,f})^k = (a^{\sharp})^k$;

(xxii) $(a^{\sharp})^ka^{\dagger}_{e,f} = a^{\dagger}_{e,f}(a^{\sharp})^k$;

(xxiii) $a^ka^{\dagger}_{e,f} = a^{\dagger}_{e,f}a^k$;

(xxiv) $a^{2k-1} = a^{\dagger}_{e,f}a^{2k+1}a^{\dagger}_{e,f}$;

(xxv) $a(a^{\dagger}_{e,f})^{k+1} = (a^{\sharp})^k = (a^{\dagger}_{e,f})^{k+1}a$;

(xxvi) $a^kaa^{\dagger}_{e,f} + a^{\dagger}_{e,f}aa^k = 2a^k$;

(xxvii) $(a^{\sharp})^{k+l-1} = (a^{\dagger}_{e,f})^l(a^{\sharp})^{k-1} = (a^{\sharp})^{k-1}(a^{\dagger}_{e,f})^l$;

(xxviii) $aa^{\dagger}_{e,f}(a^{\sharp})^{k+l-1} = (a^{\dagger}_{e,f})^k(a^{\sharp})^la = (a^{\sharp})^la(a^{\dagger}_{e,f})^k$,

(xxix) a *is weighted EP with weights* f *and* e;

(xxx) a *is both weighted EP with weights* e *and* e *and with weights* f *and* f;

(xxxi) a^k *is weighted EP with weights* e *and* f;

(xxxii) $a^{\sharp}$ *is weighted EP with weights* e *and* f;

(xxxiii) $aa^{\sharp} = aa^{\dagger}_{e,e} = aa^{\dagger}_{f,f}$ (*or* $aa^{\sharp} = a^{\dagger}_{e,e}a = a^{\dagger}_{f,f}a$);

(xxxiv) $aa^\sharp = aa^\dagger_{e,f} = aa^\dagger_{f,e}$ (*or* $aa^\sharp = a^\dagger_{f,e}a = a^\dagger_{e,f}a$);

(xxxv) $aa^\dagger_{e,f}(a^k + \lambda a^\dagger_{e,f}) = (a^k + \lambda a^\dagger_{e,f})aa^\dagger_{e,f}$;

(xxxvi) $a^\dagger_{e,f}a(a^k + \lambda a^\dagger_{e,f}) = (a^k + \lambda a^\dagger_{e,f})a^\dagger_{e,f}a$;

(xxxvii) $(a + \lambda a^\dagger_{e,e})A = (a + \lambda a^\dagger_{f,f})A = (\lambda a + a^3)A$ *and*

$(a + \lambda a^\dagger_{e,e})^{-1}(0) = (a + \lambda a^\dagger_{f,f})^{-1}(0) = (\lambda a + a^3)^{-1}(0)$;

(xxxviii) $(a + \lambda a^\dagger_{e,f})A = (\lambda a + a^3)A$ *and* $(a + \lambda a^\dagger_{e,f})^{-1}(0) = (\lambda a + a^3)^{-1}(0)$,

where $k, l \in \mathbb{N}$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$.

Proof. Let $a \in A$ that satisfies the hypothesis of the Theorem. It is not difficult to prove that if a is weighted EP with weights e and f, then statements (i)-(xiii) and (xix)-(xxxviii) hold. To prove the converse in this case, consider $L_a \colon A \to A$ and apply the left multiplication representation to the statements. According Theorem 2.8 and Remark 3.3(i), each statement is relaborated with L_a, $(L_a)^\sharp$, and $(L_a)^\dagger_{L_e,L_f}$ instead of a, $a^\sharp$ and $a^\dagger_{e,f}$ respectively. Then apply Theorems 3.5 and 3.6 to prove that $L_a \in L(A)$ is weighted EP with weights L_e and L_f (recall that according to Lemma 2.4, L_e, $L_f \in L(A)$ are invertible and positive). Finally, apply again Theorem 2.8 to prove that a is weighted EP with weights e and f.

To prove the equivalence between the condition of being weighted EP and statement (xiv), note that if $aa^\dagger_{e,f} = a^\dagger_{e,f}a$, then it is not difficlt to prove that $a = (a^2 + 1 - a^\dagger_{e,f}a)a^\dagger_{e,f} = a^\dagger_{e,f}(a^2 + 1 - aa^\dagger_{e,f})$ and $(a^2 + 1 - a^\dagger_{e,f}a)^{-1} = (a^\dagger_{e,f})^2 + 1 - a^\dagger_{e,f}a$. Clearly, statement (xiv) implies statement (xv), which in turn implies statement (xvi). On the other hand, statement (xvi) implies statement (vi).

Now consider statement (xvii). If $a^\dagger_{e,f}$ exists, then since the group inverse is unique, $a^\dagger_{e,f} = a^\sharp$. Then, according to [6, Lemma 1.4.5], statement (xvii) holds. If, on the other hand $a \in A$ satisfies the condition of statement (xvii), then applying this condition to $b = a$, a is weighted EP with weights e and f.

Finally, to prove the equivalence between statement (i) and (xviii), suppose that $a^\dagger_{e,f}$ exists. Then, as in the previous paragraph, $a^\dagger_{e,f} = a^\sharp$. Then, according to [12, Theorem 4.4], statement (xviii) holds. On the other hand, if $a^\dagger_{e,f} = f(a)$, where $f \colon U \to C$ is holomorphic ($\sigma(a) \subset U$), according to [5, Proposition 4.9, Chapter VII], a is weighted EP with weights e and f. □

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