FULL-RANK AND DETERMINANTAL REPRESENTATION OF THE DRAZIN INVERSE

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Abstract. In this article we introduce a full-rank representation of the Drazin inverse A^D of a given complex matrix A, which is based on an arbitrary full-rank decomposition of A^l, $l \geq k$, where k is the index of A. We show that the known representation of the Drazin inverse of A, devised in [7], represents a partial case of this result. Using this general representation, we introduce a determinantal representation of the Drazin inverse. More precisely, we represent elements of the Drazin inverse A^D as ratios of two expressions involving minors of the order $\operatorname{rank}(A^k)$, $k = \operatorname{ind}(A)$, taken from the matrices A and rank invariant powers A^l, $\quad l \geq k$. Also, we examine conditions for the existence of the Drazin inverse for matrices whose elements are taken from an integral domain. Finally, a few correlations between the minors of powers of the Drazin inverse A^D and the minors of the matrix A^k are explicitly derived.

1. Introduction

The set of all $m \times n$ complex matrices of rank r is denoted by $\mathbb{C}_r^{m\times n}$, and the set of all $m \times n$ matrices of rank r whose elements are taken from an integral domain $\mathbb{I}$ is denoted by $\mathbb{I}_r^{m\times n}$. $\operatorname{Tr}(A)$ denotes the trace of a square matrix A, the determinantal rank of A is denoted by $\rho(A)$, and $|A|$ denotes the determinant of A.

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An arbitrary matrix $A \in \mathbb{C}_r^{n\times n}$ has the Drazin inverse, provided that there exists the unique matrix $A^D \in \mathbb{C}_r^{n\times n}$, satisfying

$$A^{k+1}A^D = A^k, \quad A^D A A^D = A^D, \quad AA^D = A^D A,$$

for some non-negative integer $k = \operatorname{ind}(A) = \min_p\{p : \quad \operatorname{rank}(A^{p+1}) = \operatorname{rank}(A^p)\}$. In the case $k = 1$, the Drazin inverse is well-known as the group inverse of A and is denoted by $A^{\#}$.

For $A \in \mathbb{C}^{m\times n}$, if $G \in \mathbb{C}^{n\times m}$ satisfies $AGA = A$, $GAG = G$, then G is a reflexive g-inverse of A. G is called the Moore-Penrose inverse of A if it satisfies the equations

$$AGA = A, \quad GAG = G, \quad (AG)^* = AG, \quad (GA)^* = GA.$$

We use the following notation from [1], [12]. Let A be an $m \times n$ matrix of rank r; let $\alpha = \{\alpha_1, \dots, \alpha_p\}$ and $\beta = \{\beta_1, \dots, \beta_p\}$ be subsets of $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively, of the order $1 \le p \le \min\{m, n\}$. Then $|\, A_\beta^\alpha \,|$ denotes the minor of A determined by the rows indexed by α and the columns indexed by β.

For $1 \le p \le n$, denote the collection of strictly increasing sequences of p integers chosen from $\{1, \dots, n\}$, by

$$\mathcal{Q}_{p,n} = \{\alpha : \ \alpha = (\alpha_1, \dots, \alpha_p), \ 1 \le \alpha_1 < \cdots < \alpha_p \le n\}.$$

Let $\mathcal{N} = \mathcal{Q}_{r,m} \times \mathcal{Q}_{r,n}$. For fixed $\alpha \in \mathcal{Q}_{k,m}$, $\beta \in \mathcal{Q}_{k,n}$, $1 \le k \le r$, let

$$\mathcal{I}(\alpha) = \{I : \ I \in \mathcal{Q}_{r,m}, \ I \supseteq \alpha\}, \quad \mathcal{J}(\beta) = \{J : \ J \in \mathcal{Q}_{r,n}, \ J \supseteq \beta\},$$

$$\mathcal{N}(\alpha, \beta) = \mathcal{I}(\alpha) \times \mathcal{J}(\beta).$$

If A is a square matrix, then the coefficient of $|\, A_\beta^\alpha \,|$ in the Laplace expansion of $|A|$ is denoted by $\dfrac{\partial}{\partial |\, A_\beta^\alpha \,|}|A|$. For the special case $\alpha = \{i\}$, $\beta = \{j\}$, we get the cofactor of a_{ij}: $\dfrac{\partial}{\partial a_{ij}}\,|\, A \,|$.

We use $C_p(A)$ to denote the pth compound matrix of A with rows indexed by r-element subsets of $\{1, \dots, m\}$, columns indexed by r-element subsets of $\{1, \dots, n\}$, and the (α, β) entry defined by $|\, A_\beta^\alpha \,|$.

Also, we use the following extensions of these notions:

$$\mathcal{N}_{r_k} = \mathcal{Q}_{r_k,m} \times \mathcal{Q}_{r_k,n}, \text{ where } r_k = \operatorname{rank}(A^l), \quad l \geq k = \operatorname{ind}(A);$$

for fixed $\alpha,\ \beta \in \mathcal{Q}_{p,n}$, $1 \leq p \leq r_k$, let

$$\mathcal{I}_{r_k}(\alpha) = \{I :\ I \in \mathcal{Q}_{r_k,m},\ I \supseteq \alpha\}, \quad \mathcal{J}_{r_k}(\beta) = \{J :\ J \in \mathcal{Q}_{r_k,n},\ J \supseteq \beta\},$$

$\mathcal{N}_{r_k}(\alpha, \beta) = \mathcal{I}_{r_k}(\alpha) \times \mathcal{J}_{r_k}(\beta)$.

The paper is organized as follows. In Section 2 we introduce a full-rank representation of the Drazin inverse of a complex matrix A, by means of components from an arbitrary full-rank factorization of any matrix power A^l, $l \geq k = \operatorname{ind}(A)$. We show that the Cline's representation of the Drazin inverse [7] gives a partial case of this general representation. This full-rank representation will be used in Section 3. In Section 3 we introduce a determinantal formula for the Drazin inverse of a given complex matrix A. In other words, an arbitrary element of the Drazin inverse is characterized in terms of minors of the order $r_k = \operatorname{rank}(A^k)$, $k = \operatorname{ind}(A)$, selected from the matrices A and A^l, $l \geq k$. Such an approach, as far as we know, is not employed before. In the papers [1], [2], [3], [4], [12], [13], [14], concerning the determinantal representation of generalized inverses, only $r \times r$ minors, $r = \operatorname{rank}(A)$, are used in investigation and representation of the reflexive g-inverses, the Moore-Penrose inverse and the group inverse. In the papers [8], [9] there are investigated the determinantal representations of generalized inverses for matrices over an arbitrary field. In these papers it is allowed to use minors of the order h, $h \leq \operatorname{rank}(A)$, but its application in the representation of the Drazin inverse is not mentioned. Among the other results, we develop full-rank and determinantal representations for some expressions involving the Drazin inverse, arising from the representations of the Drazin inverse. As an consequence we obtain the known determinantal formula for the reflexive g-inverses, introduced in [2].

In the set of matrices over an integral domain $\mathbb{I}$ we give a few necessary and sufficient conditions for the existence of the Drazin inverse and its determinantal representation. These conditions are complementary with respect to the conditions for the existence of the Drazin inverse, investigated in [13].

In the last section we investigate correlatons between the minors of the order $\operatorname{rank}(A^k)$.

2. Representation of the Drazin inverse based on the full-rank decomposition

Using the known Cline's representation of the Drazin inverse [7], we introduce a full-rank representation of the Drazin inverse in terms of the full-rank factorization of the matrix powers A^l, $l \geq k = \operatorname{ind}(A)$.

Theorem 2.1. *The Drazin inverse of a given matrix $A \in \mathbb{C}_r^{n\times n}$ can be represented in the form*

$$A^D = P_{A^l}(Q_{A^l}\ A\ P_{A^l})^{-1}Q_{A^l},$$

for an arbitrary integer $l \geq k = \operatorname{ind}(A)$ and arbitrary full-rank factorization $A^l = P_{A^l}Q_{A^l}$ of the matrix power A^l.

Proof. Using the following useful property of the Moore–Penrose inverse from [7]: $(BCD)^\dagger = D^\dagger C^{-1}B^\dagger$, where B has full column rank, C is nonsingular and D has full row rank, we get

$$(A^{2l+1})^\dagger = (P_{A^l}Q_{A^l}AP_{A^l}Q_{A^l})^\dagger = (Q_{A^l})^\dagger(Q_{A^l}AP_{A^l})^{-1}(P_{A^l})^\dagger.$$

Applying this result to the known representation of the Drazin inverse from [7]: $A^D = A^l(A^{2l+1})^\dagger A^l$, we obtain the following:

$$\begin{aligned} A^D &= P_{A^l}Q_{A^l}(Q_{A^l})^\dagger(Q_{A^l}AP_{A^l})^{-1}(P_{A^l})^\dagger P_{A^l}Q_{A^l} \\ &= P_{A^l}(Q_{A^l}AP_{A^l})^{-1}Q_{A^l}. \qquad \square \end{aligned}$$

Remark 2.1. From [16] we can derive the following general representation for an arbitrary reflexive g-inverse X of A:

$$X = VQ^*(P^*UAVQ^*)^{-1}P^*U, \quad \operatorname{rank}(P^*UAVQ^*) = \operatorname{rank}(A).$$

This representation is equivalent to the following representation, introduced in [15]:

$$X = W_1(W_2AW_1)^{-1}W_2, \quad \operatorname{rank}(W_2AW_1) = \operatorname{rank}(A).$$

Therefore, the Drazin inverse and the class of reflexive g-inverses possess the same general form:

$$W_1(W_2AW_1)^{-1}W_2, \quad \text{where } W_2AW_1 \text{ is invertible .}$$

In the rest of this section we investigate representations of some expressions involving the Drazin inverse, in terms of an arbitrary full-rank factorization of rank invariant powers of A.

Theorem 2.2. *If A is an $n\times n$ complex matrix of index k and $A^l = P_{A^l}Q_{A^l}$ is an arbitrary full-rank decomposition of A^l, $l \geq k$, then*

(i) $(A^D)^l = P_{A^l}(Q_{A^l}A^lP_{A^l})^{-1}Q_{A^l} = P_{A^l}(Q_{A^l}P_{A^l})^{-2}P_{A^l}$;

(ii) $AA^D = P_{A^l}(Q_{A^l}P_{A^l})^{-1}Q_{A^l}$;

(iii) $(A^D)^\dagger = (Q_{A^l})^\dagger Q_{A^l}AP_{A^l}(P_{A^l})^\dagger$.

Proof. (i) Follows from $(A^D)^l = (A^l)^\#$ and the known representation of the group inverse: If $B = RS$ if a full-rank factorization of a square matrix B, rthen $B^\# = R(SR)^{-2}S$.

(ii) Using $AA^D = A^l(A^{2l})^\dagger A^l$ from [7], we obtain

$$AA^D = P_{A^l}Q_{A^l}(Q_{A^l})^\dagger(Q_{A^l}P_{A^l})^\dagger(P_{A^l})^\dagger P_{A^l}Q_{A^l} = P_{A^l}(Q_{A^l}P_{A^l})^{-1}Q_{A^l}.$$

(iii) P_{A^l} has full column rank, $Q_{A^l}P_{A^l}$ is invertible and Q_{A^l} has full row rank, and again applying the property $(BCD)^\dagger = D^\dagger C^{-1}B^\dagger$ from [7] we get

$$(A^D)^\dagger = \big(P_{A^l})(Q_{A^l}AP_{A^l})^{-1}Q_{A^l}\big)^\dagger = (Q_{A^l})^\dagger Q_{A^l}AP_{A^l}(P_{A^l})^\dagger \qquad \square$$

Remark 2.2. The properties of the Drazin inverse investigated in Theorem 2.3 can be verified using the canonical form representation of the Drazin inverse from [6, p. 122].

3. Determinantal representation of the Drazin inverse

In the following theorem we introduce a determinantal representation of the Drazin inverse. More precisely, elements of the Drazin inverse are expressed in terms of minors of the order $\operatorname{rank}(A^k)$, $k = \operatorname{ind}(A)$, taken from the matrices A and A^l, $\quad l \geq k$.

Theorem 3.1. *The Drazin inverse of an arbitrary matrix $A \in \mathbb{C}_r^{n\times n}$ possesses the following determinantal representation:*

$$\begin{aligned} A_{ij}^D &= \frac{\sum\limits_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)} |\,(A^l)_\alpha^\beta\,|\,\dfrac{\partial}{\partial a_{ji}}\,|\,A_\beta^\alpha\,|}{\sum\limits_{(\gamma,\delta)\in\mathcal{N}_{r_k}} |\,(A^l)_\gamma^\delta\,|\,|\,A_\delta^\gamma\,|} \\ &= \frac{\sum\limits_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)} |\,(A^l)_\alpha^\beta\,|\,\dfrac{\partial}{\partial a_{ji}}\,|\,A_\beta^\alpha\,|}{\operatorname{Tr}(C_{r_k}(A^{l+1}))}, \quad 1 \leq i,j \leq n, \end{aligned} \tag{3.1}$$

where $l \geq k = \operatorname{ind}(A)$ and $r_k = \operatorname{rank}(A^l)$.

Proof. Assume that $A = PQ$ is an arbitrary full-rank factorization of A, and $A^l = P_{A^l}Q_{A^l}$ is a full-rank factorization of A^l, $l \geq \operatorname{ind}(()A)$. According to Theorem 2.1, we get

$$A^D = \frac{P_{A^l}\operatorname{adj}(Q_{A^l}P\ \ QP_{A^l})Q_{A^l}}{|\,Q_{A^l}PQP_{A^l}\,|} \tag{3.2}$$

An application of the Cauchy-Binet theorem transforms the denominator in (3.2) as follows:

$$|\,Q_{A^l}PQP_{A^l}\,| = \sum_{\epsilon\in\mathcal{Q}_{r_k,r}} |\,(Q_{A^l}P)_\epsilon\,|\,|\,(QP_{A^l})^\epsilon\,| = \sum_{\epsilon\in\mathcal{Q}_{r_k,r}} |\,Q_{A^l}P_\epsilon\,|\,|\,Q^\epsilon P_{A^l}\,|$$

Another application of the Cauchy-Binet formula gives

$$|\,Q_{A^l}PQP_{A^l}\,| = \sum_{\epsilon\in\mathcal{Q}_{r_k,r}} \left(\sum_{\gamma\in\mathcal{Q}_{r_k,n}} |\,(Q_{A^l})_\gamma\,|\,|\,P_\epsilon^\gamma\,| \right) \left(\sum_{\delta\in\mathcal{Q}_{r_k,n}} |\,Q_\delta^\epsilon\,|\,|\,(P_{A^l})^\delta\,| \right)$$

Hence,

$$\begin{aligned}
|\,Q_{A^l}PQP_{A^l}\,| &= \sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}} |\,(A^l)^\delta_\gamma\,| \left(\sum_{\epsilon\in\mathcal{Q}_{r_k,r}} |\,P^\gamma_\epsilon\,|\,|\,Q^\epsilon_\delta\,| \right) \\
&= \sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}} |\,(A^l)^\delta_\gamma\,|\,|\,P^\gamma Q_\delta\,| = \sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}} |\,(A^l)^\delta_\gamma\,|\,|\,A^\gamma_\delta\,| \\
&= \sum_{\delta\in\mathcal{Q}_{r_k}} |\,(A^l\cdot A)^\delta_\delta\,| = \operatorname{Tr}(C_{r_k}(A^{l+1})).
\end{aligned}$$

Now we consider the overlying expression in (3.2). If the submatrix of A, generated by deleting the ith row of A (the jth column respectively), is denoted by $A^{\{i\}'}$ ($A_{\{j\}'}$ respectively), then we can write

$$(\operatorname{adj}(Q_{A^l}P\;\;QP_{A^l}))_{ij} = (-1)^{i+j}\,|\,(Q_{A^l})^{\{j\}'}P\;\;Q(P_{A^l})_{\{i\}'}\,|\,.$$

The Cauchy-Binet formula produces

$$\begin{aligned}
&(\operatorname{adj}(Q_{A^l}P\;\;QP_{A^l}))_{ij} = (-1)^{i+j} \sum_{\epsilon'\in\mathcal{Q}_{r_k-1,r}} |\,((Q_{A^l})^{\{j\}'}\cdot P)_{\epsilon'}\,|\,|\,(Q\cdot(P_{A^l})_{\{i\}'})^{\epsilon'}\,| \\
&= (-1)^{i+j} \sum_{\epsilon'\in\mathcal{Q}_{r_k-1,r}} |\,Q^{\epsilon'}\cdot(P_{A^l})_{\{i\}'}\,|\;\;|\,(Q_{A^l})^{\{j\}'}\cdot P_{\epsilon'}\,|\,.
\end{aligned}$$

Now, applying the Cauchy-Binet formula to both of the determinants contained in the last formula, we obtain

$$\begin{aligned}
(\operatorname{adj}(Q_{A^l}PQP_{A^l}))_{ij} = (-1)^{i+j} \sum_{\epsilon'\in\mathcal{Q}_{r_k-1,r}} &\left(\sum_{\beta'\in\mathcal{Q}_{r_k-1,n}} \left|Q^{\epsilon'}_{\beta'}\right| |((P_{A^l})_{\{i\}'})^{\beta'}| \right) \times \\
&\times \left(\sum_{\alpha'\in\mathcal{Q}_{r_k-1,n}} |P^{\alpha'}_{\epsilon'}|\,|((Q_{A^l})^{\{j\}'})_{\alpha'}| \right).
\end{aligned}$$

Therefore,

$$(\operatorname{adj}(Q_{A^l} PQP_{A^l}))_{ij} = \sum_{\epsilon' \in \mathcal{Q}_{r_k-1,r}} \left(\sum_{\beta' \in \mathcal{Q}_{r_k-1,n}} (-1)^i \, |\, Q^{\epsilon'}_{\beta'} \,|\, |\, ((P_{A^l})_{\{i\}'})^{\beta'} \,| \right) \times$$
$$\times \left(\sum_{\alpha' \in \mathcal{Q}_{r_k-1,n}} (-1)^j \, |\, P^{\alpha'}_{\epsilon'} \,|\, |\, ((Q_{A^l})^{\{j\}'})_{\alpha'} \,| \right).$$

Consequently,

$$(P_{A^l} \operatorname{adj}(Q_{A^l} PQP_{A^l}))_{ij} = \sum_{t=1}^{r_k} (P_{A^l})_{it} \, (\operatorname{adj}(Q_{A^l} P \; QP_{A^l}))_{tj}$$
$$= \sum_{\epsilon' \in \mathcal{Q}_{r_k-1,r}} \quad \sum_{\beta' \in \mathcal{Q}_{r_k-1,n}} \left| Q^{\epsilon'}_{\beta'} \right| \left(\sum_{t=1}^{r_k} (-1)^t (P_{A^l})_{it} \, |((P_{A^l})_{\{t\}'})^{\beta'} | \right) \times$$
$$\times \left(\sum_{\alpha' \in \mathcal{Q}_{r_k-1,n}} (-1)^j \, |\, P^{\alpha'}_{\epsilon'} \,|\, |\, ((Q_{A^l})^{\{j\}'})_{\alpha'} \,| \right)$$

If i is contained in the combination β', then

$$\sum_{t=1}^{r_k} (-1)^t (P_{A^l})_{it} \, |\, ((P_{A^l})_{\{t\}'})^{\beta'} \,| = 0.$$

If the set β' does not contain i, then $i = \beta_p$ and the system β' is denoted by

$$\beta' = \{1 \le \beta_1 < \ldots < \beta_{p-1} < \beta_{p+1} < \ldots < \beta_{r_k} \le n\}.$$

If the set β denotes the following combination:

$$\beta = \{1 \le \beta_1 < \ldots < \beta_{p-1} < i = \beta_p < \beta_{p+1} < \ldots < \beta_{r_k} \le n\}$$

we obtain the following representation for $(P_{A^l} \operatorname{adj}(Q_{A^l} P \;\; QP_{A^l}))_{ij}$:

$$\sum_{\epsilon' \in \mathcal{Q}_{r_k-1,r}} \left(\sum_{\beta \in \mathcal{J}_{r_k}(i)} (-1)^p \left| Q^{\epsilon'}_{\beta \setminus \{i\}} \right| |(P_{A^l})^{\beta}| \right) \left(\sum_{\alpha' \in \mathcal{Q}_{r_k-1,n}} (-1)^j |P^{\alpha'}_{\epsilon'}| \, |((Q_{A^l})^{\{j\}'})_{\alpha'}| \right).$$

Continuing in the same way, we get

$$
\begin{aligned}
&(P_{A^l}\operatorname{adj}(Q_{A^l}PQP_{A^l})Q_{A^l})_{ij} = \sum_{t=1}^{r_k}(P_{A^kl}\operatorname{adj}(Q_{A^l}PQP_{A^l}))_{it}\,(Q_{A^l})_{tj}\\
&= \sum_{\epsilon'\in\mathcal{Q}_{r_k-1,r}}\left(\sum_{\beta\in\mathcal{J}_{r_k}(i)}(-1)^p\left|Q^{\epsilon'}_{\beta\setminus\{i\}}\right|\,|\,(P_{A^l})^{\beta}\,|\right)\times\\
&\quad\times\left(\sum_{\alpha'\in\mathcal{Q}_{r_k-1,n}}|\,P^{\alpha'}_{\epsilon'}\,|\sum_{t=1}^{r_k}(-1)^t(Q_{A^l})_{tj}\,|\,((Q_{A^l})^{\{t\}'})_{\alpha'}\,|\right).
\end{aligned}
$$

Similarly, if j is contained in the combination α', then

$$
\sum_{t=1}^{r_k}(-1)^t(Q_{A^l})_{tj}\,|\,((Q_{A^l})_{\{t\}'})_{\alpha'}\,| = 0.
$$

Otherwise, $j=\alpha_q$ and the systems α' and α are equal to

$$
\begin{aligned}
\alpha' &= \{1\le\alpha_1<\ldots<\alpha_{q-1}<\alpha_{q+1}<\ldots<\alpha_{r_k}\le n\}\\
\alpha &= \{1\le\alpha_1<\ldots<\alpha_{q-1}<j=\alpha_q<\alpha_{q+1}<\ldots<\alpha_{r_k}\le n\}
\end{aligned}
$$

Therefore, the (i,j)th element of the matrix $P_{A^l}\operatorname{adj}(Q_{A^l}PQP_{A^l})Q_{A^l}$ is equal to

$$
\begin{aligned}
&\sum_{\epsilon'\in\mathcal{Q}_{r_k-1,r}}\left(\sum_{\beta\in\mathcal{J}_{r_k}(i)}(-1)^p\left|Q^{\epsilon'}_{\beta\setminus\{i\}}\right|\left|(P_{A^l})^{\beta}\right|\right)\left(\sum_{\alpha\in\mathcal{J}_{r_k}(j)}(-1)^q\left|P^{\alpha\setminus\{j\}}_{\epsilon'}\right|\left|(Q_{A^l})_{\alpha}\right|\right)\\
&= \sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)}|\,(A^l)^{\beta}_{\alpha}\,|\sum_{\epsilon'\in\mathcal{Q}_{r_k-1,r}}(-1)^{p+q}\left|P^{\alpha\setminus\{j\}}_{\epsilon'}\right|\left|Q^{\epsilon'}_{\beta\setminus\{i\}}\right|\\
&= \sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)}|\,(A^l)^{\beta}_{\alpha}\,|\,(-1)^{p+q}|\,P^{\alpha\setminus\{j\}}\,|\,|\,Q_{\beta\setminus\{i\}}\,|\\
&= \sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)}|\,(A^l)^{\beta}_{\alpha}\,|\,\frac{\partial}{\partial a_{ji}}\,|\,A^{\alpha}_{\beta}\,|. \qquad \square
\end{aligned}
$$

Corollary 3.1. *The determinantal representation of an arbitrary element of the Drazin inverse of a given matrix $A\in\mathbb{C}^{n\times n}_r$ possesses the form*

$$A^D_{ij} = \sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)} \lambda_{\alpha,\beta} \frac{\partial}{\partial a_{ji}} \,|\, A^\alpha_\beta \,|, \quad 1 \leq i,j \leq n, \tag{3.3}$$

where the matrix $\Lambda = (\lambda_{\alpha,\beta})$ *satisfies the following conditions*

$$\operatorname{rank}(\Lambda) = 1, \qquad \sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}} \lambda_{\alpha,\beta} \,|\, A^\alpha_\beta \,| = 1. \tag{3.4}$$

Proof. Indeed, for arbitrary $l \geq k = \operatorname{ind}(A)$ we can use

$$\begin{aligned}
\Lambda &= (|\, Q_{A^l} A P_{A^l} \,|)^{-1} \, C_{r_k}\left((A^l)^T\right) \\
&= \left(\sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}} |\,(A^l)^\delta_\gamma \,|\,|\, A^\gamma_\delta \,| \right)^{-1} C_{r_k}\left((A^l)^T\right) \\
&= \left(\operatorname{Tr}(C_{r_k}(A^{l+1})) \right)^{-1} C_{r_k}\left((A^l)^T\right). \qquad \square
\end{aligned}$$

Remark 3.1. (i) In the case $l = \operatorname{ind}(A) = 1$, the result of Theorem 3.1 reduces to the known determinantal representation of the group inverse, introduced in [13].

(ii) Representation of the Drazin inverse, given in Corollary 3.1, is similar to the following representation of the reflexive g-inverses from [12]:

Proposition 3.1. *Let* $A \in \mathbb{C}^{m\times n}_r$. *Then* $G = (g_{ij})$ *is a reflexive* g*-inverse of* A *if and only if*

$$g_{ij} = \sum_{(\alpha,\beta)\in\mathcal{N}(j,i)} \lambda_{\alpha,\beta} \frac{\partial}{\partial a_{ji}} \,|\, A^\alpha_\beta \,|,$$

where $\lambda_{\alpha,\beta} \in \mathbb{C}$ *and* $(\alpha,\beta) \in \mathcal{N}$ *satisfy*

$$\sum_{(\alpha,\beta)\in\mathcal{N}} \lambda_{\alpha,\beta} \,|\, A^\alpha_\beta \,| = 1,$$

and the rank of the matrix $\Lambda = (\lambda_{\alpha,\beta})$ *is*

$$\operatorname{rank}(\Lambda) = 1.$$

(iii) The generalized inverses over an arbitrary field are investigated in [8] and [9]. In these papers it is allowed to use minors of the order $h \leq \operatorname{rank}(A)$

from a given matrix A in the known determinantal representation of the Moore-Penrose inverse. The order h of the minors is the greatest integer satisfying

$$N_h(A) = \sum_{(\alpha,\beta)\in N_h} |\,\overline{A}_\beta^\alpha\,|\,|\,A_\beta^\alpha\,| \neq 0.$$

The algebraic complement of the order h, corresponding to a_{ij} is defined by

$$A_{ij}^{(h)} = \sum_{(\alpha,\beta)\in N_h(j,i)} |\,\overline{A}_\beta^\alpha\,| \frac{\partial}{\partial a_{ij}} |\,A_\beta^\alpha\,|\,.$$

The result of Theorem 3.1 is a continuation of these results in the following sense: the determinantal representation of the Drazin inverse can be generated in the case $h = r_k = \operatorname{rank}(A^l)$, $l \geq k = \operatorname{ind}(A)$, substituting the minors $|\,\overline{A}_\beta^\alpha\,|$ of the order h by the minors of the order r_k from $(A^l)^T$.

In the light of Proposition 3.1 , it seems interesting to state the following problem, representing the dual result to Corollary 3.1.

Problem 3.1. *If each element of the matrix X can be represented by* (3.3) *and* (3.4)*, then does it follow that $X = A^D$?*

In the following theorem we introduce a determinantal representation of a few expressions involving the Drazin inverse.

Theorem 3.2. *For a given matrix $A \in \mathbb{C}_r^{n\times n}$ with the Drazin inverse of the index k, we obtain the following determinantal representations:*

(i)
$$\begin{aligned}(AA^D)_{ij} &= \frac{\sum\limits_{\alpha\in\mathcal{J}_{r_k}(i),\ j\notin\alpha} |\,(Q_{A^l})_\alpha\,|\,|\,(P_{A^l})^\alpha\,|}{\sum\limits_{\gamma\in\mathcal{Q}_{r_k,n}} |\,(Q_{A^l})_\gamma\,|\,|\,(P_{A^l})^\gamma\,|} \\ &= \frac{\sum\limits_{\alpha\in\mathcal{J}_{r_k}(i),\ j\notin\alpha} |\,(Q_{A^l})_\alpha\,|\,|\,(P_{A^l})^\alpha\,|}{\operatorname{Tr}(C_{r_k}(A^l))}, \quad 1 \leq i,j \leq n.\end{aligned}$$

(ii)
$$\begin{aligned}(A^D(A^D)^\dagger)_{ij} &= \frac{\sum\limits_{\alpha\in\mathcal{J}_{r_k}(i),\ j\notin\alpha} |\,(\overline{P_{A^l}})^\alpha\,|\,|\,(P_{A^l})^\alpha\,|}{\sum\limits_{\gamma\in\mathcal{Q}_{r_k,n}} |\,(\overline{P_{A^l}})^\gamma\,|\,|\,(P_{A^l})^\gamma\,|}\\ &= \frac{\sum\limits_{\alpha\in\mathcal{J}_{r_k}(i),\ j\notin\alpha} |\,(\overline{P_{A^l}})^\alpha\,|\,|\,(P_{A^l})^\alpha\,|}{\mathrm{Tr}(C_{r_k}(P_{A^l}(P_{A^l})^*))}, \quad 1\le i,j\le n.\end{aligned}$$

(iii)
$$\begin{aligned}((A^D)^\dagger A^D)_{ij} &= \frac{\sum\limits_{\alpha\in\mathcal{J}_{r_k}(i),\ j\notin\alpha} |\,(\overline{Q_{A^l}})_\alpha\,|\,|\,(P_{A^l})_\alpha\,|}{\sum\limits_{\gamma\in\mathcal{Q}_{r_k,n}} |\,(\overline{Q_{A^l}})_\gamma\,|\,|\,(Q_{A^l})_\gamma\,|}\\ &= \frac{\sum\limits_{\alpha\in\mathcal{J}_{r_k}(i),\ j\notin\alpha} |\,(\overline{Q_{A^l}})_\alpha\,|\,|\,(P_{A^l})_\alpha\,|}{\mathrm{Tr}(C_{r_k}((Q_{A^l})^*Q_{A^l}))}, \quad 1\le i,j\le n.\end{aligned}$$

Proof. (i) We start from the representation $AA^D = P_{A^l}(Q_{A^l}P_{A^l})^{-1}Q_{A^l}$, developed in the part (ii) of Theorem 2.3. Using the methods from [8], it is an exercise to prove

$$\left((Q_{A^l}P_{A^l})^{-1}Q_{A^l}\right)_{tj} = \frac{\sum\limits_{\alpha\in\mathcal{J}_{r_k}(j)} |\,(Q_{A^l})_\alpha\,|\,\dfrac{\partial}{\partial(P_{A^l})_{jt}}\,|\,(P_{A^l})^\alpha\,|}{\sum\limits_{\gamma\in\mathcal{Q}_{r_k,n}} |\,(Q_{A^l})_\gamma\,|\,|\,(P_{A^l})^\gamma\,|}, \quad \left(\begin{smallmatrix}1\le t\le r_k\\ 1\le j\le n\end{smallmatrix}\right).$$

Now, for arbitrary $1\le i,j\le n$ we get

$$\begin{aligned}(AA^D)_{ij} &= \sum_{t=1}^{r_k}(P_{A^l})_{it}\left((Q_{A^l}P_{A^l})^{-1}Q_{A^l}\right)_{tj}\\ &= \frac{\sum\limits_{\alpha\in\mathcal{J}_{r_k}(j),\ i\notin\alpha} |\,(Q_{A^l})_\alpha\,|\,\sum\limits_{t=1}^{r_k}(P_{A^l})_{it}\dfrac{\partial}{\partial(P_{A^l})_{jt}}\,|\,(P_{A^l})^\alpha\,|}{\sum\limits_{\gamma\in\mathcal{Q}_{r_k,n}} |\,(Q_{A^l})_\gamma\,|\,|\,(P_{A^l})^\gamma\,|}\\ &= \frac{\sum\limits_{\alpha\in\mathcal{J}_{r_k}(i),\ j\notin\alpha} |\,(Q_{A^l})_\alpha\,|\,|\,(P_{A^l})^\alpha\,|}{\sum\limits_{\gamma\in\mathcal{Q}_{r_k,n}} |\,(Q_{A^l})_\gamma\,|\,|\,(P_{A^l})^\gamma\,|}.\end{aligned}$$

In the rest of this part of the proof we use the following:

$$\sum_{\gamma\in\mathcal{Q}_{r_k,n}} |\,(Q_{A^l})_\gamma\,|\,|\,(P_{A^l})^\gamma\,| = \sum_{\gamma\in\mathcal{Q}_{r_k,n}} |\,(A^l)_\gamma^\gamma\,| = \mathrm{Tr}(C_{r_k}(A^l)).$$

(ii), (iii) Using the known result from [7]:

$$A^D(A^D)^\dagger = A^l(A^l)^\dagger, \qquad (A^D)^\dagger A^D = (A^l)^\dagger A^l,$$

we get

$$\begin{aligned} A^D(A^D)^\dagger &= P_{A^l}(P_{A^l})^\dagger = P_{A^l}\left((P_{A^l})^* P_{A^l}\right)^{-1}(P_{A^l})^* \\ (A^D)^\dagger A^D &= (Q_{A^l})^\dagger Q_{A^l} = (Q_{A^l})^*\left(Q_{A^l}(Q_{A^l})^*\right)^{-1} Q_{A^l}. \end{aligned}$$

Now, the proof can be completed in the same way as in the part (i) $\square$

It seems interesting to replace the components P_{A^l} and Q_{A_l} from the full-rank factorization $A^l = P_{A^l}Q_{A_l}$ by two appropriate full rank matrices $P_S \in \mathbb{C}_{r_k}^{m\times r_k}$ and $Q_T \in \mathbb{C}_{r_k}^{r_k\times n}$. This idea leads to the following determinantal representation of the subclass of $\{2\}$-inverses.

Theorem 3.3. *Assume that $A \in \mathbb{C}_r^{m\times n}$ has the index k and $r_k = \mathrm{rank}(A^k)$. Let $P_S \in \mathbb{C}_{r_k}^{m\times r_k}$ and $Q_T \in \mathbb{C}_{r_k}^{r_k\times n}$ satisfy $\mathrm{rank}(Q_T A P_S) = r_k$. Then the matrix G, defined by*

$$\begin{aligned} G_{ij} &= \frac{\displaystyle\sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)} |\,(P_SQ_T)_\alpha^\beta\,|\,\frac{\partial}{\partial a_{ji}}\,|\,A_\beta^\alpha\,|}{\displaystyle\sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}} |\,(P_SQ_T)_\gamma^\delta\,|\,|\,A_\delta^\gamma\,|} \\ &= \frac{\displaystyle\sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)} |\,(P_SQ_T)_\alpha^\beta\,|\,\frac{\partial}{\partial a_{ji}}\,|\,A_\beta^\alpha\,|}{\mathrm{Tr}(C_{r_k}(P_SQ_TA))}, \quad 1\le i\le n;\ 1\le j\le m \end{aligned}$$

satisfies $G \in A\{2\}$.

Proof. Using the principles from Theorem 3.1, one can verify

$$G = P_S(Q_TAP_S)^{-1}Q_T.$$

Consider two full rank matrices $Q_S \in \mathbb{C}_{r_k}^{r_k \times n}$ and $P_T \in \mathbb{C}_{r_k}^{n \times r_k}$, such that $\operatorname{rank}(Q_T A P_S) = r_k$. Then $P_S Q_S$ is the full-rank factorization of the matrix $S = P_S Q_S \in \mathbb{C}_{r_k}^{n \times n}$ and $P_T Q_T$ is the full-rank factorization of the matrix $T = P_T Q_T \in \mathbb{C}_{r_k}^{n \times n}$. According to the known result about $\{1,2\}$-inverses and full-rank factorizations from [5], we get

$$G = P_S Q_S Q_S^{(1,2)} (Q_T A P_S)^{-1} P_T^{(1,2)} P_T Q_T = S(TAS)^{(1,2)} T.$$

Using the general solution of the equation (2) from [16, p. 56], it is easy to conclude that $G \in A\{2\}$. $\square$

Example 3.1. Let us consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 2 \\ 1 & -1 & 1 & 1 \end{bmatrix}.$$

It is easy to verify the following: $r = \operatorname{rank}(A) = 3$, $\operatorname{rank}(A^2) = \operatorname{rank}(A^3) = 2$. This implies $k = \operatorname{ind}(A) = 2$, $r_k = \operatorname{rank}(A^2) = 2$.

Let the matrices P_S and Q_T be chosen as follows:

$$P_S = \begin{bmatrix} 1 & -5 \\ 1 & 1 \\ -2 & -8 \\ 1 & 7 \end{bmatrix}, \qquad Q_T = \begin{bmatrix} 1 & -\frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix}.$$

Applying the result of Theorem 3.3 we obtain the following $\{2\}$-inverse of A:

$$X = \begin{bmatrix} 0 & 1 & 1 & 3 \\ -\frac{1}{10} & \frac{1}{2} & \frac{2}{5} & \frac{11}{10} \\ \frac{3}{10} & -\frac{1}{2} & -\frac{1}{5} & -\frac{3}{10} \\ -\frac{1}{5} & 0 & -\frac{1}{5} & -\frac{4}{5} \end{bmatrix}.$$

In the case

$$P_S = P_{A^2} = \begin{bmatrix} -4 & 3 \\ 1 & 0 \\ -5 & 4 \\ -4 & 3 \end{bmatrix}, \qquad Q_T = Q_{A^2} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

we obtain the following Drazin inverse of A:

$$A^D = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{7}{4} & \frac{5}{2} & -\frac{5}{2} & \frac{7}{4} \\ \frac{5}{4} & \frac{3}{2} & -\frac{3}{2} & \frac{5}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The determinantal representation of the reflexive g-inverses, introduced in [2] can be derived as a consequence of Theorem 3.3.

Corollary 3.2. *The class of reflexive g-inverses of $A \in \mathbb{C}_r^{m\times n}$ is characterized by the following determinantal formula*

$$G_{ij} = \frac{\sum\limits_{(\alpha,\beta)\in\mathcal{N}(j,i)} |\, H_\alpha^\beta \,| \dfrac{\partial}{\partial a_{ji}} |\, A_\beta^\alpha \,|}{\sum\limits_{(\gamma,\delta)\in\mathcal{N}} |\, H_\gamma^\delta \,|\,|\, A_\delta^\gamma \,|}, \quad 1 \le i \le n;\ \ 1 \le j \le m$$

where $H \in \mathbb{C}_r^{m\times n}$ satisfies $\operatorname{rank}(AH) = \operatorname{rank}(HA) = r$.

Proof. Using the following known general solution of the system of equations (1), (2) from [15], [16]:

$$G = W_1(W_2AW_1)^{-1}W_2, \quad W_1 \in \mathbb{C}_r^{n\times r}, \quad W_2 \in \mathbb{C}_r^{r\times n}$$

we conclude that G can be derived from the set of $\{2\}$-inverses, defined in Theorem 3.3, for the case $r_k = r$, $P_S = W_1$, $Q_T = W_2$. This implies the following determinantal representation for G:

$$G_{ij} = \frac{\sum\limits_{(\alpha,\beta)\in\mathcal{N}(j,i)} |\, (W_1W_2)_\alpha^\beta \,| \dfrac{\partial}{\partial a_{ji}} |\, A_\beta^\alpha \,|}{\sum\limits_{(\gamma,\delta)\in\mathcal{N}} |\, (W_1W_2)_\gamma^\delta \,|\,|\, A_\delta^\gamma \,|}, \quad 1 \le i \le n;\ \ 1 \le j \le m$$

Now, W_1, W_2 can be considered as the factors of the full-rank factorization of the matrix $H = W_1W_2 \in \mathbb{C}_r^{m\times n}$. In [2] it is proved that the conditions $\operatorname{rank}(AH) = \operatorname{rank}(HA) = r$ are equivalent to $\operatorname{rank}(W_2AW_1) = r$, which ensures invertibility of the matrix W_2AW_1. □

Now we give a few necessary and sufficient conditions for the existence of the Drazin inverse, for matrices whose elements are taken from an integral domain $\mathbb{I}$. These results are additional with respect to the conditions for the existence of the Drazin inverse, proposed in [13]. Derivation of the determinantal formula for the Drazin inverse can be transferred from the set of complex matrices. Remark that in [13] the Drazin inverse is investigated only through the representation $A^D = A^k(A^{2k+1})^{(1)}A^k$, where $k = \operatorname{ind}(A)$ and $(A^{2k+1})^{(1)}$ stands for an arbitrary g-inverse of A.

Theorem 3.4. *Let A be an $n\times n$ matrix of rank r over $\mathbb{I}$, $A = PQ$ be a full-rank factorization of A. Let $k = \operatorname{ind}(A)$ and assume that l is an arbitrary integer satisfying $l \geq k$. Consider the sequences P_i, Q_i, $i = 1,\dots,l$ defined as follows:*

$$A=PQ=P_1Q_1, \quad P = P_1 \in \mathbb{C}_{r_1}^{n\times r_1},\ Q = Q_1 \in \mathbb{C}_{r_1}^{r_1\times n},$$

$$Q_iP_i = P_{i+1}Q_{i+1}, \quad P_{i+1} \in \mathbb{C}_{r_{i+1}}^{r_i\times r_{i+1}},\ Q_{i+1} \in \mathbb{C}_{r_{i+1}}^{r_{i+1}\times r_i}, \quad i=1,\dots,l$$

such that $P_{i+1}Q_{i+1}$ is a full-rank factorization for Q_iP_i, $i = 1,\dots,l$. Then $A^l = P_{A^l}Q_{A^l}$ is the full-rank factorization of A^l, where $P_{A^l} = P_1\cdots P_l$, $Q_{A^l} = Q_l\cdots Q_1$, $l \geq k$. Also, the following conditions are equivalent:

(i) *A^D exists, and $k = \operatorname{ind}(A) < \infty$.*

(ii) *Q_kP_k is the invertible matrix over $\mathbb{I}$.*

(iii) *$Q_{A^k}AP_{A^k}$ is the invertible matrix over $\mathbb{I}$.*

(iv) *$u = \sum_{(\gamma,\delta)\in\mathcal{N}_{r_k}} |(A^k)_\gamma^\delta||A_\delta^\gamma| = \operatorname{Tr}(C_{r_k}(A^{k+1}))$ is invertible in $\mathbb{I}$.*

Moreover, the Drazin inverse, if it exists, is given by the following full-

rank and determinantal representations

$$A^D = P_{A^l}(Q_{A^l} A P_{A^l})^{-1} Q_{A^l};$$

$$(A^D)_{ij} = u^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)} |\,(A^l)^{\beta}_{\alpha}\,| \frac{\partial}{\partial a_{ji}} |\,A^{\alpha}_{\beta}\,| \tag{3.5}$$

Proof. (i) $\Leftrightarrow$ (ii): This is a known result [7, Theorem 1]

(ii) $\Leftrightarrow$ (iii): The matrix $Q_k P_k$ is invertible if and only if $(Q_k P_k)^{k+1} = Q_{A^k} A P_{A^k}$ is invertible.

(iii) $\Leftrightarrow$ (iv): A square matrix over a ring is invertible if and only if its determinant is invertible in the ring [10], [11]. Hence, $Q_{A^k} A P_{A^k}$ is invertible matrix in $\mathbb{I}$ if and only if $|\,Q_{A^k} A P_{A^k}\,|$ is invertible in $\mathbb{I}$. From Theorem 3.1 we obtain

$$|\,Q_{A^k} A P_{A^k}\,| = \sum_{(\gamma,\delta)\in\mathcal{N}_k} |\,(A^k)^{\delta}_{\gamma}\,|\,|\,A^{\gamma}_{\delta}\,| = \operatorname{Tr}(C_{r_k}(A^{k+1}))$$

and complete this part of the proof.

(iii) $\Rightarrow$ (i): If $Q_{A^k} A P_{A^k}$ is invertible, then $\operatorname{rank}(Q_{A^k} A P_{A^k}) = r_k$. This means the following:

$$\operatorname{rank}(P_{A^k}) = \operatorname{rank}(Q_{A^k}) = r_k$$

Consequently,

$$\operatorname{rank}(A^k) = \operatorname{rank}(P_{A^k} Q_{A^k}) \le r_k \tag{3.6}$$

Also, using

$$\begin{aligned}\operatorname{Tr}(C_{r_k}(Q_{A^k} A P_{A^k})) &= \operatorname{Tr}(C_{r_k}(Q_{A^k}) \cdot C_{r_k}(A) \cdot C_{r_k}(P_{A^k})) \\ &= \operatorname{Tr}(C_{r_k}(A) \cdot C_{r_k}(P_{A^k}) \cdot C_{r_k}(Q_{A^k})) \\ &= \operatorname{Tr}(C_{r_k}(A P_{A^k} Q_{A^k})) = \operatorname{Tr}(C_{r_k}(A^{k+1}))\end{aligned}$$

we conclude $\operatorname{rank}(A^{k+1}) = \operatorname{rank}(Q_{A^k} A P_{A^k}) = r_k$, which means $\operatorname{rank}(A^k) \ge r_k$. Using this result and (3.6) we conclude $r_k = \operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$, which means $k = \operatorname{ind}(A) < \infty$.

Finally, the representations (3.5) can be developed applying the method used in the set of complex matrices. □

As an easy consequence of Theorem 3.4 we obtain the following result related to the conditions for the existence and representations of the Drazin inverse. Some of these conditions are complementary with respect to the results from [13].

Corollary 3.3. *Let A be an $n \times n$ matrix of rank r over $\mathbb{I}$ and $A = PQ$ be its full-rank factorization of A. Then the following conditions are equivalent:*

(i) $A^{\#}$ *exists.*

(ii) QP *is the invertible matrix over* $\mathbb{I}$*.*

(iii) QAP *is the invertible matrix over* $\mathbb{I}$*.*

(iv) $u = \sum\limits_{(\gamma,\delta)\in\mathcal{N}} | A_\gamma^\delta | \, | A_\delta^\gamma | = \left(\sum\limits_{\gamma\in\mathcal{Q}_{r,n}} | A_\gamma^\gamma | \right)^2 = \operatorname{Tr}(C_r(A^2))$ *is invertible in* $\mathbb{I}$*.*

Moreover, the group inverse, if it exists, is given by the following general and determinantal representation:

$$A^D = P(QAP)^{-1}Q = P(QP)^{-2}Q;$$

$$(A^{\#})_{ij} = u^{-1} \sum_{(\alpha,\beta)\in\mathcal{N}(j,i)} | A_\alpha^\beta | \frac{\partial}{\partial a_{ji}} | A_\beta^\alpha | \tag{3.7}$$

Remark 3.1. Equivalence of the conditions (i) and (iv) and the determinantal representation of the group inverse are the known results in [13].

Using $(A^D)^l = (A^l)^{\#}$ from [7], the known results about the existence and the determinantal representation of the group inverse from [13], together with the results of Corollary 3.3, we obtain a few conditions for the existence of $(A^D)^l$ and its determinantal representation.

Corollary 3.4. *Let A be an $n \times n$ matrix over $\mathbb{I}$, such that $\rho(A^l) = r_k$, where $l \geq \operatorname{ind}(A)$ is an arbitrary integer. Then the following conditions are equivalent:*

(i) $(A^D)^l$ *exists.*

(ii) $C_{r_k}(A^l)$ *has a group inverse.*

(iii) $u = \left(\sum_{\gamma \in \mathcal{Q}_{r_k,n}} |\,(A^l)^\gamma_\gamma\,| \right)^2 = \mathrm{Tr}(C_r(A^{2l}))$ *is invertible in* $\mathbb{I}$.

(iv) $\rho(A^l) = \rho(A^{2l})$ *and* A^{2l} *is regular.*

(v) $Q_{A^l}P_{A^l}$ *is the invertible matrix over* $\mathbb{I}$.

(vi) $Q_{A^l}AP_{A^l}$ *is the invertible matrix over* $\mathbb{I}$.

Moreover, if $(A^D)^l$, *exists, then it is given by*

$$\left((A^D)^l\right)_{ij} = u^{-2} \sum_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)} |\,(A^l)^\beta_\alpha\,|\, \frac{\partial}{\partial (A^l)_{ji}}\, |\,(A^l)^\alpha_\beta\,|$$

4. Minors of the Drazin inverse

Now we examine properties of the minors of the Drazin inverse. Our motivation for this investigation is based on the following results. It is well known that if A is an $m\times n$ matrix of rank r over an integral domain $\mathbb{I}$, then $A^\dagger$ and A^* have the proportional $r \times r$ minors [1]. In [2] it is proved that A admits a g-inverse whose $r \times r$ minors are proportional to the corresponding minors of a given $m \times n$ matrix H if and only if $\mathrm{Tr}(C_r(AH))$ is invertible. Similarly, if A is an $n \times n$ matrix, then $A^\#$ and A have the proportional minors of the order r [13]. In Theorem 4.4 we generalize the result from [13], and show that the Drazin inverse A^D and A^l, $l \geq k = \mathrm{ind}(A)$ have the proportional minors of the order $r_k = \mathrm{rank}(A^k)$. Firstly we prove several useful relations between the minors selected from the powers of the Drazin inverse and the corresponding minors taken from the Drazin inverse.

Lemma 4.1. *Let* $A \in \mathbb{C}^{n\times n}_r$ *has a Drazin inverse* A^D *of index* k *and* $r_k = \mathrm{rank}(A^k)$. *Then for each* $p \geq 1$ *the following two identities are valid:*

(i) $C_{r_k}((A^D)^{p+1}) = C_{r_k}((A^D)^p)\,\mathrm{Tr}(C_{r_k}(A^D))$;

(ii) $C_{r_k}((A^D)^p) = C_{r_k}(A^D)\left[\mathrm{Tr}(C_{r_k}(A^D))\right]^{p-1}$;

(iii) *There exist constants* c_i, d_i, $1 \le i \le \binom{n}{r_k}$, *such that*

$$\left(C_{r_k}((A^D)^p)\right)_{ij} = c_i d_j \cdot \left(C_{r_k}((A^D)^p)\right)_{11}, \quad 1 \le i,j \le \binom{n}{r_k}.$$

Proof. The proof proceeds by the induction on p. Indeed, in the case $p=1$ the statement (ii) is evident. Since $\mathrm{rank}(A^D) = \mathrm{rank}(A^k)$ [7], we conclude $\mathrm{rank}(C_{r_k}(A^D)) = 1$. Consequently, there exist constants c_i, d_j which satisfy

$$\left(C_{r_k}(A^D)\right)_{ij} = c_i d_j \left(C_{r_k}(A^D)\right)_{11}, \quad 1 \le i,j \le \binom{n}{r_k}.$$

Hence, (iii) is also satisfied in the case $p = 1$.

Finally, the statement (i) can be verified for the case $p = 1$ as follows:

$$\begin{aligned}
\left(C_{r_k}((A^D)^2)\right)_{ij} &= \sum_{t=1}^{\binom{n}{r_k}} \left(C_{r_k}(A^D)\right)_{it} \left(C_{r_k}(A^D)\right)_{tj} \\
&= \sum_{t=1}^{\binom{n}{r_k}} c_i d_t \left(C_{r_k}(A^D)\right)_{11} c_t d_j \left(C_{r_k}(A^D)\right)_{11} \\
&= c_i d_j \left(C_{r_k}(A^D)\right)_{11} \sum_{t=1}^{\binom{n}{r_k}} c_t d_t \left(C_{r_k}(A^D)\right)_{11} \\
&= \left(C_{r_k}(A^D)\right)_{ij} \sum_{t=1}^{\binom{n}{r_k}} \left(C_{r_k}(A^D)\right)_{tt} \\
&= \left(C_{r_k}(A^D)\right)_{ij} \mathrm{Tr}(C_{r_k}(A^D)).
\end{aligned}$$

Suppose that the identities (i), (ii) and (iii) are already true for any number p less than u. We prove (i), (ii) and (iii) for $p = u+1$. Indeed, using $C_{r_k}((A^D)^{p+1}) = C_{r_k}((A^D)^p) C_{r_k}(A^D)$ and the inductive hypothesis for (i) in the case $p = u$, we get

$$\begin{aligned}
C_{r_k}((A^D)^{u+1}) &= C_{r_k}(A^D) C_{r_k}((A^D)^u) \\
&= C_{r_k}(A^D) C_{r_k}((A^D)^{u-1}) \,\mathrm{Tr}(C_{r_k}(A^D)) \\
&= C_{r_k}((A^D)^u) \,\mathrm{Tr}(C_{r_k}(A^D)).
\end{aligned}$$

Therefore, the statement (i) is valid.

An application of the proposition (ii) for the case $p = u$ in the statement (i) gives us that

$$C_{r_k}((A^D)^{u+1}) = C_{r_k}(A^D)\left[\operatorname{Tr}(C_{r_k}(A^D))\right]^{u-1}\operatorname{Tr}(C_{r_k}(A^D)),$$

which confirms (ii) for $p = u + 1$.

Lastly, from the inductive hypothesis for (iii) we obtain

$$\begin{aligned}\left(C_{r_k}((A^D)^{u+1})\right)_{ij} &= \left(C_{r_k}((A^D)^u)\right)_{ij}\operatorname{Tr}(C_{r_k}(A^D))\\ &= c_i d_j\left(C_{r_k}((A^D)^u)\right)_{11}\operatorname{Tr}(C_{r_k}(A^D)).\end{aligned}\tag{4.1}$$

Using (i) and (4.1) we obtain

$$\left(C_{r_k}((A^D)^{u+1})\right)_{ij} = c_i d_j\left(C_{r_k}((A^D)^{u+1})\right)_{11}.$$

This verifies (iii) in the case $p = u + 1$ and completes the proof. □

Theorem 4.1. *The Drazin inverse, the powers of the Drazin inverse of a given matrix $A \in \mathbb{C}_r^{n\times n}$ and the matrix power A^k, $k = \operatorname{ind}(A)$, have proportional minors of the order $r_k = \operatorname{rank}(A^k)$, as follows:*

(i) $C_{r_k}(A^D) = \left[\operatorname{Tr}(C_{r_k}(A^D))\right]^{1-k}\left[\operatorname{Tr}(C_{r_k}(A^k))\right]^{-2} C_{r_k}(A^k);$

(ii) $C_{r_k}((A^D)^p) = \left[\operatorname{Tr}(C_{r_k}(A^D))\right]^{p-k}\left[\operatorname{Tr}(C_{r_k}(A^k))\right]^{-2} C_{r_k}(A^k);$

(iii) $C_r((A^{\#})^p) = \left[\operatorname{Tr}\left(\left[\operatorname{Tr}(C_r(A))\right]^{-2} C_r(A)\right)\right]^{p-1}\left[\operatorname{Tr}(C_r(A))\right]^{-2} C_r(A),$
where $p \geq 1$ is an arbitrary integer.

Proof. (i) Applying the known result from [7]: $(A^D)^k = (A^k)^{\#}$, we obtain

$$C_{r_k}((A^D)^k) = C_{r_k}((A^k)^{\#}).$$

Using $C_{r_k}((A^k)^{\#}) = \left(C_{r_k}(A^k)\right)^{\#}$ and $C_{r_k}((A^D)^k) = \left(C_{r_k}(A^D)\right)^k$ we get

$$\left(C_{r_k}(A^k)\right)^{\#} = \left((C_{r_k}(A))^D\right)^k.\tag{4.2}$$

Now, using $(C_{r_k}(A))^k = C_{r_k}(A^k)$ and the following known result about the group inverse from [13]:

$$\operatorname{Tr}(C_r(A^{\#})) = \left[\operatorname{Tr}(C_r(A))\right]^{-2} C_r(A),\tag{4.3}$$

we can write

$$((C_{r_k}(A))^k)^{\#} = (C_{r_k}(A^k))^{\#} = \left[\operatorname{Tr}(C_{r_k}(A^k))\right]^{-2} C_{r_k}(A^k), \tag{4.4}$$

It is easy to show that $(C_{r_k}(A))^D = C_{r_k}(A^D)$. Using this result together with (4.2) and (4.4), we obtain

$$((C_{r_k}(A))^D)^k = (C_{r_k}(A^D))^k = C_{r_k}\left((A^D)^k\right) = \left[\operatorname{Tr}(C_{r_k}(A^k))\right]^{-2} C_{r_k}(A^k).$$

Consequently, $(A^D)^k$ and A^k have proportional minors of the order r_k. Now, applying proposition (i) of Lemma 4.1, we get

$$C_{r_k}(A^D) = \left[\operatorname{Tr}(C_{r_k}(A^D))\right]^{1-k} \left[\operatorname{Tr}(C_{r_k}(A^k))\right]^{-2} C_{r_k}(A^k).$$

(ii) Follows from the just proved result and from the part (i) of Lemma 4.1.

(iii) In the case $k = 1$ an application of (4.3) to the result (i) of Lemma 4.1 leads to $\operatorname{Tr}(C_r(A^{\#})) = [\operatorname{Tr}(C_r(A))]^{-2} C_r(A)$, and produce the following

$$\begin{aligned} C_r((A^{\#})^p) &= \left[\operatorname{Tr}(C_r(A^{\#}))\right]^{p-1} [\operatorname{Tr}(C_r(A))]^{-2} C_r(A) \\ &= \left[\operatorname{Tr}\Big([\operatorname{Tr}(C_r(A))]^{-2} C_r(A)\Big)\right]^{p-1} [\operatorname{Tr}(C_r(A))]^{-2} C_r(A). \qquad \square \end{aligned}$$

Remark 4.1. (i) In the case $k = \operatorname{ind}(A) = 1$, part (i) of Theorem 3.3 produces the known result (4.3). This result is also obtained in the case $p = 1$ from the part (iii) of Theorem 3.3.

(ii) From Theorem 3.3 and Theorem 4.1 we conclude that G is the Drazin inverse of A if and only if the following two statements are valid: (i) G is a $\{2\}$-inverse of A and (ii) G and A^k, $k = \operatorname{ind}(A)$, have proportional minors of the order $r_k = \operatorname{rank}(A^l)$, $l \geq k$.

Example 4.1. Consider the following matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 2 \\ 1 & -1 & 1 & 1 \end{bmatrix}.$$

The minors of the order $r_k = 2$, selected from A^D (derived in Example 3.1) are contained in the matrix

$$C_2(A^D) = \begin{bmatrix} \frac{3}{8} & -\frac{3}{8} & 0 & 0 & -\frac{3}{8} & \frac{3}{8} \\ \frac{1}{8} & -\frac{1}{8} & 0 & 0 & -\frac{1}{8} & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{8} & \frac{3}{8} & 0 & 0 & \frac{3}{8} & -\frac{3}{8} \\ -\frac{1}{8} & \frac{1}{8} & 0 & 0 & \frac{1}{8} & -\frac{1}{8} \end{bmatrix}.$$

Also,

$$C_2(A^2) = \begin{bmatrix} 3 & -3 & 0 & 0 & -3 & 3 \\ 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 4 & 0 & 0 & 4 & -4 \\ -3 & 3 & 0 & 0 & 3 & -3 \\ -1 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

This confirms the following result, which follows from part (i) of Theorem 4.1:

$$C_2(A^D) = \left[\,\mathrm{Tr}(C_{r_k}(A^D))\right]^{-1} \left[\,\mathrm{Tr}(C_2(A^2))\right]^{-2} C_2(A^2) = 2 * 1/16 * C_2(A^2).$$

Finally, using

$$C_2((A^D)^2) = \begin{bmatrix} \frac{3}{16} & -\frac{3}{16} & 0 & 0 & -\frac{3}{16} & \frac{3}{16} \\ \frac{1}{16} & -\frac{1}{16} & 0 & 0 & -\frac{1}{16} & \frac{1}{16} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{16} & \frac{3}{16} & 0 & 0 & \frac{3}{16} & -\frac{3}{16} \\ -\frac{1}{16} & \frac{1}{16} & 0 & 0 & \frac{1}{16} & -\frac{1}{16} \end{bmatrix}$$

which confirms the following computation, implied by the part (ii) of Theorem 3.4:

$$C_2((A^D)^2) = \left[\,\mathrm{Tr}(C_2(A^2))\right]^{-2} C_2(A^2) = 1/16 * C_2(A^2).$$

From Theorem 3.1 and Theorem 4.1 we conclude the following.

Corollary 4.1. *Consider $A \in \mathbb{C}_r^{n\times n}$ of the index k. Then the Drazin inverse of A can be expressed in terms of one's own minors and the minors of A, as follows:*

$$A^D_{ij} = \frac{\sum\limits_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)} |\,(A^D)^\beta_\alpha\,| \dfrac{\partial}{\partial a_{ji}} |\,A^\alpha_\beta\,|}{\sum\limits_{(\gamma,\delta)\in\mathcal{N}_{r_k}} |\,(A^D)^\delta_\gamma\,|\,|\,A^\gamma_\delta\,|}$$

$$= \frac{\sum\limits_{(\alpha,\beta)\in\mathcal{N}_{r_k}(j,i)} |(A^D)|^\beta_\alpha \dfrac{\partial}{\partial a_{ji}} |A^\alpha_\beta|}{\mathrm{Tr}(C_{r_k}(A^D A))}, \quad 1 \le i, j \le n,$$

where $r_k = \mathrm{rank}(A^k)$.

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