APPLICATIONS OF THE GROETCH THEOREM

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Abstract. In this paper we investigate general representations of various classes of generalized inverses of bounded operators over Hilbert spaces, based on the full-rank factorization of operators. Using these general representations we introduce a generalization of the Groetch representation theorem for the Moore-Penrose inverse. As corollaries, we derive a few iterative methods for computing reflexive g-inverses. In a particular case we get the main result from [9]. The present method is compared with [6].

1. Introduction

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ denote arbitrary Banach spaces and $B(\mathcal{X}_1, \mathcal{X}_2)$ denote the set of all bounded operators from $\mathcal{X}_1$ into $\mathcal{X}_2$. For an arbitrary operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$, we use $\mathcal{N}(A)$ to denote its kernel, and $\mathcal{R}(A)$ to denote its image. An operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$ is g-invertible, provided that there exists some $X \in B(\mathcal{X}_2, \mathcal{X}_1)$, such that $AXA = A$. In this case X is called a g-inverse of A. If X satisfies both of the equations $AXA = A$ and $XAX = X$, then X is called a reflexive g-inverse of A. It is well–known that an operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$ has a g-inverse if and only if $\mathcal{R}(A)$ is closed, and $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are complemented subspaces of $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively. An arbitrary right inverse and an arbitrary left inverse of A are denoted by A_r^{-1} and A_l^{-1}, respectively.

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We say that $A \in B(\mathcal{X})$ has the Drazin inverse, if there exists an operator $A^D \in B(\mathcal{X})$, such that A^D satisfies the equation (2) and the equations

$$(1^k) \quad A^{k+1}A^D = A^k, \qquad (5) \quad A^DA = AA^D,$$

for some non-negative integer k. Let us mention that the Drazin inverse, if it exists, is unique. The smallest k in the previous definition is called the index of A and denoted by $\mathrm{ind}(A)$. In the case $\mathrm{ind}(A) = 1$ the Drazin inverse is known as the group inverse of A, denoted by $A^{\#}$.

The full rank factorization of matrices is well-known and frequently used in representations of pseudoinverses [1, 7, 8, 10]. The following analogy of the full rank factorization for matrices is established in [2], [3]:

Let $A \in B(\mathcal{X}_1, \mathcal{X}_2)$. If there exist a Banach space $\mathcal{X}_3$ and operators $Q \in B(\mathcal{X}_1, \mathcal{X}_3)$ and $P \in B(\mathcal{X}_3, \mathcal{X}_2)$, such that P is left invertible, Q is right invertible and

$$A = PQ, \tag{1.1}$$

then we say that (1.1) is the full-rank decomposition of A.

It is well-known that an operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$ has the full-rank decomposition, if and only if A is g-invertible. In this case $\mathcal{X}_3$ is isomorphic to $\mathcal{R}(A)$, and $\mathcal{R}(A) = \mathcal{R}(P)$ [3].

In the case when $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, it is well-known that an operator $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has a g-inverse if and only if $\mathcal{R}(A)$ is closed. We consider the following equations in X:

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

For a subset $\mathcal{S}$ of the set $\{1, 2, 3, 4\}$, the set of operators obeying the conditions contained in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. An operator in $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and is denoted by $A^{(\mathcal{S})}$. If $\mathcal{R}(A)$ is closed, the set $A\{1, 2, 3, 4\}$ consists of a single element, the Moore-Penrose inverse of A, denoted by $A^\dagger$.

A basic tool used in this paper is the following general representation theorem for the Moore-Penrose inverse of a bounded linear operator [3], [4], [5]:

Theorem 1.1. *Let $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range. Then* [5, p. 45]

$$T^\dagger = \tilde{T}^{-1} T^*, \quad \text{where } \tilde{T} = T^*T|_{\mathcal{R}(T^*)}. \tag{1.2}$$

Moreover, if Ω is an open set with $\sigma(\tilde{T}) \subset \Omega \subset (0, \infty)$, and $\{S_\beta(x)\}_\beta$ is a family of continuous real valued functions on Ω, with $\lim\limits_{\beta} S_\beta(x) = \frac{1}{x}$ uniformly on $\sigma(\tilde{T})$, then [3, p. 42]*,* [4]*,* [5, p. 57]

$$T^\dagger = \lim_{\beta} S_\beta(\tilde{T}) T^*, \tag{1.3}$$

where the convergence is in the uniform topology for $B(\mathcal{H}_2, \mathcal{H}_1)$. Furthermore,

$$\|S_\beta(\tilde{T})T^* - T^\dagger\| \leq \sup_{x \in \sigma(\tilde{T})} |x S_\beta(x) - 1| \cdot \|T^\dagger\|.$$

We investigate general representations of bounded operators on Hilbert spaces, based on the full-rank factorization (1.1). These representations are extensions of known results from [2], [7], [8] and [10].

Using these general representations together with the Groetch representation theorem for the Moore-Penrose inverse of a bounded operator on Hilbert spaces, we introduce representations for various subsets of the set of all reflexive g-inverses of a bounded operator. Using this extension of the Groetch representation theorem, as particular cases, we derive a few iterative methods for computing g-inverses. As a partial result we get an improvement of the hyper-power iterative method, which is investigated in [9] for operators acting on finite dimensional complex Hilbert spaces. This method is not known for matrices before.

2. Results

Firstly we state the following general representations based on the full-rank factorization of operators. These representations are known for matrices (see [7], [8] and [10]). For bounded operators between Hilbert spaces it is known a representation of the Moore-Penrose inverse, introduced in [2].

Lemma 2.1. *Let $A = PQ$ be a full–rank decomposition of $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ according to* (1.1). *Then:*

(a) $X \in A\{1,2\}$ *if and only if there exist operators* $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$ *and* $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$, *such that* QW_1 *and* W_2P *are invertible in* $B(\mathcal{H}_3)$. *In such a case,* X *possesses the following general representation*

$$X = Q_r^{-1} P_l^{-1}, \quad Q_r^{-1} = W_1(QW_1)^{-1}, \quad P_l^{-1} = (W_2P)^{-1}W_2. \tag{2.1}$$

(b) $X \in A\{1,2,3\}$ *if and only if there exists an operator* $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, *such that* QW_1 *is invertible in* $B(\mathcal{H}_3)$. *In the case when it exists, a general representation for* X *is as follows:*

$$X = W_1(QW_1)^{-1}(P^*P)^{-1}P^*. \tag{2.2}$$

(c) $X \in A\{1,2,4\}$ *if and only if there exists an operator* $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$, *such that* W_2P *is invertible in* $B(\mathcal{H}_3)$. *In this case*

$$X = Q^*(QQ^*)^{-1}(W_2P)^{-1}W_2.$$

(d) $A^\dagger = Q^\dagger P^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = Q^*(P^*AQ^*)^{-1}P^*$ [2].

(e) *Let* $A : \mathcal{H}_1 \to \mathcal{H}_2$ *and* $X : \mathcal{H}_2 \to \mathcal{H}_1$. *Then* $X \in A\{2\}$ *if and only if there exist operators*

$$C \in B(\mathcal{H}_4, \mathcal{H}_1),\ D \in B(\mathcal{H}_2, \mathcal{H}_3),\ W_1 \in B(\mathcal{H}_5, \mathcal{H}_4),\ W_2 \in B(\mathcal{H}_3, \mathcal{H}_5),$$

such that DAC *is g-invertible and* W_2DACW_1 *is invertible and* X *posseses the following general form:*

$$X = CW_1(W_2DACW_1)^{-1}W_2D. \tag{2.3}$$

Proof. **(a)** This statement can be proved as in [8, Theorem 2.1.1 and Lemma 2.5.2].

(b) If X has the form (2.2), then it is easy to verify $X \in A\{1,2,3\}$. We need to prove that the form (2.2) holds for all $\{1,2,3\}$ inverses of A. Indeed, if $X \in A\{1,2,3\}$, then $X = Q_r^{-1}P_l^{-1}$, and from the equation (3) it follows that $(PP_l^{-1})^* = PP_l^{-1}$. Thus $P^*PP_l^{-1} = P^*$. The operator P^*P is invertible, so that $P_l^{-1} = (P^*P)^{-1}P^*$. The right inverse of Q retains the general form $Q_r^{-1} = W_1(QW_1)^{-1}$ given in (2.1). Consequently,

$$X = W_1(QW_1)^{-1}(P^*P)^{-1}P^*.$$

(c) This part of the proof can be proved in the same way as **(b)**.

(d) Follows from **(b)** and **(c)** (also, this fact is proved in [2]).

(e) If X possesses the form (2.3), it is not difficult to verify $X \in A\{2\}$. On the other hand, using the method from [8, Theorem 3.4.1], it is easy to verify that $X \in A\{2\}$ if and only if there exist operators C and D, such that DAC is g-invertible and

$$X = C(DAC)^{(1,2)}D, \quad C \in B(\mathcal{H}_4, \mathcal{H}_1), \quad D \in B(\mathcal{H}_2, \mathcal{H}_3).$$

According to part **(a)**, $X \in A\{2\}$ if and only if there exist operators $W_1 \in B(\mathcal{H}_5, \mathcal{H}_4)$ and $W_2 \in B(\mathcal{H}_3, \mathcal{H}_5)$, such that W_2DACW_1 is invertible, and X possesses the form (2.3). □

Lemma 2.2. *Let $\mathcal{X}$ be a Banach space. If $A \in B(\mathcal{X})$, $l \geq k = \operatorname{asc}(A) = \operatorname{des}(A) < \infty$ and $A^l = P_{A^l}Q_{A^l}$ is the full-rank decomposition of A^l, then*

$$A^D = P_{A^l}(Q_{A^l}AP_{A^l})^{-1}Q_{A^l}.$$

Proof. If $\operatorname{asc}(A) = \operatorname{des}(A) = k < \infty$, then it is well-known that $\mathcal{N}(A^l) = \mathcal{N}(A^k)$ and $\mathcal{R}(A^l) = \mathcal{R}(A^k)$ for all $l \geq k$,

$$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2, \tag{2.4}$$

where $\mathcal{X}_1 = \mathcal{N}(A^l)$ and $\mathcal{X}_2 = \mathcal{R}(A^l)$, $A(\mathcal{X}_i) \subset \mathcal{X}_i$ for $i = 1, 2$, $A_1 = A|_{\mathcal{X}_1}$ is nilpotent and $A_2 = A|_{\mathcal{X}_2}$ is invertible (A is not nilpotent) [3], [4]. We can write

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad A^D = \begin{bmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{bmatrix}$$

with respect to the decomposition (2.4) (see [3]). Since $\mathcal{N}(A^l)$ and $\mathcal{R}(A^l)$ are complementary and closed subspaces of $\mathcal{X}$, it follows that A^l is g-invertible, so there exists the full-rank decomposition $A^l = P_{A^l} Q_{A^l}$, where $P_{A^l} \in B(\mathcal{Z}, \mathcal{X})$ is left invertible and $Q_{A^l} \in B(\mathcal{X}, \mathcal{Z})$ is right invertible, for some Banach space $\mathcal{Z}$. By the isomorphism theorem [3], we can take that $\mathcal{Z} = \mathcal{X}_2$. We conclude that P_{A^l} and Q_{A^l} have the following representations with respect to (2.4):

$$P_{A^l} = \begin{bmatrix} M \\ \tilde{P} \end{bmatrix} \qquad \text{and} \qquad Q_{A^l} = [\, N \quad \tilde{Q} \,],$$

where $\tilde{P}, \tilde{Q} \in B(\mathcal{X}_2)$, $M \in B(\mathcal{X}_2, \mathcal{X}_1)$, $N \in B(\mathcal{X}_1, \mathcal{X}_2)$. Now, P_{A^l} is left invertible and Q_{A^l} is right invertible, so P_{A^l} and Q_{A^l} are g-invertible operators, $\mathcal{N}(P_{A^l}) = \{0\}$ and $\mathcal{R}(Q_{A^l}) = \mathcal{X}_2$. It follows that $\mathcal{R}(P_{A^l}) = \mathcal{R}(A^l) = \mathcal{X}_2$ and $\mathcal{N}(Q_{A^l}) = \mathcal{N}(A^l) = \mathcal{X}_1$, so $M = 0$, $N = 0$ and

$$P_{A^l} = \begin{bmatrix} 0 \\ \tilde{P} \end{bmatrix} \qquad \text{and} \qquad Q_{A^l} = [\, 0 \quad \tilde{Q} \,].$$

It is easy to verify that $\tilde{P}$ is left invertible and $\tilde{Q}$ is right invertible in $B(\mathcal{X}_2)$. But

$$\begin{bmatrix} 0 & 0 \\ 0 & A_2^l \end{bmatrix} = A^l = P_{A^l} Q_{A^l} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}\tilde{Q} \end{bmatrix},$$

so $A_2^l = \tilde{P}\tilde{Q}$. Since A_2^l is invertible, it follows that $\tilde{P}$ and $\tilde{Q}$ are invertible in $B(\mathcal{X}_2)$.

Now, $Q_{A^l} A P_{A^l} = \tilde{Q} A_2 \tilde{P}$ is invertible in $B(\mathcal{X}_2)$, so

$$A^D = \begin{bmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}(\tilde{Q} A_2 \tilde{P})^{-1} \tilde{Q} \end{bmatrix} = P_{A^l} (Q_{A^l} A P_{A^l})^{-1} Q. \qquad \square$$

Remark 2.1. The result of part **(e)** of Lemma 2.1 is an extension of the analogous result, introduced in [10, Theorem 2.1], stated for the set of complex matrices. Also, the result of Lemma 2.2 is an extension of an analogous result [10, Theorem 2.2], which is derived for complex matrices.

Our main aim is an application of considered general representations in a generalization of the Groetch representation theorem.

We begin with the result which enable us to get various reflexive generalized inverses of the considered operator, changing initial operators W_1 and W_2.

Theorem 2.1. *Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, $A = PQ$ be the full–rank decomposition of A and $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$. Suppose that QW_1 is right invertible, W_2P is left invertible, $W = W_2AW_1$ and $\tilde{W} = W^*W|_{\mathcal{R}(W^*)}$. If Ω is an open set with $\sigma(\tilde{W}) \subset \Omega \subset (0, \infty)$, and $\{S_\beta(x)\}_\beta$ is a family of continuous real valued functions on Ω, with $\lim_\beta S_\beta(x) = \frac{1}{x}$ uniformly on $\sigma(\tilde{W})$, then:*

$$X = \lim_\beta W_1 \left[S_\beta(\tilde{W}) \right] W^* W_2 \in A\{1, 2\},$$

where the convergence is in the uniform topology for $B(\mathcal{H}_2, \mathcal{H}_1)$.

Furthermore,

$$\|W_1 S_\beta(\tilde{W}) W^* W_2 - X\| \leq \|W_1\| \sup_{x \in \sigma(\tilde{W})} |x S_\beta(x) - 1| \cdot \|W^\dagger\| \|W_2\|.$$

Proof. Since $W = (W_2P)(QW_1)$, QW_1 is onto, W_2P is one-to-one and $\mathcal{R}(W_2P)$ is closed, it follows that $\mathcal{R}(W) = \mathcal{R}(W_2P)$, so we may apply Theorem 1.1 for W instead of T. We conclude

$$X = \lim_\beta W_1 \left[S_\beta(\tilde{W}) \right] W^* W_2 = W_1 (W_2 A W_1)^\dagger W_2 = W_1 ((W_2P)(QW_1))^\dagger W_2.$$

Operators W_2P and QW_1 form the full-rank decomposition for W, and applying the part **(d)** of Lemma 2.1 we immediately obtain $((W_2P)(QW_1))^\dagger =$

$(QW_1)^\dagger(W_2P)^\dagger$. Since $(QW_1)^\dagger$ is the right inverse of QW_1 and $(W_2P)^\dagger$ is the left inverse for W_2P, we easily conclude that

$$X = W_1(QW_1)^\dagger(W_2P)^\dagger W_2 \in A\{1,2\}. \qquad \square$$

Using Lemma 2.1, similar results can be stated for $\{i,j,k\}$ generalized inverses. For example, if $W_1 = Q^*$ then $X \in A\{1,2,3\}$. Also, if $W_2 = P^*$ then $X \in A\{1,2,4\}$. To avoid repetition we omit the proof.

Applying Lemma 2.1, Lemma 2.2 and the method from Theorem 2.1, we get the following representations of $\{2\}$, $\{1,2\}$, $\{1,2,3\}$, $\{1,2,4\}$ inverses, the Moore-Penrose inverse and the Drazin inverse.

Corollary 2.1. *Let $A \in B(\mathcal{H}_1,\mathcal{H}_2)$ has closed range and $A = PQ$ be the full–rank decomposition of A according to* (1.1). *Let $\{S_\beta(x)\}_\beta$ be a family of continuous real valued functions on $(0,+\infty)$, with $\lim\limits_\beta S_\beta(x) = \frac{1}{x}$ uniformly on all compact subsets of $(0,+\infty)$. Then:*

- **(a)** *$X \in A\{1,2\}$ if and only if there exist operators $W_1 \in B(\mathcal{H}_3,\mathcal{H}_1)$, $W_2 \in B(\mathcal{H}_2,\mathcal{H}_3)$, such that QW_1 and W_2P are invertible, and*

$$X = \lim_\beta W_1 \left[S_\beta(\tilde{W})\right] W^*W_2 = W_1\tilde{W}^{-1}W^*W_2, \quad W = W_2AW_1.$$

- **(b)** *$X \in A\{1,2,3\}$ if and only if there exists $W_2 \in B(\mathcal{H}_2,\mathcal{H}_3)$ such that W_2P is invertible and*

$$X = \lim_\beta Q^* \left[S_\beta(\widetilde{W_2AQ^*})\right] (W_2AQ^*)^*W_2 = Q^*(\widetilde{W_2AQ^*})^{-1}(W_2AQ^*)^*W_2.$$

- **(c)** *$X \in A\{1,2,4\}$ if and only if there exists $W_1 \in B(\mathcal{H}_3,\mathcal{H}_1)$ such that QW_1 is left invertible and*

$$X = \lim_\beta W_1 \left[S_\beta(\widetilde{P^*AW_1})\right] (P^*AW_1)^*P^* = W_1(\widetilde{P^*AW_1})^{-1}(P^*AW_1)^*P^*.$$

- **(d)** $A^\dagger = \lim\limits_\beta Q^* \left[S_\beta(\widetilde{P^*AQ^*})\right] (P^*AQ^*)^*P^* = Q^*(\widetilde{P^*AQ^*})^{-1}(P^*AQ^*)^*P^*.$

(e) *$X \in A\{2\}$ if and only if there exist operators*

$$C \in B(\mathcal{H}_4, \mathcal{H}_1),\ D \in B(\mathcal{H}_2, \mathcal{H}_3),\ W_1 \in B(\mathcal{H}_5, \mathcal{H}_4),\ W_2 \in B(\mathcal{H}_3, \mathcal{H}_5),$$

such that DAC is g-invertible, W_2DACW_1 is invertible and

$$X = \lim_\beta CW_1 \left[S_\beta(W_2\widetilde{DAC}W_1) \right] (W_2DACW_1)^* W_2 D.$$

(f) *If $l \geq k = \operatorname{ind}(A)$ and $Q_{A^l}AP_{A^l}$ is nonsingular, then*

$$A^D = \lim_\beta P_{A^l} \left[S_\beta(\widetilde{Q_{A^l}AP_{A^l}}) \right] (Q_{A^l}AP_{A^l})^* Q;$$

The convergence is in the uniform topology for $B(\mathcal{H}_2, \mathcal{H}_1)$.

Our aim is to use various initial conditions for W_1 and W_2, so we need the next result.

Theorem 2.2. *Let $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, let Ω be an open set with $\sigma(T^*T|_{\mathcal{R}(T^*)}) \cup \sigma(TT^*|_{\mathcal{R}(T)}) \subset \Omega \subset (0, \infty)$, and let $\{S_\beta(x)\}_\beta$ be a family of continuous real valued functions on Ω, with $\lim_\beta S_\beta(x) = \frac{1}{x}$ uniformly on $\sigma(T^*T|_{\mathcal{R}(T^*)}) \cup \sigma(TT^*|_{\mathcal{R}(T)})$. Then*

$$\lim_\beta T^* \left[S_\beta(TT^*|_{\mathcal{R}(T)}) \right] = \lim_\beta \left[S_\beta(T^*T|_{\mathcal{R}(T^*)}) \right] T^* = T^\dagger.$$

Proof. Using the Weierstrass Approximation Theorem, we get that the operator $S_\beta(T^*T|_{\mathcal{R}(T^*)})$ is selfadjoint on $\mathcal{R}(T^*)$ and $S_\beta(TT^*|_{\mathcal{R}(T)})$ is selfadjoint on $\mathcal{R}(T)$. By Theorem 1.1 we get

$$\begin{aligned} \lim_\beta T^* \left[S_\beta(TT^*|_{\mathcal{R}(T)}) \right] &= \lim_\beta \left(\left[S_\beta(TT^*|_{\mathcal{R}(T)}) \right] T \right)^* = \left((T^*)^\dagger \right)^* = T^\dagger \\ &= \lim_\beta \left[S_\beta(T^*T|_{\mathcal{R}(T^*)}) \right] T^*. \quad \square \end{aligned}$$

In the following theorem we obtain a few additional initial conditions for the operators W_1 and W_2, which produce various subsets of $\{i, j, k\}$ generalized inverses.

Theorem 2.3. *Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, $A = PQ$ be a full-rank decomposition of A, $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$ and $W = W_2 A W_1 \in B(\mathcal{H}_3)$.*

(a) *If W_2 is unitary, QW_1 is right invertible and S_β is a family possessing the properties from Theorem 1.1 with $T = AW_1$, then*

$$\lim_\beta W_1 \left[S_\beta(\tilde{W}) \right] W^* W_2 = W_1 (AW_1)^\dagger \in A\{1,2,3\}.$$

(b) *If W_1 is unitary, $W_2 P$ is left invertible and S_β is a family which satisfies conditions of Theorem 1.1 for the operator $T = A^* W_2^*$, then*

$$\lim_\beta W_1 W^* \left[S_\beta(\tilde{W}) \right] W_2 = (W_2 A)^\dagger W_2 \in A\{1,2,4\}.$$

(c) *If both W_1 and W_2 are unitary and S_β has the properties from* **(a)** *and* **(b)***, then*

$$\begin{aligned} A^\dagger &= \lim_\beta W_1 \left[S_\beta(\tilde{W}) \right] W^* W_2 = W_1 (AW_1)^\dagger \\ &= \lim_\beta W_1 W^* \left[S_\beta(\tilde{W}) \right] W_2 = (W_2 A)^\dagger W_2. \end{aligned}$$

(d) *If* **(a)** *is valid and $W_1 = Q^*$, then*

$$\lim_\beta W_1 \left[S_\beta(\tilde{W}) \right] W^* W_2 = Q^* (AQ^*)^\dagger = A^\dagger.$$

(e) *If* **(b)** *is valid and $W_2 = P^*$, then*

$$\lim_\beta W_1 W^* \left[S_\beta(\tilde{W}) \right] W_2 = (P^* A)^\dagger P^* = A^\dagger.$$

Proof. **(a)** The operator W_2 is unitary, which implies

$$W^* W = (AW_1)^* A W_1, \quad W^* W_2 = (AW_1)^*.$$

Since $W^* = (AW_1)^* W_2^*$ and W_2 is invertible, it follows that $\mathcal{R}(W^*) = \mathcal{R}((AW_1)^*)$. Using Theorem 1.1 we obtain

$$X = \lim_\beta W_1 \left[S_\beta(\tilde{W}) \right] W^* W_2 = W_1 (AW_1)^\dagger \in A\{2,3\}.$$

We need to prove $W_1(AW_1)^\dagger \in A\{1\}$. Note that $X = W_1[P(QW_1)]^\dagger$. Now, P is left invertible, and QW_1 is right invertible, so $P(QW_1)$ is given as the full-rank factorization. Using the result from Lemma 2.1 **(d)** or [2], we get $X = W_1(QW_1)^\dagger P^\dagger$. Now, the equation (1) can be easily verified.

(b) We use **(a)** with: A^* instead of A, W_2^* instead of W_1 and W_1^* instead of W_2. Note that W_1^* is unitary and $(W_2P)^* = P^*W_2^*$ is right invertible. In this case we have

$$W_1W^* = A^*W_2^*, \quad WW^* = W_2A(W_2A)^*, \quad \mathcal{R}(W_2A) = \mathcal{R}(W)$$

which implies $\widetilde{W^*} = (\widetilde{W_2A})^*$. Using the Weierstrass Approximation Theorem, we get that $S_\beta(W_2A(W_2A)^*|_{\mathcal{R}(W_2A)})$ is selfadjoint, so

$$\begin{aligned}&\lim_\beta (A^*W_2^*)\left[S_\beta(W_2A(W_2A)^*|_{\mathcal{R}(W_2A)})\right]W_2 =\\ &=\lim_\beta \left\{W_2^*\left[S_\beta((A^*W_2^*)^*A^*W_2^*|_{\mathcal{R}[(A^*W_2^*)^*]})\right](A^*W_2^*)^*\right\}^* = (W_2^*(A^*W_2^*)^\dagger)^*.\end{aligned}$$

By **(a)** we know that $W_2^*(A^*W_2^*)^\dagger \in A^*\{1,2,3\}$, so

$$(W_2^*(A^*W_2^*)^\dagger)^* = (W_2A)^\dagger W_2 \in A\{1,2,4\}.$$

(c) It is enough to prove that the limits from **(a)** and **(b)** are equal. If W_2 is unitary, from the proof of **(a)** we get $\tilde{W} = \widetilde{AW_1}$. If W_1 is unitary, from the proof of **(b)** we get $\widetilde{W^*} = (\widetilde{W_2A})^*$. Now, by Theorem 2.4, and using the parts **(a)** and **(b)** of this proof, we get:

$$\begin{aligned}W_1(AW_1)^\dagger &= \lim_\beta W_1\left[S_\beta((AW_1)^*AW_1|_{\mathcal{R}[(AW_1)^*]})\right](AW_1)^*W_2^*W_2\\ &= \lim_\beta W_1\left[S_\beta(W^*W|_{\mathcal{R}(W^*)})\right]W^*W_2 = W_1W^\dagger W_2\\ &= \lim_\beta W_1W^*\left[S_\beta(WW^*|_{\mathcal{R}(W)})\right]W_2 = \lim_\beta A^*W_2^*\left[S_\beta(WW^*|_{\mathcal{R}(W)})\right]W_2\\ &= \lim_\beta A^*W_2^*S_\beta(W_2A(W_2A)^*|_{\mathcal{R}(W_2A)})W_2 = (W_2A)^\dagger W_2. \quad \square\end{aligned}$$

Finally, as corollaries, we introduce a few iterative methods for computing reflexive g-inverses.

Corollary 2.2. *Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, $A = PQ$ is the full–rank decomposition of A and let $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$ be two operators, such that QW_1 is right invertible and W_2P is left invertible. Let $W = W_2AW_1$ and $\tilde{W} = W^*W|_{\mathcal{R}(W^*)}$. Then the following representations of the reflexive g-inverses are convergent in the uniform topology for $B(\mathcal{H}_2, \mathcal{H}_1)$:*

(a) $A^{(1,2)} = W_1 \left[\int\limits_0^\infty e^{-W^*Wu} W^* du \right] W_2$;

(b) $A^{(1,2)} = \alpha W_1 \sum\limits_{k=0}^{\infty} (I - \alpha W^*W)^k \, W^*W_2$, *where* $0 < \alpha < 2\|W\|^{-2}$;

(c) $A^{(1,2)} = W_1 \lim\limits_{t \to 0_+} (tI + W^*W)^{-1} \, W^*W_2$;

(d) $A^{(1,2)} = W_1 \sum\limits_{k=0}^{\infty} \dfrac{1}{k+1} \left(\prod\limits_{j=0}^{k-1} \left(I - \dfrac{1}{j+1} W^*W \right) \right) W^*W_2$;

(e) $A^{(1,2)} = W_1 \lim\limits_{t \to 0_+} \sum\limits_{k=0}^{\infty} \dfrac{1}{\Gamma(1+tk)} \left[I - W^*W \right]^k W^*W_2$;

(f) $A^{(1,2)} = W_1 \left(W^* + \lim\limits_{t \to 0_+} \sum\limits_{k=1}^{\infty} e^{-tk \log k} \left[I - W^*W \right]^k W^* \right) W_2$;

(g) $A^{(1,2)} = W_1 \lim\limits_{t \to 0_+} \sum\limits_{k=0}^{\infty} \dfrac{\Gamma(1 + (1-t)k)}{\Gamma(1+k)} \left[I - W^*W \right] W^*W_2$.

Also, as a corollary, we get the next generalization of the main result form [9].

Corollary 2.3 ([9, Lemma 2.1, Theorem 2.1])**.** *Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has closed range, $A = PQ$ is the full–rank decomposition of A and let $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$ be two operators, such that QW_1 is invertible and W_2P is invertible. Let $W = W_2AW_1$. Then the class of $\{1,2\}$ inverses of A can be generated by changing the values of the operators W_1, W_2 in the following two iterative processes:*

$$Y_0 = Y_0' = \alpha (W_2AW_1)^*, \qquad 0 < \alpha \leq 2\|W\|^{-2},$$

$$\begin{cases} T_k = I_X - Y_k W, \\ Y_{k+1} = (I_X + T_k + \cdots + T_k^{q-1}) Y_k, \\ X_{k+1} = W_1 Y_{k+1} W_2 \end{cases} \quad \begin{cases} T'_k = I_Y - W Y'_k, \\ Y'_{k+1} = Y'_k (I_Y + T'_k + \cdots + {T'_k}^{q-1}), \\ X'_{k+1} = W_1 Y'_{k+1} W_2 \quad k = 0, 1, \ldots \end{cases}$$

Moreover, the following statements are valid:

(a) *If* W_2 *is unitary, then* $X_k \to X = W_1(AW_1)^\dagger \in A\{1,2,3\}$ *as* $k \to \infty$.

(b) *If* W_1 *is unitary then* $X'_k \to X = (W_2 A)^\dagger W_2 \in A\{1,2,4\}$ *as* $k \to \infty$.

(c) *If* **(a)** *and* **(b)** *are valid, then* $X_k \to A^\dagger$.

(d) *If* **(a)** *is valid and* $W_1 = Q^*$, *then* $X_k \to X = A^\dagger$.

(e) *If* **(b)** *is valid and* $W_2 = P^*$, *then* $X'_k \to X = A^\dagger$.

Remark 2.1. In [6] it is also introduced a modification of the hyper-power method, which generates the class of all $\{1,2\}$-inverses for operators on Banach spaces. Using the method from [6] for Hilbert spaces operators, it is not clear how to choose the initial values to get $\{1,2,3\}$, $\{1,2,4\}$-inverses. Also, our method is applicable for various classes of $\{S_\beta\}$ families.

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