Inner generalized inverses with prescribed idempotents

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Abstract

We define and characterize inner generalized inverses with prescribed idempotents. These classes of generalized inverses are natural algebraic extension of generalized inverses of linear operators with prescribed range and kernel. We consider the reverse order rule for inner generalized inverses of elements of a ring, some perturbation bounds and we construct an iterative method for computing a $(p, q)-$inner inverse in Banach algebras.

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1 Introduction

In this paper we investigate generalized inverses in rings with respect to prescribed idempotents, and we are particulary interested in inner generalized inverses. We find the motivation for this paper in [10], [12] and [20].

Let $\mathcal{R}$ be an associative ring with the unit 1. An element $p \in \mathcal{R}$ is an idempotent if $p^2 = p$. The set of all idempotents in $\mathcal{R}$ is denoted by $\mathcal{R}^\bullet$. An element $a \in \mathcal{R}$ is inner generalized invertible (inner regular) if there exists an element $a^- \in \mathcal{R}$ such that $aa^-a = a$ holds. In this case a^- is an inner generalized inverse of a. If $a^-aa^- = a^-$ holds then a^- is an outer generalized inverse of a, and a is outer generalized invertible (outer regular). If a^- is either inner or outer generalized inverse of a, then the elements a^-a and $1 - aa^-$ are idempotents corresponding to a and a^-. This is the approach taken in [10], where outer generalized inverses with prescribed idempotents are considered. See [9] for a general overview of problems on generalized inverses in rings, and many recent results on this topic. If a^- is both inner and outer generalized inverse of a, then it is a reflexive generalized inverse of a. If a^- is an inner generalized inverse of a, then a^-aa^- is a reflexive generalized inverse of a. Thus, inner regularity implies outer regularity of a. In general, an element $a \in \mathcal{R}$ need not to be outer or inner invertible, even in Banach algebras. If an inner generalized inverse with prescribed idempotents exists, it is not necessarily unique. On the other hand, the outer generalized inverse with prescribed idempotents is unique in the case when it exists [10].

The motivation for defining generalized inverses with prescribed idempotents arise from the definition of the Moore-Penrose inverse of an element in a ring with involution.

If $\mathcal{R}$ is a ring with involution, then the (unique, if it exists) Moore-Penrose inverse of a, denoted by $a^\dagger$, satisfies

$$aa^\dagger a = a \quad a^\dagger aa^\dagger = a^\dagger \quad (aa^\dagger)^* = aa^\dagger \quad (a^\dagger a)^* = a^\dagger a. \tag{1}$$

The set of all Moore-Penrose invertible elements of $\mathcal{R}$ is denoted by $\mathcal{R}^\dagger$.

In the case $a \in \mathcal{A}$ and $\mathcal{A}$ is a C^*-algebra, the element $a^\dagger$ exists if and only if a is inner invertible [14].

Further, we are interested in the reverse order law for inner invertible elements. If $a, b \in \mathcal{R}$ are invertible, then the rule $(ab)^{-1} = b^{-1}a^{-1}$ is called the reverse order law for the ordinary inverse. The same rule does not hold in general for the generalized inverse, even in the case of the Moore-Penrose inverse of complex matrices. There are many equivalent (or sufficient) conditions such that the reverse order rule holds for some generalized inverses (see [1, 2, 3, 13, 15, 21]). In this paper we prove some results related to the reverse order rule for inner generalized inverses.

The paper is organized as follows. In Section 2 we define and investigate conditions on the existence of an inner generalized inverse with prescribed idempotents in rings and in rings with involution. We define the proper splitting of a. Also, we prove some results concerning the reverse order law for inner invertible elements. In Section 3 a generalization of the condition number in a normed algebra is given. Also, we construct an iterative method for computing an inner (p, q)-inverse.

2 Inner generalized inverses in rings

Let $a \in \mathcal{R}$ and let $p, q \in \mathcal{R}^\bullet$ be given. Following [20] and [10], consider the following equations in $\mathcal{R}$:

$$\begin{array}{llll} (1) & axa = a, & (2) & xax = x, \\ (3p) & xa = p, & (4q) & ax = 1 - q. \end{array}$$

Let $\eta \subset \{1, 2, 3p, 4q\}$. If $x \in \mathcal{R}$ satisfies equations (i) for all $i \in \eta$, then x is called an η-inverse of a, frequently denoted as a^{η}. The set of all such x is denoted by $a\eta$.

Notice that if $x \in a\{2, 3p, 4q\}$, then x is unique, known as the outer inverse of a with prescribed idempotents p and q (see [10],[9]).

For an inner invertible element $a \in \mathcal{R}$ we have

$$a\{1\} = \{a^- \in \mathcal{R} \,|\, aa^-a = a\} \tag{2}$$

and as its special subclass

$$a\{1, 3p, 4q\} = \{a^- \,|\, a^- \in a\{1\},\ a^-a = p,\ 1 - aa^- = q\}. \tag{3}$$

$\mathcal{R}^-$ denotes the set of all inner invertible elements in $\mathcal{R}$, and $\mathcal{R}^{-1}$ denotes the set of all invertible elements in $\mathcal{R}$. In general, $\mathcal{R}^{-1} \subset \mathcal{R}^-$, and if $\mathcal{R}$ is a ring with involution then $\mathcal{R}^{-1} \subset \mathcal{R}^{\dagger} \subset \mathcal{R}^-$.

Definition 2.1. *Let $a \in \mathcal{R}$ and let $p, q \in \mathcal{R}^{\bullet}$. An element $c \in \mathcal{R}$ satisfying*

$$aca = a, \qquad ca = p, \qquad ac = 1 - q,$$

is called a $(p, q)-$inner inverse of a (or an inner inverse of a with prescribed idempotents p and q). That is, $c \in a\{1, 3p, 4q\}$.

In general, if an inner inverse of a with prescribed idempotents exists, it is not necessarily unique (see [11] for the case of linear bounded operators on Banach spaces). In order to establish the uniqueness, we consider an extra equation in $\mathcal{R}$: if $r \in \mathcal{R}$ is given, then we require that $c \in \mathcal{R}$ satisfies

$$(5r) \quad r = c - cac. \tag{4}$$

Now we can state the following result.

Theorem 2.1. *There exists at most one element in the set $a\{1, 3p, 4q, 5r\}$.*

Proof. If $a^-, a^= \in a\{1, 3p, 4q, 5r\}$, then

$$a^- - a^= = r + a^-aa^- - (r + a^=aa^=) = a^-aa^- - a^=aa^= = a^=aa^- - a^=aa^- = 0,$$

because $aa^{=} = aa^{-} = 1 - q$ and $a^{-}a = a^{=}a = p$. $\square$

From now on, we use $\hat{a}_r$ to denote the unique element of the set $a\{1, 3p, 4q, 5r\}$, in the case when the last set is non-empty.

Lemma 2.1. *Let $a, r \in \mathcal{R}$ and $p, q \in \mathcal{R}^{\bullet}$ be such that $\hat{a}_r$ exists. Then*

$$r = (1 - \hat{a}_r a)\hat{a}_r(1 - a\hat{a}_r). \tag{5}$$

We prove the following result. If we know one $(p, q)-$inner inverse of a, then we can describe all of them.

Lemma 2.2. *Let $a, r \in \mathcal{R}$ and $p, q \in \mathcal{R}^{\bullet}$ be such that $\hat{a}_r$ exists. Then*

$$\begin{aligned} a\{1, 3p, 4q\} &= \{\hat{a}_r - r + s \,|\, s = (1 - \hat{a}_r a)u(1 - a\hat{a}_r),\ u \in \mathcal{R}\} \\ &= \{\hat{a}_r a\hat{a}_r + s \,|\, s = (1 - \hat{a}_r a)u(1 - a\hat{a}_r),\ u \in \mathcal{R}\} \end{aligned} \tag{6}$$

Proof. Let $c \in a\{1, 3p, 4q\}$. Then $aca = a, ca = p, ac = 1 - q$, and let $s = c - cac$. So, we get $c = \hat{a}_r - r + s$ where $s = (1 - \hat{a}_r a)c(1 - a\hat{a}_r)$.

On the other hand, notice that $s = u - ua\hat{a}_r - \hat{a}_r au + \hat{a}_r aua\hat{a}_r$, and let $b = \hat{a}_r - r + s = \hat{a}_r a\hat{a}_r + u - ua\hat{a}_r - \hat{a}_r au + \hat{a}_r aua\hat{a}_r$. Then we obtain $ba = p + ua - ua - \hat{a}_r aua + \hat{a}_r aua = p$, and consequently, $aba = b$. Moreover, $ab = a\hat{a}_r + au - aua\hat{a}_r - au + aua\hat{a}_r = 1 - q$. Finally,

$$\begin{aligned} b - bab \quad &= \hat{a}_r a\hat{a}_r + u - ua\hat{a}_r - \hat{a}_r au + \hat{a}_r aua\hat{a}_r \\ &\quad -\hat{a}_r a(\hat{a}_r a\hat{a}_r + u - ua\hat{a}_r - \hat{a}_r au + \hat{a}_r aua\hat{a}_r) \\ &= \hat{a}_r a\hat{a}_r + u - ua\hat{a}_r - \hat{a}_r au + \hat{a}_r aua\hat{a}_r - \hat{a}_r a\hat{a}_r - \hat{a}_r au \\ &\quad +\hat{a}_r aua\hat{a}_r + \hat{a}_r a\hat{a}_r au - \hat{a}_r a\hat{a}_r aua\hat{a}_r \\ &= u - ua\hat{a}_r - \hat{a}_r au + \hat{a}_r aua\hat{a}_r - \hat{a}_r au + \hat{a}_r aua\hat{a}_r \\ &\quad +\hat{a}_r au - \hat{a}_r aua\hat{a}_r \\ &= s. \end{aligned}$$

From Theorem 2.1 it follows that $b = \hat{a}_s$. $\square$

Notice that if $\hat{a}_0$ exists, then it is a reflexive generalized inverse with prescribed idempotents p and q, so it is unique [10].

Lemma 2.3. *Let $a \in \mathcal{R}$, and $p, q \in \mathcal{R}^{\bullet}$. The following statements are equivalent:*

(1) $a\{1, 3p, 4q\} \neq \emptyset$.

(2) *$ap = a$, and there exists some $c \in \mathcal{R}$ such that $cap = p$ and $ac = 1 - q$.*

Proof. (1) $\Longrightarrow$ (2): Obvious.

(2) $\Longrightarrow$ (1) : If (2) holds, then we have $aca = ac(ap) = a(cap) = ap = a$, that is $c \in a\{1\}$. Also, $ca = cap = p$ and $ac = 1 - q$. $\square$

We need the notion of the group inverse in a ring. Let $a \in \mathcal{R}$. An element $b \in \mathcal{R}$ is a group inverse of a, provided that the following hold:

$$aba = a, \ bab = b, \ ab = ba.$$

In this case the element a is group invertible. If a group inverse of a exists, then it is unique and denoted by a^g. The set of all group invertible elements in $\mathcal{R}$ is denoted by $\mathcal{R}^g$. If a is invertible, then a is group invertible and $a^{-1} = a^g$. Hence, $\mathcal{R}^{-1} \subset \mathcal{R}^g$.

Now, we prove the following result, which is a generalization of a result from [7].

Theorem 2.2. *Let $a, c, r \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ be such that $\hat{a}_r$ exists. If $\hat{c}_0$ is the reflexive $(1-q, 1-p)-$inverse of c then $ac, ca \in \mathcal{R}^g$, and*

$$\hat{a}_r = r + c(ac)^g = r + (ca)^g c.$$

Proof. Since $\hat{a}_r = a^-$ and $\hat{c}_0 = c^-$ exist, we have $c^-a^-ac = c^-pc = c^-cc^-c = c^-c = 1 - q = aa^- = aa^-aa^- = apa^- = acc^-a^-$. Also $acc^-a^-ac = (1-q)ac = ac$, and $c^-a^-acc^-a^- = c^-a^-$. Hence, $(ac)^g = c^-a^-$. Similarly, we obtain $(ca)^g = a^-c^-$. Now it easily follows $r + cc^-a^- = r + pa^- = r + a^-aa^- = a^-$. $\square$

We also prove the following result.

Theorem 2.3. *Let $a, b, r, s \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ such that $\hat{a}_r$ and $\hat{b}_s$ exist. Then $b\hat{a}_r a = a\hat{a}_r b$, and there exists some c which is a $(p, q)-$inner inverse of the element $b\hat{a}_r$.*

Moreover,

$$1 + \hat{a}_r(b - a), \, 1 + (b - a)\hat{a}_r \in \mathcal{R}^{-1}$$

and

$$\hat{b}_r = (1 + \hat{a}_r(b - a))^{-1}\hat{a}_r = \hat{a}_r(1 + (b - a)\hat{a}_r)^{-1}. \tag{7}$$

Proof. Let $a^- = \hat{a}_r$, and $b^- = \hat{b}_s$. Then we have $ba^-a = bb^-b = aa^-b$. Also, b^- is a $(p, q)-$inner inverse of ba^-a, since $(ba^-a)b^-(ba^-a) = ba^-a$, $b^-ba^-a = p^2 = p$ and $ba^-ab^- = aa^-bb^- = 1-q$. Now, from $a^- \in a\{1, 3p, 4q\}$

and $b^- \in b\{1, 3p, 4q\}$ and because $a^-ab^-a = b^-bb^-a = b^-aa^-a = b^-a$ we get

$$\begin{aligned}
&(1 + a^-(b-a))(1 + (b^- - a^-)a) = (1 + a^-b - a^-a)(1 + b^-a - a^-a) \\
&= 1 + a^-b - a^-a + b^-a + a^-bb^-a - a^-ab^-a - a^-a - a^-ba^-a + a^-aa^-a \\
&= 1 + (a^-b - a^-ba^-a) + (-a^-a + a^-aa^-a) + (b^-a - a^-ab^-a) \\
&\quad +(a^-bb^-a - a^-a) \\
&= 1.
\end{aligned} \tag{8}$$

Similarly, we get $(1 + (b^- - a^-)a)(1 + a^-(b-a)) = 1$.

So, there exists $(1 + \hat{a}_r(b-a))^{-1} = 1 + (\hat{b}_s - \hat{a}_r)a$. Note that, $\hat{b}_s a$ is unique for any choice on s. So

$$\begin{aligned}
(1 + a^-(b-a))^{-1}a^- &= (1 + (\hat{b}_s - \hat{a}_r)a)\hat{a}_r \\
= \hat{a}_r + \hat{b}_s a\hat{a}_r - \hat{a}_r a\hat{a}_r &= r + \hat{b}_r a\hat{a}_r = r + \hat{b}_r b\hat{b}_r = \hat{b}_r.
\end{aligned}$$

that is, the equation (7) follows. □

Now, let $\mathcal{R}$ be a ring with involution. We need a structure more similar to C^*-algebras, so we can apply our results to linear bounded operators on Hilbert spaces.

Definition 2.2. *Let $p \in \mathcal{R}^\bullet$ be such that $p^* = p$. Then p is called a projection.*

The following result is a generalization of a result for Moore-Penrose invertible elements in rings with involution(Theorem 1.4.2 in [9]).

Theorem 2.4. *Let $a \in \mathcal{R}$ and p, q be projections in $\mathcal{R}$. If there exists $\hat{a}_r$, then the following statements are satisfied:*

(1) $(\hat{a}_r)^* \in a^*\{1, 3(1-q), 4(1-p), 5r^*\}$ *that is* $(\hat{a}_r)^* = (\widehat{a^*})_{r^*}$;

(2) $\hat{a}_r\hat{a}_r^*$ *is a* $(p, 1-p)-$*inner inverse of* a^*a;

(3) $\hat{a}_r^*\hat{a}_r$ *is a* $(1-q, q)-$*inner inverse of* aa^*;

(4) $a[\hat{a}_r(\hat{a}_r)^*] = [(\hat{a}_r)^*\hat{a}_r]a \in a^*\{1, 2, 3(1-q), 4(1-p)\}$;

Proof. Let there exists $\hat{a}_r$, that is

$$a\hat{a}_r a = a,\ \ \hat{a}_r a = p,\ \ a\hat{a}_r = 1 - q \ \text{ and } \ r = \hat{a}_r - \hat{a}_r a\hat{a}_r.$$

(1) We have $a^*(\hat{a}_r)^*a^* = (a\hat{a}_r a)^* = a^*$, also $(\hat{a}_r)^*a^* = (a\hat{a}_r)^* = (1-q)^* = 1-q$ and $a^*(\hat{a}_r)^* = (\hat{a}_r a)^* = p^* = p$. Also $(\hat{a}_r)^* - (\hat{a}_r)^*a^*(\hat{a}_r)^* = (\hat{a}_r - \hat{a}_r a\hat{a}_r)^* = r^*$. That is $(\hat{a}_r)^*$ is $(1-q, 1-p)-$inner inverse of a^*.

(2) Now, $a^*a(\hat{a}_r\hat{a}_r^*)a^*a = a^*a\hat{a}_r(a\hat{a}_r)^*a = a^*a$. Also $(\hat{a}_r\hat{a}_r^*)a^*a = \hat{a}_r(a\hat{a}_r)^*a = \hat{a}_r a = p$ and $a^*a(\hat{a}_r\hat{a}_r^*) = a^*(a\hat{a}_r)^*\hat{a}_r^* = (\hat{a}_r a\hat{a}_r a)^* = (\hat{a}_r a)^* = p$.

(3) Similarly.

(4) First, notice that $a\hat{a}_r\hat{a}_r^* = (a\hat{a}_r)^*\hat{a}_r^* = (\hat{a}_r a\hat{a}_r)^*$. The rest is a direct consequence. □

From the last results we actually see that the following chain of equivalences hold

$$a \in \mathcal{R}^- \Longleftrightarrow a^* \in \mathcal{R}^- \Longleftrightarrow aa^* \in \mathcal{R}^- \Longleftrightarrow a^*a \in \mathcal{R}^-$$

Definition 2.3. *An element $a \in \mathcal{R}$ is left $*$-cancellable if $a^*ax = 0$ implies $ax = 0$. Analogously, $a \in \mathcal{R}$ is right $*-$cancellable if $xaa^* = 0$ implies $xa = 0$. Finally, $a \in \mathcal{R}$ is $*-$cancellable if it is both left and right $*-$cancellable.*

In C^*-algebras, every element is $*-$cancellable. See [17, 16] to notice the connection between Moore-Penrose invertibility and $*-$cancellability.

We give some analogous connections between inner invertibility and $*-$cancellability of an element in a ring with involution.

Theorem 2.5. *Let $a \in \mathcal{R}$, and let p and q be projections in $\mathcal{R}$. Then the following statements are equivalent:*

(1) *There exists a^- which is a $(p, q)-$inner inverse of a;*

(2) *a is $*-$cancellable and there exist $(p, 1-p)-$inner inverse of a^*a and $(1-q, q)-$inner inverse of aa^*.*

Moreover, if any of the above condition is true, then any $(p, q)-$inner inverse of a is given with

$$a^- = ba^*aa^*c$$

where b and c are $(p, 1-p)$ and $(1-q, q)$ $-$ inner inverses of a^*a and aa^* respectively.

Proof.

(1) $\implies$ (2) : Let there exists a^-, a $(p,q)-$inner inverse of a, and let $a^*ax = 0$. Then $ax = aa^-ax = (aa^-)^*ax = (a^-)^*a^*ax = 0$. Also, if $xaa^* = 0$, then we have $xa = xaa^-a = xa(a^-a)^* = xaa^*(a^-)^* = 0$.

From Theorem 2.4 it follows that a^*a and aa^* are $(p, 1-p)$ and $(1-q, q)-$inner invertible, respectively.

(2) $\implies$ (1) : Let a be $*-$cancellable, and let there exist $b \in (a^*a)\{1, 3p, 4(1-p)\}$ and $c \in (aa^*)\{1, 3(1-q), 4q\}$. Denote $d = ba^*aa^*c$ and $x_1 = (ba^*aa^*ca-1)a^*$. Then since a is left $*-$cancelable and

$$\begin{array}{rcl} a^*ax_1 & = & (a^*aba^*a)a^*caa^* - a^*aa^* \\ & = & a^*(aa^*caa^*) - a^*aa^* \\ & = & a^*aa^* - a^*aa^* \\ & = & 0, \end{array} \tag{9}$$

it follows that

$$0 = ax_1 = (aba^*aa^*c - 1)aa^*.$$

Now since a is right $*-$cancelable we get $aba^*aa^*ca = a$, which proves the inner invertibility of a with an inner inverse ba^*aa^*c.

Now again from right $*-$cancelability of a and because

$$\begin{array}{rcl} (da-p)a^* & = & daa^* - ba^*aa^* \\ & = & ba^*(aa^*caa^*) - ba^*aa^* \\ & = & 0, \end{array}$$

we get $da = p$. In the similar way, using left $*-$cancelability of a we get $ad = 1 - q$. □

Theorem 2.6. *Let $\mathcal{R}$ be a ring with involution, let $a \in \mathcal{R}$ and let $p \in \mathcal{R}^\bullet$. Then the following statements are equivalent:*

(1) *There exists an inner inverse a^- of a such that $a^-a = p$;*

(2) *a is left $*-$cancelable and there exists a $(p, 1-p)-$inner inverse of a^*a.*

Proof. Using technique of the proof of Theorem 2.5, the result can be obtained.

□

Analogously we get the following result.

Theorem 2.7. *Let $\mathcal{R}$ be a ring with involution and let $a \in \mathcal{R}$. Then the following statements are equivalent:*

(1) *There exists an inner inverse a^- of a such that $aa^- = 1 - q$;*

(2) *a is right $*-$cancelable and there exists a $(1-q, q)-$inner inverse of aa^*.*

Now, we prove some results concerning the reverse order law for inner generalized inverses in a ring.

Theorem 2.8. *Let $a, b \in \mathcal{R}$, $c = ab$, and let $p, q \in \mathcal{R}^\bullet$, such that $a\{1, 3p\} \neq \emptyset$ and $b\{1, 4q\} \neq \emptyset$. Then the following statements are equivalent:*

(1) *$b^- a^- \in c\{1\}$ for some $a^- \in a\{1, 3p\}$ and for some $b^- \in b\{1, 4q\}$;*

(2) *$r = p(1-q) \in \mathcal{R}^\bullet$;*

(3) *$b^- a^- \in c\{1\}$ for all $a^- \in a\{1, 3p\}$ and all $b^- \in b\{1, 4q\}$.*

Moreover, in any of these cases, the corresponding idempotents w and v of c are given by

$$w = (b^- a^-)(ab) = b^- rb \quad \text{and} \quad 1 - v = abb^- a^- = ara^-.$$

Proof. (1) $\Rightarrow$ (2): Let $abb^- a^- ab = ab$ for some $a^- \in a\{1, 3p\}$ and $b^- \in b\{1, 4q\}$. We multiply $abb^- a^- ab = ab$ by a^- from the left side and by b^- from the right side. Thus, we obtain that $a^- abb^- = p(1-q) = r$ is an idempotent.

(2) $\Rightarrow$ (3): Let $a^- \in a\{1, 3p\}$ and $b^- \in b\{1, 4q\}$ be arbitrary. Now we have the following chain of implications and equivalencies:

$$\begin{aligned} & p(1-q)p(1-q) = p(1-q) \\ \Longleftrightarrow \quad & a^- a(1-q)pbb^- = p(1-q) \\ \Longrightarrow \quad & a(1-q)pb = ap(1-q)b \\ \Longleftrightarrow \quad & abb^- a^- ab = ab \\ \Longleftrightarrow \quad & (3). \end{aligned}$$

(3) $\Rightarrow$ (1): Obvious.

Finally, we obtain

$$w = (b^- a^-)(ab) = b^- pbb^- b = b^- p(1-q)b = b^- rb$$

and

$$1 - v = abb^- a^- = a(1-q)a^- = aa^- a(1-q)a^- = ara^-.$$

$\square$

Recall that $x^\circ = \{y \in \mathcal{R} : xy = 0\}$ and $x\mathcal{R} = \{xy : y \in \mathcal{R}\}$ are right ideals of $\mathcal{R}$. Now, we can easily prove the following result.

Theorem 2.9. *Suppose that the conditions of Theorem 2.8 are satisfied. Then* $(1-v)^\circ = (ra^-)^\circ$ *and* $w\mathcal{R} = b^- r\mathcal{R}$.

Previous Theorem 2.8 and Theorem 2.9 are proved in [20] (Theorem 2.3 and Theorem 2.4) for complex matrices.

We use $\mathcal{R}^\dagger$ to denote the set of all Moore-Penrose invertible elements in $\mathcal{R}$. Now, we prove the following result.

Theorem 2.10. *Let* $\mathcal{R}$ *be a ring with involution,* $a, b \in \mathcal{R}^\dagger$, $1 - q = bb^\dagger$ *and* $p = a^\dagger a$. *Then the following statements are equivalent:*

(1) $b^- a^- \in (ab)\{1,2\}$ *for some* $a^- \in a\{2, 3p\}$ *and for some* $b^- \in b\{2, 4q\}$.

(2) $xy \in (ab)\{1,2\}$ *for all* $y \in a\{2, 3p\}$ *and for all* $x \in b\{2, 4q\}$.

Proof. (1) $\Rightarrow$ (2): Suppose that there exists some a^-, b^- such that the following hold:

$$\begin{aligned} a^- aa^- = a^-,\ a^- a = a^\dagger a, &\qquad b^- bb^- = b^-,\ bb^- = bb^\dagger, \\ abb^- a^- ab = ab, &\qquad b^- a^- abb^- a^- = b^- a^-. \end{aligned}$$

Also, suppose that for $x, y \in \mathcal{R}$ we have

$$yay = y,\ ya = a^\dagger a,\ xbx = x,\ bx = bb^\dagger.$$

We obtain the following:

$$abxyab = abb^\dagger a^\dagger ab = abb^- a^- ab = ab,$$

so $xy \in (ab)\{1\}$.

Since $x = xbb^-$ and $y = a^- ay$, we have

$$s = xyabxy = (xbb^-)(a^- ay)ab(xbb^-)(a^- ay).$$

Also,

$$ayabxb = aa^{\dagger}abb^{\dagger}b = ab,$$

so we get

$$s = (xbb^{-}a^{-})ayabxb(b^{-}a^{-}ay) = xb(b^{-}a^{-})ab(b^{-}a^{-})ay = (xbb^{-})(a^{-}ay) = xy.$$

Consequently, $xy \in (ab)\{2\}$.

(2) $\Rightarrow$ (1): Obvious. $\square$

Previous result is an algebraic version of [20] (Theorem 5.4).

Definition 2.4. *Let $p, q \in \mathcal{R}^{\bullet}$, and let $a \in \mathcal{R}$ be such that there exists a $(p,q)-$inner inverse of a, and $u, v \in \mathcal{R}$. Then the splitting $a = u - v$ such that there exists a $(p,q)-$inner inverse of u, is called a $(p,q)-$splitting of a induced by its inner inverse.*

See [11], [19] and references therein for various types of splitting of matrices and operators.

Theorem 2.11. *Let $p, q \in \mathcal{R}^{\bullet}$, $r \in \mathcal{R}$ and let $a \in \mathcal{R}$ such that $a = u - v$ is a (p,q)-splitting of a induced by its inner inverse. Then for $\hat{a}_r$, the following statements are satisfied:*

(1) *There exists $\hat{u}_r$;*

(2) $\hat{a}_r - \hat{u}_r = u^{=}va^{=} = a^{=}vu^{=}$ *for arbitrary elements* $a^{=} \in a\{1, 3p, 4q\}$ *and* $u^{=} \in u\{1, 3p, 4q\}$*;*

(3) $\hat{a}_r = (1 - \hat{u}_r v)^{-1}\hat{u}_r = \hat{u}_r(1 - v\hat{u}_r)^{-1}$*;*

(4) $\hat{u}_r = (1 + \hat{a}_r v)^{-1}\hat{a}_r = \hat{a}_r(1 + v\hat{a}_r)^{-1}$.

Proof. Let $a = u - v$ and let $\hat{u}_s$ be the $(p,q)-$inner inverse of u such that $s = \hat{u}_s - \hat{u}_s u\hat{u}_s$.

(1) Obvious, taking into account that $\hat{u}_r = r - s + \hat{u}_s$, (see the equality (6));

(2) If we denote $a^{=} = \hat{a}_k$ and $u^{=} = \hat{u}_s$ then

$$\begin{aligned} \hat{a}_r - \hat{u}_r &= (\hat{a}_r - r) - (\hat{u}_r - r) \\ &= \hat{a}_r a\hat{a}_r - \hat{u}_r u\hat{u}_r \\ &= a^{=}aa^{=} - u^{=}uu^{=} \\ &= u^{=}ua^{=} - u^{=}aa^{=} \\ &= u^{=}va^{=}. \end{aligned}$$

In the same manner, we get $\hat{a}_r - \hat{u}_r = a^{=}vu^{=}$;

(3) Using $u\hat{a}_r a = a\hat{a}_r u$ with Theorem 2.3, we get the results;

(4) This part follows from (3).

$\square$

3 Inner generalized inverses in Banach algebras

Now, let $\mathcal{R}$ be a normed algebra. If a is invertible, then the condition number of a is defined as $k(a) = \|a\| \cdot \|a^{-1}\|$. The condition number of a is related with the sensitivity of the equation $ax = b$ for perturbations of a. If a is not invertible then the generalized condition number can be used.

Definition 3.1. *The generalized condition number $k_{p,q,(r)}(a)$ of a, is defined with*

$$k_{p,q,(r)}(a) = \|a\| \cdot \|\hat{a}_r\|. \tag{10}$$

Now, we prove the following result.

Theorem 3.1. *Let $a, b, r \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ such that there exist $\hat{a}_r$ and $\hat{b}_r$. Then the following results hold:*

(1) $\hat{a}_r - \hat{b}_r = \hat{b}_r(b-a)\hat{a}_r = \hat{a}_r(b-a)\hat{b}_r$;

(2) *If $\mathcal{R}$ is a Banach algebra and $\|\hat{a}_r\| \cdot \|b-a\| < 1$, then*

$$\frac{\|\hat{a}_r(b-a)\|}{k_{p,q,(r)}(a)(1+\|\hat{a}_r\|\|b-a\|)} \leq \frac{\|\hat{b}_r - \hat{a}_r\|}{\|\hat{a}_r\|}$$
$$\leq \frac{\|\hat{a}_r(b-a)\|}{1-\|\hat{a}_r(b-a)\|} \leq \frac{k_{p,q,(r)}(a)\|b-a\|/\|a\|}{1-k_{p,q,(r)}\|b-a\|/\|a\|}.$$

(3) *If $\mathcal{R}$ is a normed algebra, then*

$$\frac{\|\hat{a}_r\|}{1+\|\hat{a}_r(b-a)\|} \leq \|\hat{a}_r\| \leq \frac{\|\hat{a}_r\|}{1-\|\hat{a}_r(b-a)\|}.$$

Proof.

(1) Obvious from Theorem 2.11;

(2) From Theorem 2.3 and equality (7) we have

$$\begin{aligned}
\hat{b}_r - \hat{a}_r &= (1 + \hat{a}_r(b-a))^{-1}\hat{a}_r - \hat{a}_r \\
&= \left(\sum_{n=0}^{\infty}(-1)^n(\hat{a}_r(b-a))^n - 1\right)\hat{a}_r \\
&= \sum_{n=1}^{\infty}(-1)^n(\hat{a}_r(b-a))^n\hat{a}_r.
\end{aligned}$$

That is, the second and the third inequality hold.

On the other hand,

$$\begin{aligned}
\hat{a}_r(b-a) &= \hat{a}_r b - \hat{a}_r a \\
&= \hat{a}_r b - \hat{b}_r b = (\hat{a}_r - \hat{b}_r)b = \hat{a}_r(b-a)\hat{b}_r b \\
&= \hat{a}_r(b-a)\hat{a}_r a = \hat{a}_r(b-a)\hat{a}_r(1+(b-a)\hat{a}_r)^{-1}(1+(b-a)\hat{a}_r)a \\
&= \hat{a}_r(b-a)\hat{b}_r(1+(b-a)\hat{a}_r)a \\
&= (\hat{a}_r - \hat{b}_r)(1+(b-a)\hat{a}_r)a.
\end{aligned}$$

So, the first inequality is true.

(3) Obvious. □

Theorem 2.11 and Theorem 3.1 extends some results from [11] and [12].

Let a^- and b^- be $(p,q)-$inner inverses of a and b respectively. Now we are going to characterize elements a and b such that $b^-b = a^-a + u$ and $bb^- = aa^- + v$ for $1-u^2, 1-v^2 \in \mathcal{R}^{-1}$.

Such a characterization of outer invertible elements is proved in [18] and for Drazin invertible elements in [4].

Theorem 3.2. [18] *Let $u \in \mathcal{R}$ be such that $1-u^2 \in \mathcal{R}^{-1}$, and let $p, m, p+u \in \mathcal{R}^\bullet$. Then the following conditions are equivalent:*

(1) $m = p + u$

(2) $p(1+u)(1-m) = (1-p)(1-u)m;$

(3) $m(1-u)(1-p) = (1-m)(1+u)p.$

Theorem 3.3. *Let $a, b, u, v \in \mathcal{R}$ be such that a^- and b^- are any $(p,q)-$inner inverses of a and b respectively and $1-u^2, 1-v^2 \in \mathcal{R}^{-1}$. Then the following conditions are equivalent:*

(1) $b^-b = a^-a + u$ *and* $bb^- = aa^- + v$;

(2) $au + vb = b - a - a(a^- - b^-)b$ *and* $ub^- + a^-v = b^-bb^- - a^-aa^- - a^-(a-b)b^-$.

Proof. The proof uses Theorem 3.2 and is similar with the proof of Theorem 3.2 in [18]. □

Let $\mathcal{R}$ be a complex Banach algebra with unit 1. Many papers deal with the computing of the outer inverse of a given element in $\mathcal{R}$ (see [5], [8] and [6]). Here, we give a method for computing the inner inverse with prescribed idempotents of an element a in Banach algebra.

We state an auxiliary result.

Lemma 3.1. *Let $a, p \in \mathcal{R}$, such that $pa = ap$ and $p^2 = p$. Then the element a is invertible in $\mathcal{R}$ if and only if ap is invertible in $p\mathcal{R}p$ and $a(1-p)$ is invertible in $(1-p)\mathcal{R}(1-p)$. In this case*

$$a^{-1} = [ap]^{-1}_{p\mathcal{R}p} + [a(1-p)]^{-1}_{(1-p)\mathcal{R}(1-p)}.$$

Now, we prove the following result.

Theorem 3.4. *Let $p, q \in \mathcal{R}^\bullet$ and $a, c \in \mathcal{R}$ such that, $a^- \in a\{1, 3p, 4q, 5r\}$ and $c^- \in c\{1, 2, 3(1-q), 4(1-p)\}$. Also, suppose that $x_0 = x_0(1-q)$, and $\beta \neq 0$ is a complex number. Define the sequence $(x_k)_k$ in $\mathcal{R}$ in the following way*

$$x_{k+1} = x_k + \beta(1 - x_k a)c, \qquad k = 0, 1, 2, \ldots. \tag{11}$$

If $(1 - q - \beta ac)^k \to 0$, then $x_k \to a^-aa^-$, that is

$$r + x_{k+1} \to a^-.$$

The opposite implication holds if $x_0 - a^-$ is not the left topological divisor of zero.

Proof. The elements a^- and c^- satisfy

$$aa^-a = a,\ cc^-c = c,\ c^-cc^- = c^-,\ a^-a = p = cc^-,\ aa^- = c^-c = 1 - q.$$

Note that $x_0 = x_0(1-q)$ by the assumption. If we suppose that $x_{k-1} = x_{k-1}(1-q)$ than we have

$$x_k(1-q) = (x_{k-1} + \beta(1 - x_{k-1}a)c)(1-q) = x_{k-1} + \beta(1 - x_{k-1}a)c(1-q) = x_k.$$

Then we get

$$\begin{aligned}
x_{k+1} - a^- a a^- &= x_k - a^- a a^- + \beta(1 - x_k a)c \\
&= (x_k - a^-) + \beta(1 - x_k a)cc^- c \\
&= (x_k - a^-)aa^- - \beta(x_k a - a^- a)c \\
&= (x_k - a^- a a^-)(1 - q) - \beta(x_k - a^- a a^-)ac \\
&= (x_k - a^- a a^-)(1 - q - \beta ac) \\
&= \dots \\
&= (x_0 - a^- a a^-)(1 - q - \beta ac)^{k+1} \\
&= (x_0 - a^-)(1 - q - \beta ac)^{k+1}
\end{aligned} \tag{12}$$

So, since $(1 - q - \beta ac)^k \to 0$ we get the result $x_{k+1} \to a^- a a^-$, that is

$$r + x_{k+1} \to a^-.$$

If $x_0 - a^-$ is not a topological divisor of zero, then the opposite implication also holds.

□

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