Moore–Penrose inverse in rings with involution

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Abstract

We study the Moore–Penrose inverse (MP-inverse) in the setting of rings with involution. The results include the relation between regular, MP-invertible and well-supported elements. We present an algebraic proof of the reverse order rule for the MP-inverse valid under certain conditions on MP-invertible elements. Applications to C^*-algebras are given.

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1 Introduction and preliminaries

In this paper we study the Moore–Penrose inverse in rings with involution. Whereas in C^*-algebras or $*$-reducing algebras $a^*a = 0$ always implies $a = 0$, in this paper we only consider $*$-cancellability as a local property.

The paper is motivated by the work of Harte and Mbekhta [11, 12] in C^*-algebras and Koliha and Patricio [18] in rings with involution. (See also the recent paper of Fernandez-Miranda and Labrousse [9].) We relate the concept of a well-supported element in a ring with involution (see [1] for a C^*-algebra definition) to the regularity of the element and the existence of the Moore–Penrose inverse.

In Section 3 we give applications of our results to C^*-algebras, in particular to the characterization of stable rank 1 and real rank 1 (see [15]).

In Section 4 we study the reverse order rule for the product of Moore–Penrose invertible elements in the setting of rings with involution, extending the known results for matrices [4] and Hilbert space operators [2, 3] and [14]. We then apply this result to obtain the reverse order rule for the weighted Moore–Penrose inverse in C^*-algebras in Section 5, generalizing the matrix results of Sun and Wei [23].

Throughout this paper, $\mathcal{R}$ will be a ring with a unit $1 \neq 0$ and an involution $a \mapsto a^*$ satisfying

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*.$$

By $\mathcal{R}^{-1}$ we denote the group of invertible elements in $\mathcal{R}$, and by $\mathcal{R}^{\mathsf{sa}}$ the set of all self-adjoint elements of $\mathcal{R}$ ($a^* = a$). An element $a \in \mathcal{R}$ is *regular* (in the sense of von Neumann) if $a \in a\mathcal{R}a$. The set of all regular elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^-$.

An element $f \in \mathcal{R}$ is *idempotent* if $f^2 = f$. A self-adjoint idempotent is a *projection*. The idempotents $f, g \in \mathcal{A}$ are *equivalent*, written $f \sim g$, if there exist elements $a, b \in \mathcal{A}$ such that $f = ba$ and $g = ab$. Any regular element a generates equivalent idempotents: If $a = aba$, then $f = ba$ and $g = ab$ are equivalent idempotents. Idempotents $f, g \in \mathcal{R}$ are *mutually orthogonal*, written $f \perp g$, if $fg = 0 = gf$.

The usual notation for the commutator of u and v is used: $[u, v] = uv - vu$. In this paper we shall frequently use the fact that the *product of two selfadjoint elements u and v is self-adjoint if and only if* $[u, v] = 0$.

Definition 1. We say that $a \in \mathcal{R}$ is *Moore–Penrose invertible* (or MP-*invertible*), if there exists $b \in \mathcal{R}$ such that the following hold [20]:

$$aba = a, \; bab = b, \; (ab)^* = ab, \; (ba)^* = ba. \tag{1}$$

Any b that satisfies (1) is called a *Moore–Penrose inverse* of a.

It is well known that the Moore–Penrose inverse is unique when it exists; here is a quick argument based on the observation that $[xa, ya] = [ax, ay] = 0$ if x, y are two candidates for a Moore–Penrose inverse for a:

$$x = xax = (xa)(ya)x = (ya)(xa)x = yax = y(ay)(ax) = y(ax)(ay) = yay = y.$$

We will denote the Moore–Penrose inverse of a by $a^\dagger$. We point out some properties of the Moore–Penrose inverse that follow from the definition. Clearly, a is MP-invertible if and only if a^* is MP-invertible; in this case

$$(a^*)^\dagger = (a^\dagger)^*.$$

If a is MP-invertible, then so are a^*a and aa^*, while

$$(a^*a)^\dagger = a^\dagger (a^*)^\dagger, \quad (aa^*)^\dagger = (a^*)^\dagger a^\dagger.$$

Definition 2. An element $a \in \mathcal{R}$ is *left $*$-cancellable* if $a^*ax = a^*ay$ implies $ax = ay$, it is *right $*$-cancellable* if $xaa^* = yaa^*$ implies $xa = ya$, and *$*$-cancellable* if it is both left and right cancellable. We observe that a is left $*$-cancellable if and only if a^* is right $*$-cancellable.

In a C^*-algebra, every element is $*$-cancellable: If $a^*az = 0$, then $\|az\|^2 = \|(az)^*az\| = \|z^*a^*az\| = 0$; similarly $zaa^* = 0$ implies $za = 0$. A ring $\mathcal{R}$ is called *$*$-reducing* if every element of $\mathcal{R}$ is $*$-cancellable. This is equivalent to the implication $a^*a = 0 \implies a = 0$ for all $a \in \mathcal{R}$.

If $T : X \to Y$ and $S : Y \to Z$ are bounded and linear operators between normed spaces, then we may classify the par (S,T) as left skew exact if there is the equality

$$(ST)^{-1}\{0\} = \{0\}. \tag{2}$$

The condition (2) is equivalent to the condition $S^{-1}\{0\} \cap T(X) = \{0\}$ which holds for left*-cancellable $a \in A$ if we take $S = L_{a^*}$ and $T = L_a$. Hence, for left*-cancellable $a \in \mathcal{A}$, the pair (L_{a^*}, L_a) is left skew exact. When $\mathcal{A} = B(H)$ is an algebra of bounded operators on Hilbert space, then for arbitrary $a \in \mathcal{A}$ the pair (L_{a^*}, L_a) is left skew exact (for more details concerning exactness see [19]) .

Definition 3. The Drazin inverse of $a \in \mathcal{A}$ is the element $a^D \in \mathcal{A}$ which satisfies

$$a^D a a^D = a^D, \quad a a^D = a^D a, \quad a^{k+1} a^D = a^k \tag{3}$$

for some nonnegative integer k. The least such k is the index of a, denoted by $ind(a)$.

Drazin inverse of a is unique if it exists. When $ind(a) = 1$ then the Drazin inverse a^D is called the group inverse. It is well known that a^D double commutes with a, that is, $[a, x] = 0 \implies [a^D, x] = 0$. Also, when $a = a^*$ and a is Drazin invertible then $ind(a) \leq 1$.

The basic existence theorem for the Moore–Penrose inverse in the setting of rings with involution was given in [22, Theorem 8.25] (see also [18, Theorem 5.3]):

Proposition 1. *Let $\mathcal{R}$ be a ring with involution and let $a \in \mathcal{R}$. Then the following are equivalent*:

(a) *a is MP-invertible.*

(b) *a is left $*$-cancellable and a^*a is group invertible.*

(c) *a is right $*$-cancellable and aa^* is group invertible.*

(d) *a is $*$-cancellable and both a^*a and aa^* are group invertible.*

The MP-inverse of a is given by

$$a^\dagger = (a^*a)^D a^* = a^*(aa^*)^D.$$

To gain access to the circle of ideas connected with the positivity of elements of the form a^*a in C^*-algebras, we coin the following term.

Definition 4. A ring $\mathcal{R}$ with involution has the *Gelfand–Naimark property* (*GN-property*) if

$$1 + x^*x \in \mathcal{R}^{-1} \quad \text{for all } x \in \mathcal{R}. \tag{4}$$

It is well known that C^*-algebras possess the GN-property.

2 Existence of the MP-inverse

We extend the definition [1, Definition 6.5.3] from C^*-algebras to rings with involution.

Definition 5. An element a of a ring $\mathcal{R}$ with involution is *well-supported* if there exists a self-adjoint idempotent p such that

$$ap = a, \quad a^*a + 1 - p \in \mathcal{R}^{-1}. \tag{5}$$

The idempotent p is called the *support* of a.

(The second condition in [1, Definition 4.3.3] is the invertibility of a^*a in $p\mathcal{R}p$; this is easily seen to be equivalent to $a^*a + 1 - p \in \mathcal{R}^{-1}$.)

Notice that if a is a linear bounded operator on a Hilbert space H, then a is well-supported in the ring of all linear bounded operators on H if and only if the range of a is closed. In this case p is the orthogonal projection from H onto the range of a^*.

We observe that a support p of $a \in \mathcal{R}$ satisfies $p^0 = a^0$, where

$$a^0 = \{x \in \mathcal{R} : ax = 0\}.$$

Indeed, if $ax = 0$, then $a^*apx = 0$ and $(a^*a + 1 - p)px = 0$, which implies $px = 0$. Conversely, $px = 0$ implies $ax = apx = 0$. From $p^0 = a^0$ we deduce that the support is unique: Suppose p, q are two supports for a. Then $p^0 = a^0 = q^0$. From $1 - p \in p^0 \subset q^0$ we obtain $q = qp$. Interchanging the roles of p and q we get $p = pq$. Taking adjoints, we get $p = p^* = qp = q$. The support p is in the double commutant of $\{a, a^*\}$, that is, $[a, x] = 0 = [a^*, x]$ implies $[p, x] = 0$. This can be also deduced from the equation $p^0 = a^0$.

Theorem 1. *Let $\mathcal{R}$ be a ring with involution. An element $a \in \mathcal{R}$ is MP-invertible if and only if a is left $*$-cancellable and well-supported. The support p of a is given by $p = a^\dagger a$.*

Proof. Suppose that a is left $*$-cancellable and well-supported with the support p. We observe that $[a^*a, p] = 0$ and $a^*ap = a^*a$. Set

$$b = (a^*a + 1 - p)^{-1}p.$$

Then $a^*ab = ba^*a = p$, $a^*ab^2 = (a^*ab)b = pb = b$, and $(a^*a)^2b = a^*a(a^*ab) = a^*ap = a^*a$. This proves $b = (a^*a)^D$. Hence by Proposition 1, a is MP-invertible with

$$a^\dagger = (a^*a + 1 - p)^{-1}p.$$

Conversely, let a be MP-invertible. Set $p = a^\dagger a$. Then $ap = aa^\dagger a = a$. By Proposition 1, a^*a is group invertible. Then

$$((a^*a)^D + 1 - p)(a^*a + 1 - p) = (a^*a)^D a^*a + 1 - p = a^\dagger a + 1 - p = 1,$$

which shows that $a^*a + 1 - p$ is invertible. Hence a is well-supported with the support $p = a^\dagger a$. $\square$

In analogy with a support we can introduce a *co-support* of a as a projection $q \in \mathcal{R}$ satisfying

$$qa = a, \quad aa^* + 1 - q \in \mathcal{R}^{-1}. \tag{6}$$

An element $a \in \mathcal{R}$ is MP-invertible if and only if it has a co-support q and is right $*$-cancellable. In this case $q = aa^\dagger$.

Theorem 2. *Let $\mathcal{R}$ be a ring with involution satisfying the GN-property. Then $a \in \mathcal{R}$ is MP-invertible if and only if a is regular.*

Proof. Any MP-invertible element a is regular as $a = aa^\dagger a$.

Suppose that a is regular, that is, $aba = a$ for some $b \in \mathcal{R}$. The elements $f = ba$ and $g = ab$ are idempotents, and

$$s = 1 + (g^* - g)^*(g^* - g) \in \mathcal{R}^{-1}, \quad t = 1 + (f^* - f)^*(f^* - f) \in \mathcal{R}^{-1}$$

in view of the GN-property. Define

$$p = gg^*s^{-1} \quad \text{and} \quad q = f^*ft^{-1}.$$

After some algebra (see, for instance, [1, Proposition 4.6.2] for details) we obtain

$$\begin{aligned} p^2 = p = p^*, \quad pg = g, \quad gp = p, \\ q^2 = q = q^*, \quad fq = f, \quad qf = q. \end{aligned}$$

From $af = a$ and $ga = a$ we obtain $aq = a$ and $pa = a$, respectively. Set $c = qbp$. Then

$$\begin{aligned} ac &= aqbp = abp = gp = p \in \mathcal{R}^{\mathsf{sa}}, \\ ca &= qbpa = qba = qf = q \in \mathcal{R}^{\mathsf{sa}}, \\ aca &= (ac)a = pa = a, \\ cac &= (ca)c = qc = c. \end{aligned}$$

This proves that c is the Moore–Penrose inverse of a. □

Regular elements in rings with involution need not be $*$-cancellable. The preceding theorem together with Proposition 1 shows that in a ring with the GN-property, regularity does imply $*$-cancellability.

3 Applications to C^*-algebras

Let $\mathcal{A}$ be a unital C^*-algebra. Then $\mathcal{A}$ is a $*$-reducing ring with the GN-property, and we can apply to it the results of the preceding section. The denseness of the set of all well-supported elements in $\mathcal{A}$ plays an important role in the theory of stable rank of C^*-algebras. From Theorems 1 and 2 we obtain the following result.

Proposition 2. *In a unital C^*-algebra $\mathcal{A}$ the following conditions on $a \in \mathcal{A}$ are equivalent*:

(a) *a is well-supported,*

(b) *a is Moore–Penrose invertible,*

(c) *a is regular.*

This has implications for characterizations within C^*-algebra theory involving well-supported elements as the simple algebraic property of regularity does not involve involution. It is also convenient to have on hand the other characterization of well-supported elements as the Moore–Penrose invertible elements of $\mathcal{A}$.

Let $\mathcal{A}$ be a unital C^*-algebra. The *stable rank* of $\mathcal{A}$ is the least positive integer such that the elements $(x_1, \ldots, x_n) \in \mathcal{A}^n$ with $\sum_{i=1}^n x_i^* x_i \in \mathcal{A}^{-1}$ are dense in $\mathcal{A}^n$. A parallel theory of *real rank* was introduced in [5]: It is the least nonnegative integer n for which the elements of the form $(x_0, x_1, \ldots, x_n) \in (\mathcal{A}^{\mathsf{sa}})^{n+1}$ with $\sum_{i=1}^n x_i^2 \in \mathcal{A}^{-1}$ are dense in $(\mathcal{A}^{\mathsf{sa}})^{n+1}$. For instance, $\mathcal{A}$ has stable rank 1 if the invertible elements are dense in $\mathcal{A}$; it has real rank 0 if the invertible self-adjoint elements are dense in $\mathcal{A}^{\mathsf{sa}}$; it has real rank less than or equal to 1 if the elements $x \in \mathcal{A}$ for which $x^*x + xx^* \in \mathcal{A}^{-1}$ are dense in $\mathcal{A}$.

Let $\mathrm{ex}(\mathcal{A})$ be the set of all extreme points of the closed unit ball of $\mathcal{A}$. A C^*-algebra $\mathcal{A}$ is called *extremally rich* (see [5]) if the set $\mathcal{A}^{-1}\mathrm{ex}(\mathcal{A})\mathcal{A}^{-1}$ is dense in $\mathcal{A}$. From Proposition 2 we obtain the following result (see [15, Proposition 3.2]).

Proposition 3. *Let $\mathcal{A}$ be a unital C^*-algebra. If the $\mathcal{A}$ is extremally rich, then the set of all regular elements of $\mathcal{A}$ is dense in $\mathcal{A}$.*

Proof. From [5, Theorem 1.1] it follows that if x is an extreme point of $\mathcal{A}$, then x^*x is invertible or 0 is an isolated spectral point of x^*x. Then x is Moore–Penrose invertible by [16, Theorem 1.1], and hence x is regular by Proposition 2. Hence $\mathcal{A}^{-1}\mathrm{ex}(\mathcal{A})\mathcal{A}^{-1} \subset \mathcal{A}^-$, and the result follows. □

The following result is obtained from Proposition 2 and [1, Theorem 6.5.6].

Proposition 4. *Let $\mathcal{A}$ be a unital C^*-algebra. Then the regular self-adjoint elements of $\mathcal{A}$ are dense in $\mathcal{A}^{\mathsf{sa}}$ if and only if real rank of $\mathcal{A}$ is 0.*

We say that $\mathcal{A}$ has *cancellation of idempotents* if

$$f \perp h \quad g \perp h, \quad f + h \sim g + h \implies f \sim g.$$

From Proposition 2 and [15, Theorem 3.8] we get the following.

Proposition 5. *Let $\mathcal{A}$ be a unital C^*-algebra. Then the following are equaivalent.*

(a) *$\mathcal{A}$ has stable rank* 1.

(b) *The regular elements of $\mathcal{A}$ are dense in $\mathcal{A}$ and $\mathcal{A}$ has cancellation of projections.*

Proposition 2 combined with [15, Proposition 3.6] yields the following result.

Proposition 6. *Let $\mathcal{A}$ be a unital C^*-algebra in which regular elements are dense in $\mathcal{A}$. Then $\mathcal{A}$ has real rank less than or equal to* 1.

4 Reverse order rule for the Moore–Penrose inverse

If a, b are invertible in a semigroup with the unit, then the rule $(ab)^{-1} = b^{-1}a^{-1}$ is known as the reverse order rule for the ordinary inverse. In the case of the Moore–Penrose inverse in a ring with involution, the rule $(ab)^\dagger = b^\dagger a^\dagger$ is not always satisfied. Greville [10] proved that $(ab)^\dagger = b^\dagger a^\dagger$ holds for complex matrices if and only if $a^\dagger a$ commutes with bb^* and $bb^\dagger$ commutes with aa^* (see also Boullion and Odell [4]).

Bouldin [2, 3] and Izumino [14] generalized this result for closed range operators on Hilbert spaces. Their proofs are based on operator theoretical methods and use properties of ranges of operators and gaps between subspaces.

In this section we give a proof of the reverse order rule for the Moore–Penrose inverse in the setting of rings with involution. Our proof is close to the one given by Boullion and Odell in [4] for matrices, but is purely algebraic and, unlike the proof in [4], it avoids any reference to ranges of transformations.

We mention that multiple matrix products are considered in [13] and [24]. The weighted Moore-Penrose inverse is investigated in [23]. Reverse order rule involving ranks of various types of matrices is studied in [7] and [25]. More general reverse order rule for generalized inverses are also investigated in [8, 26]. Recently, several papers on generalized inverses in rings, or Banach and C^*-algebras with strong emphasis on algebraic methods, appeared [16, 17, 18].

Now we prove the main result of this section.

Theorem 3. *Let $\mathcal{R}$ be a ring with involution, let $a, b \in \mathcal{R}$ be MP-invertible and let $(1-a^\dagger a)b$ be left $*$-cancellable. Then the following condition are equivalent:*

(a) *ab is MP-invertible and* $(ab)^\dagger = b^\dagger a^\dagger$,

(b) $[a^\dagger a, bb^*] = 0$ *and* $[bb^\dagger, a^*a] = 0$.

Proof. (b) $\implies$ (a): Suppose that (b) holds. By part (c) of Lemma, we have $[a^\dagger a, (bb^*)^\dagger] = 0$ and $[bb^\dagger, (a^*a)^\dagger] = 0$. From part (b) of Lemma, $[a^\dagger a, bb^\dagger] = [a^\dagger a, (bb^*)(bb^*)^\dagger] = 0$. Then

$$\begin{aligned} abb^\dagger a^\dagger ab &= a(bb^\dagger)(a^\dagger a)b = a(a^\dagger a)(bb^\dagger)b = ab, \\ b^\dagger a^\dagger abb^\dagger a^\dagger &= b^\dagger (a^\dagger a)(bb^\dagger)a^\dagger = b^\dagger (bb^\dagger)(a^\dagger a)a^\dagger = b^\dagger a^\dagger, \end{aligned}$$

which implies that $b^\dagger a^\dagger$ is a reflexive (inner and outer) generalized inverse of ab. Further,

$$abb^\dagger a^\dagger = abb^\dagger (a^*a)^\dagger a^* = a(a^*a)^\dagger bb^\dagger a^* = (a^*)^\dagger bb^\dagger a^* = (abb^\dagger a^\dagger)^*$$

and

$$b^\dagger a^\dagger ab = b^*(bb^*)^\dagger a^\dagger ab = b^* a^\dagger a(bb^*)^\dagger b = b^* a^\dagger a(b^\dagger)^* = (b^\dagger a^\dagger ab)^*.$$

(a) $\implies$ (b): The left hand side of

$$b^\dagger a^\dagger ab = (b^*b)^\dagger (b^* a^\dagger ab)$$

is self-adjoint, which implies

$$[(b^*b)^\dagger, b^* a^\dagger ab] = 0.$$

Further,

$$\begin{aligned} abb^*b &= abb^\dagger a^\dagger abb^*b = ab(b^*b)^\dagger (b^* a^\dagger ab)b^*b \\ &= ab(b^* a^\dagger ab)(b^*b)^\dagger b^*b = abb^* a^\dagger abb^\dagger b \\ &= abb^* a^\dagger ab, \end{aligned}$$

where we used the equation $(b^*b)^\dagger b^*b = b^\dagger b$. Hence $abb^*(1 - a^\dagger a)b = 0$, and

$$abb^*(1 - a^\dagger a)(1 - a^\dagger a)b = ab((1 - a^\dagger a)b)^*(1 - a^\dagger a)b = 0.$$

Using the hypothesis that $(1 - a^\dagger a)b$ is left $*$-cancellable, we get

$$abb^*(1 - a^\dagger a) = 0 \quad \text{and} \quad abb^* = abb^* a^\dagger a.$$

Now we find that

$$a^\dagger abb^* = a^\dagger abb^* a^\dagger a = a^\dagger ab(a^\dagger ab)^*$$

is self-adjoint, implying

$$[a^\dagger a, bb^*] = 0.$$

To prove the second result of (b), notice that by taking adjoints in $(ab)^\dagger = b^\dagger a^\dagger$ we obtain

$$(b^* a^*)^\dagger = (a^*)^\dagger (b^*)^\dagger.$$

From the first part of the implication (a) $\Longrightarrow$ (b), we get

$$[(b^*)^\dagger b^*, a^* a] = 0,$$

which is equivalent to

$$[bb^\dagger, a^* a] = 0. \qquad \square$$

Let us remark that condition (ii) of the preceding theorem can be expressed in several equivalent ways, as for instance in [4].

If $\mathcal{R} = \mathcal{B}(X)$ is the space of all bounded linear operators on a Hilbert space H, it is known that $A \in \mathcal{B}(X)$ is Moore–Penrose invertible if and only if the range of A is closed. If $A, B \in \mathcal{B}(X)$ are two closed range operators such that

$$[A^\dagger A, BB^*] = 0 \quad \text{and} \quad [B^\dagger B, A^* A] = 0,$$

then our theorem implies that AB is also a closed range operator.

Remark 1. The preceding theorem holds in $*$-reducing rings without the hypothesis that $(1 - a^\dagger a)b$ is left $*$-cancellable, which is then automatically satisfied. Hence we recover the results of Bouldin [2, 3] and Izumino [14] for Hilbert space operators. The results of Greville [10] are obtained as a special case of our Theorem, without the hypotheses of Moore–Penrose invertibility and $*$-cancellability, which are always true for matrices. Notice that results obtained in [13] hold in $*$-reducible ring providing that the implication $uv = 1 \implies vu = 1$ is satisfied.

5 The weighted MP-inverse in C^*-algebras

In this section we consider the so-called weighted MP-inverse. It was introduced by Chipman [6] for matrices, who used positive definite weight matrices, and extended by Prasad and Bapat [21] to include invertible, not necessarily positive definite weights.

Definition 6. Let $\mathcal{R}$ be a ring with involution and e, f two invertible elements in $\mathcal{R}$. We say that an element $a \in \mathcal{R}$ has a *weighted MP-inverse* with weights e, f if there exists $b \in \mathcal{R}$ such that

$$aba = a, \quad bab = b, \quad (eba)^* = eba, \quad (fab)^* = fab. \tag{7}$$

An element $a \in \mathcal{R}$ can have at most one weighted MP-inverse with given weights e, f: Suppose that $c \in \mathcal{R}$ is another such weighted MP-inverse for a. Then $(ab)^* = eabe^{-1}$ and $(ac)^* = eace^{-1}$. We also have $abac = ac$ and $acab = ab$. Taking adjoints we obtain $ca = ba$. Finally, $b = bab = bac = cac = c$. The unique weighted MP-inverse with weights e, f will be denoted by $a^\dagger_{e,f}$ if it exists.

Prasad and Bapat [21, Theorem 3 and Theorem 8] found necessary and sufficient conditions for the existence of the weighted MP-inverse for matrices. They showed that the conditions that a^*ea and $af^{-1}a^*$ are self-adjoint are necessary for the existence of $a^\dagger_{e,f}$. It therefore makes sense to assume that e and f are self-adjoint. In the next theorem we prove the existence of the weighted MP-inverse in a C^*-algebra $\mathcal{A}$ under the hypothesis that e, f are positive invertible elements in $\mathcal{A}$.

Suppose $e \in \mathcal{A}$ is positive and invertible. Then the mapping $x \mapsto x^{*e} = e^{-1}x^*e$ is an involution on $\mathcal{A}$. Further, for any $x \in \mathcal{A}$ define $\|x\|_e = \|e^{1/2}xe^{-1/2}\|$. We can verify that $\mathcal{A}_e = (\mathcal{A}, {}^{*e}, \|\cdot\|_e)$ is a unital C^*-algebra with the involution $x \mapsto x^{*e}$ and the norm $\|\cdot\|_e$. Conditions (7) can be rewritten as

$$aba = a, \quad bab = b, \quad (ba)^{*e} = ba, \quad (ab)^{*f} = ab. \tag{8}$$

We can then prove the following theorem.

Theorem 4. *Let $\mathcal{A}$ be a unital C^*-algebra and let e, f be positive invertible elements of $\mathcal{A}$. If $a \in \mathcal{A}$ is regular, then the unique weighted MP-inverse $a^\dagger_{e,f}$ exists.*

Proof. Since a is regular in the C^*-algebra $\mathcal{A}_e$, a has the MP-inverse $u \in \mathcal{A}_e$ satisfying

$$aua = a, \quad uau = u, \quad (ua)^{*e} = ua, \quad (au)^{*e} = au.$$

Similarly, a has the MP-inverse v in the C^*-algebra $\mathcal{A}_f$ satisfying

$$ava = a, \quad vav = v, \quad (va)^{*f} = va, \quad (av)^{*f} = av.$$

It is then straightforward to verify that $b = uav$ satisfies (8) and is therefore the required MP-inverse $a^\dagger_{e,f}$. □

It is useful to express the weighted MP-inverse in terms of the ordinary MP-inverse.

Theorem 5. *Let $\mathcal{A}$ be a unital C^*-algebra and let e, f be positive invertible elements of $\mathcal{A}$. If $a \in \mathcal{A}$ is regular, then*

$$a^{\dagger}_{e,f} = e^{-1/2}(f^{1/2}ae^{-1/2})^{\dagger}f^{1/2}. \tag{9}$$

Proof. Since a is regular with an inner inverse a^-, then so is $a_1 = f^{1/2}ae^{-1/2}$ with an inner inverse $e^{1/2}a^-f^{-1/2}$. Hence $a_1^{\dagger}$ exists. Write $b = e^{-1/2}a_1^{\dagger}f^{1/2}$. Then $a = f^{-1/2}a_1e^{1/2}$, $ba = e^{-1/2}a_1^{\dagger}a_1e^{1/2}$ and $bab = e^{1/2}a_1^{\dagger}a_1a_1^{\dagger}f^{1/2} = b$. Similarly we verify that $ab = f^{-1/2}a_1a_1^{\dagger}e^{1/2}$ and $aba = f^{-1/2}a_1a_1^{\dagger}a_1e^{1/2} = a$. Further,

$$\begin{aligned}(ba)^{*e} &= e^{-1}e^{1/2}(a_1^{\dagger}a_1)^*e^{-1/2}e = e^{-1/2}a_1^{\dagger}a_1e^{1/2} = ba,\\ (ab)^{*f} &= f^{-1}f^{1/2}(a_1a_1^{\dagger})^*f^{-1/2}f = f^{-1/2}a_1a_1^{\dagger}f^{1/2} = ab.\end{aligned}$$

This proves $b = a^{\dagger}_{e,f}$. □

Before we can give the reverse order rule for the weighted MP-inverse we introduce the weighted involution in $\mathcal{A}$. Let e, f be positive invertible elements of $\mathcal{A}$ and define $x^{*e,f} = e^{-1}x^*f$. It can be checked that this defines an involution on $\mathcal{A}$ satisfying $(a^{*e,f})^* = (a^*)^{*e,f}$. Note that the B^*-identity need not hold for this involution.

Theorem 6. *Let $\mathcal{A}$ be a unital C^*-algebra and e, f, h positive invertible elements of $\mathcal{A}$. It $a, b \in \mathcal{A}$ are regular, the following conditions are equivalent*:

(a) *ab is regular and $(ab)^{\dagger}_{e,h} = b^{\dagger}_{e,f}a^{\dagger}_{f,h}$.*

(b) *$[a^{\dagger}_{f,h}\, a, bb^{*e,f}] = 0$.*

Proof. Let $a_1 = h^{1/2}af^{-1/2}$ and $b_1 = f^{1/2}be^{-1/2}$. Then $a_1b_1 = h^{1/2}abe^{-1/2}$. According to Theorem 5,

$$(ab)^{\dagger}_{e,h} = e^{-1/2}(a_1b_1)^{\dagger}h^{1/2}, \qquad b^{\dagger}_{e,f}a^{\dagger}_{f,h} = e^{-1/2}b_1^{\dagger}a_1^{\dagger}h^{1/2},$$

provided a_1, b_1, a_1b_1 are regular, which occurs if and only if a, b, ab are regular, respectively. Hence the equation $(ab)^{\dagger}_{e,h} = b^{\dagger}_{e,f}a^{\dagger}_{f,h}$ holds if and only if $(a_1b_1)^{\dagger} = b_1^{\dagger}a_1^{\dagger}$, and we can apply Theorem 3 to finish the proof.

The following equations show that (b) is equivalent to $[a_1^\dagger a_1, b_1 b_1^*] = 0$:

$$a_1^\dagger a_1 = (h^{1/2} a f^{-1/2})^\dagger h^{1/2} a f^{-1/2} = f^{1/2} a_{f,h}^\dagger a f^{-1/2},$$
$$b_1 b_1^* = (f^{1/2} b e^{-1/2})(e^{-1/2} b^* f^{1/2}) = f^{1/2} b b^{*e,f} f^{-1/2}.$$

The proof now follows from Theorem 3. □

Necessary and sufficient conditions for the reverse order rule for the weighted MP-inverse for matrices were recently given by Sun and Wei in [23] in terms of the inclusion of matrix ranges (column spaces).

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