GENERAL REPRESENTATIONS OF PSEUDO INVERSES

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Abstract. In this paper we investigate genereal representations of various classes of generalized inverses for bounded operators over Hilbert and Banach spaces. These representations are expressed by means of the full-rank decomposition of bounded operators and adequately selected operators.

1. Introduction

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ denote arbitrary Banach spaces and $B(\mathcal{X}_1, \mathcal{X}_2)$ denote the set of all bounded operators from $\mathcal{X}_1$ into $\mathcal{X}_2$. For an arbitrary operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$, we use $\mathcal{N}(A)$ to denote its kernel, and $\mathcal{R}(A)$ to denote its image.

For $A \in B(\mathcal{X}_1, \mathcal{X}_2)$ we say that an operator $X \in B(\mathcal{X}_2, \mathcal{X}_1)$ is a generalized inverse of A, provided that some of the following equations are satisfied:

$$(1) \quad AXA = A, \qquad (2) \quad XAX = X$$

If X satisfies the equation (1), then X is called a g-inverse of A. If X satisfies the equations (1) and (2), then it is called a reflexive g-inverse of A.

It is well–known that an operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$ has a g-inverse if and only if $\mathcal{R}(A)$ is closed, and $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, are complemented subspaces of $\mathcal{X}_1$ and $\mathcal{X}_2$.

The notion of the full–rank decomposition for complex matrices is well-known and frequently used. Recall the definition of the full rank factorization for a bounded operator acting on Banach spaces from [2] and [3]:

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Let $A \in B(\mathcal{X}_1, \mathcal{X}_2)$. If there exist: a Banach space $\mathcal{X}_3$ and operators $Q \in B(\mathcal{X}_1, \mathcal{X}_3)$ and $P \in B(\mathcal{X}_3, \mathcal{X}_2)$, such that P is left invertible, Q is right invertible and

$$A = PQ, \tag{1.1}$$

then we say that (1.1) is the full-rank decomposition of A.

It is well-known that an operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$ has the full-rank decomposition, if and only if A is g-invertible. In this case $\mathcal{X}_3$ is isomorphic to $\mathcal{R}(A)$, and $\mathcal{R}(A) = \mathcal{R}(P)$ [3].

We say that $A \in B(\mathcal{X})$ has the Drazin inverse, if there exists an operator $A^D \in B(\mathcal{X})$, such that A^D satisfies the equation (2) and the equations

$$(1^k) \quad A^{k+1}A^D = A^k, \qquad (5) \quad A^D A = AA^D,$$

for some non-negative integer k. Let us mention that the Drazin inverse, if it exists, is unique. The smallest k in the previous definition is called the index of A and denoted by $\mathrm{ind}(A)$. In the case $\mathrm{ind}(A) = 1$ the Drazin inverse is known as the group inverse of A, denoted by $A^{\#}$.

Recall that $\mathrm{asc}(A)$ (respectively $\mathrm{des}(A)$), the ascent (respectively descent) of A, is the smallest non-negative integer n, such that $\mathcal{N}(A^n) = \mathcal{N}(A^{n+1})$ (respectively $\mathcal{R}(A^n) = \mathcal{R}(A^{n+1})$). If no such n exists, then $\mathrm{asc}(A) = \infty$ (respectively $\mathrm{des}(A) = \infty$) [4]. It is well-known that A has the Drazin inverse, if and only if the ascent and descent of A are finite (hence, equal to $\mathrm{ind}(A)$) [3], [4].

In the case when $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, it is well-known that an operator $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has a g-inverse if and only if $\mathcal{R}(A)$ is closed. Among the equations (1), (2) we also consider the following equations in X:

$$(3) \quad (AX)^* = AX, \qquad (4) \quad (XA)^* = XA.$$

For a subset $\mathcal{S}$ of the set $\{1, 2, 3, 4\}$, the set of operators obeying the conditions contained in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. An operator in $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and is denoted by $A^{(\mathcal{S})}$. If $\mathcal{R}(A)$ is closed, the set $A\{1, 2, 3, 4\}$ consists of a single element, the Moore-Penrose inverse of A, denoted by $A^\dagger$.

We also consider the following equations, which define the weighted Moore-Penrose inverse:

$$(3M) \qquad (MAX)^* = MAX \qquad\qquad (4N) \qquad (NXA)^* = NXA;$$

$$(3M') \qquad (AX)^*M = MAX \qquad\qquad (4N') \qquad (XA)^*N = NXA,$$

where $M \in B(\mathcal{H}_2)$, $N \in B(\mathcal{H}_1)$ are positive or invertible. Any solution of the equations (1), (2), (3M) and (4N), when it exists, will be denoted by $A^{\dagger}_{M,N}$. Similarly, any solution of the equations (1), (2), (3M') and (4N'), when it exists, will be denoted by $A^{\dagger}_{M',N'}$.

We investigate general representations and conditions for the existence of generalized inverses of bounded linear operatos on Hilbert spaces, arising from the factorization (1.1). As a related result we investigate some representations of a generalized inverse $A^{(2)}_{T,S}$. Obtained representations are generalizations of the analogous results available in the literature for marices. We also introduce a general representation and conditions for thye existence of the Drazin inverse of a bounded operator on a Banach space. There representatons are based on the full-rank decomposition of A^l, where $l \geq \operatorname{ind}(A)$. Such an approach in representation of the Drazin inverse is not employed before even for complex matrices.

2. Results

Firstly, we investigate general representations of $\{1,2\}$-inverses, $\{1,2,3\}$, $\{1,2,4\}$-inverses, the Moore-Penrose and the weighted Moore-Penrose for operators in arbitrary Hilbert spaces.

We shall frequently use the following observation. If $S \in B(\mathcal{H}_1, \mathcal{H}_2)$ is onto, then SS^* is invertible and $S^{\dagger}$ is the right inverse of S. Analogously, if $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ is one-to-one with closed range, then T^*T is invertible and $T^{\dagger}$ is the left inverse of T.

In the beginning, we state an analogy of the well-known result from [10, p. 20, 28].

Lemma 2.1. *If $A = PQ$ is the full-rank decomposition of $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ according to* (1.1)*, then:*

(a) *Any right inverse of Q can be represented in the following form: $Q_r^{-1} = W_1(QW_1)^{-1}$, for an arbitrary operator $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$ such that QW_1 is invertible.*
(b) *Any left inverse of P can be represented in the following form: $P_l^{-1} = (W_2P)^{-1}W_2$, for an arbitrary operator $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$ such that W_2P is invertible.*
(c) *Any reflexive generalized inverse X of A has the form $X = Q_r^{-1}P_l^{-1}$ for an arbitrary right inverse Q_r^{-1} of Q and an arbitrary left inverse P_l^{-1} of P.*

In the literature there are known general representations for various classes of generalized inverses, for the set of complex matrices. The general representation of $\{1,2\}$ inverses for matrices is investigated in [9] and [10, p. 20, 28]. The general representations of $\{1,2,3\}$ and $\{1,2,4\}$ inverses for matrices are investigated in [9]. In [6] there is given a general representation and conditions for the existence of the group inverse for a given complex matrix. The general representation of the Moore-Penrose inverse is given in [2], for arbitrary Hilbert spaces.

In the following theorem we give general representations of $\{1,2\}$, $\{1,2,3\}$ and $\{1,2,4\}$ inverses for an arbitrary bounded operator on Hilbert spaces. As a consequence we obtain the known representation of the Moore-Penrose inverse from [2].

Theorem 2.1. *Let $A = PQ$ be a full–rank decomposition of $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ according to* (1.1)*. Then:*

(a) *$X \in A\{1,2\}$ if and only if there exist operators $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$ and $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$, such that QW_1 and W_2P are invertible in $B(\mathcal{H}_3)$. In such a case, X possesses the following general representation*

$$X = W_1(QW_1)^{-1}(W_2P)^{-1}W_2 \tag{2.1}$$

(b) *$X \in A\{1,2,3\}$ if and only if there exists an operator $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, such that QW_1 is invertible in $B(\mathcal{H}_3)$. In the case when it exists, a general representation for X is as follows:*

$$X = W_1(QW_1)^{-1}(P^*P)^{-1}P^*. \tag{2.2}$$

(c) *$X \in A\{1,2,4\}$ if and only if there exists an operator $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$, such that W_2P is invertible in $B(\mathcal{H}_3)$. In this case*

$$X = Q^*(QQ^*)^{-1}(W_2P)^{-1}W_2.$$

(d) $A^\dagger = Q^\dagger P^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = Q^*(P^*AQ^*)^{-1}P^*.$

Proof. **(a)** Follows from Lemma 2.1.

(b) If X has the form (2.2), then it is easy to verify $X \in A\{1,2,3\}$. We need to prove that the form (2.2) holds for all $\{1,2,3\}$ inverses of A. Indeed, if $X \in A\{1,2,3\}$, then $X = Q_r^{-1}P_l^{-1}$, and from the equation (3) it follows that $(PP_l^{-1})^* = PP_l^{-1}$. Thus $P^*PP_l^{-1} = P^*$. Operator P^*P is invertible, so that $P_l^{-1} = (P^*P)^{-1}P^*$. The right inverse of Q retains the general form $Q_r^{-1} = W_1(QW_1)^{-1}$ from Lemma 2.1. Consequently,

$$X = W_1(QW_1)^{-1}(P^*P)^{-1}P^*.$$

The proof of the statement **(c)** is similar as the proof of **(b)**. Also, **(d)** follows from **(b)** and **(c)**. For the part **(d)** see also [2]. $\square$

Now, we shall consider the weighted Moore-Penrose inverse under the various hypothesis. The weighted Moore-Penrose inverse is investigated in [1], [7] and [10] for the set of complex matrices and in [8] for matrices over an integral domain. If M and N are positive, then $A^\dagger_{M,N}$ and $A^\dagger_{M',N'}$ always exist [1], [10], and $A^\dagger_{M,N} = A^\dagger_{M',N'}$. In [8] and [7] it is derived a representation and conditions for the existence of $A^\dagger_{M,N}$ and $A^\dagger_{M',N'}$, respectively, under the more general assumptions that the matrices M and N are invertible (not necessary positive).

Theorem 2.2. *Let $A = PQ$ be a full–rank decomposition of A according to* (1.1). *Then:*

- **(a)** *If $M \in B(\mathcal{H}_2)$ and $N \in B(\mathcal{H}_1)$ are invertible operators, then $A^\dagger_{M,N}$ exists if and only if P^*MP and $QN^{-1}Q^*$ are invertible selfadjoint operators. In that case*

$$\begin{aligned} A^\dagger_{M,N} &= N^{-1}Q^*(QN^{-1}Q^*)^{-1}(P^*MP)^{-1}P^*M^* \\ &= N^{-1}Q^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^*. \end{aligned} \tag{2.3}$$

- **(b)** *Let $M \in B(\mathcal{H}_2)$ and $N \in B(\mathcal{H}_1)$ be invertible operators, such that $QN^{-1}Q^*$ is left invertible and P^*MP right invertible. Then $A^\dagger_{M',N'}$ exists if and only if $QN^{-1}Q^*$ and P^*MP are invertible and*

$$\begin{aligned} E &= N^{-1}Q^*(QN^{-1}Q^*)^{-1} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}, \\ F &= (P^*MP)^{-1}P^*M = ((MP)^*P)^{-1}(MP)^*. \end{aligned} \tag{2.4}$$

  In this case is $A^\dagger_{M',N'} = Q_r^{-1}P_l^{-1}$, where $Q_r^{-1} = E$ and $P_l^{-1} = F$.

- **(c)** *If $M \in B(\mathcal{H}_2)$ and $N \in B(\mathcal{H}_1)$ are positive and invertible operators, then*

$$\begin{aligned} A^\dagger_{M,N} = A^\dagger_{M',N'} &= (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^* \\ &= N^{-1}Q^*(QN^{-1}Q^*)^{-1}(P^*MP)^{-1}P^*M. \end{aligned} \tag{2.5}$$

Proof. **(a)** If M and N are invertible operators and $A^{\dagger}_{M,N}$ exists, using the principles from [8], from (3M) and (4N) we get

$$P^*MPP_l^{-1} = P^*M^*, \qquad Q_r^{-1}Q(N^{-1})^*Q^* = N^{-1}Q^*, \tag{2.6}$$

$$P^*MP = P^*M^*P, \qquad QN^{-1}Q^* = Q(N^{-1})^*Q^*. \tag{2.7}$$

From (2.7) we conclude that P^*MP and $QN^{-1}Q^*$ are selfadjoint operators. We now prove that $P^*MP = P^*M^*P$ and $QN^{-1}Q^* = Q(N^{-1})^*Q^*$ are invertible. Indeed, from (1) and $(3M)$ we get the following equation (see [8]):

$$(QXM^{-1}X^*Q^*)(P^*M^*P) = I.$$

This means that P^*M^*P is left invertible. Also, since P^*M^*P is selfadjoint, we conclude that P^*M^*P is invertible. Similarly, (1) and $(4N)$ imply the following

$$(QN^{-1}Q^*)(P^*X^*N^*XP) = I,$$

which means that $QN^{-1}Q^*$ is right invertible, so it is also invertible.

Using invertibility of $QN^{-1}Q^*$ and P^*M^*P, from (2.6) it follows

$$P_l^{-1} = (P^*M^*P)^{-1}(MP)^*, \quad Q_r^{-1} = N^{-1}Q^*(Q(N^{-1})^*Q^*)^{-1}. \tag{2.8}$$

Now, the representations (2.3) follows from $A^{\dagger}_{M,N} = Q_r^{-1}P_l^{-1}$, (2.8) and (2.7).

(b) Suppose that $M \in B(\mathcal{H}_2)$ and $N \in B(\mathcal{H}_1)$ are invertible operators, such that $QN^{-1}Q^*$ is left invertible and P^*MP right invertible and $A^{\dagger}_{M',N'}$ exists. From the equations (1) and $(3M')$ in the same way as in [7] we get $(QXM^{-1}X^*Q^*)(P^*MP) = I$, which means that P^*MP is left invertible. Similarly, from (1) and $(4N')$ we obtain $(QN^{-1}Q^*)(P^*X^*NXP) = I$, which implies the right invertibility of $QN^{-1}Q^*$. According to the assumptions, we conclude that P^*MP and $QN^{-1}Q^*$ are invertible. The identities (2.4) can be proved using the method from [7].

On the other hand, if P^*MP and $QN^{-1}Q^*$ are invertible and (2.4) holds, one can verify that $Q_r^{-1}P_l^{-1}$ (where $Q_r^{-1} = E$ and $P_l^{-1} = F$) satisfies the equations which define $A^{\dagger}_{M',N'}$.

(c) Firstly, we prove that $Q(QN^{-1})^*$ and $(MP)^*P$ are positive and invertible in $B(\mathcal{H}_3)$. If $x \in \mathcal{H}_3$ and $\|x\| = 1$, then

$$(Q(QN^{-1})^*x, x) = (N^{-1}Q^*x, Q^*x) > 0.$$

Suppose that $\inf_{\|x\|=1}(Q(QN^{-1})^*x,x)=0$. Then there exists a sequence of unit vectors $(x_n)_n$ in $\mathcal{H}_3$, such that $\lim_n(N^{-1}Q^*x_n,Q^*x_n)=0$. Since N^{-1} is positive and invertible, it follows that there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$, such that $\lim_k Q^*x_{n_k}=0$. Now, it follows that Q^* is not one-to-one with closed range, so Q is not onto. We get the contradiction, so $Q(QN^{-1})^*$ is positive and invertible in $B(\mathcal{H}_3)$. Analogously, we can prove that $(MP)^*P$ is positive and invertible in $B(\mathcal{H}_3)$.

The rest of the proof follows from parts **(a)** and **(b)**. $\square$

Now, we consider $\{2\}$-generalized inverses with prescribed range and kernel. Fundamental results for matrices can be found in [1] and [5].

Let $A\in B(\mathcal{H}_1,\mathcal{H}_2)$ and $X\in B(\mathcal{H}_2,\mathcal{H}_1)$ be a $\{2\}$-inverse of A, such that $\mathcal{R}(X)=T$ is a closed subspace of $\mathcal{H}_1$ and $\mathcal{N}(X)=S$ is a closed subspace of $\mathcal{H}_2$. Then we write $X=A^{(2)}_{T,S}$. For given closed subspaces T of $\mathcal{H}_1$ and S of $\mathcal{H}_2$, it is a natural question when $A^{(2)}_{T,S}$ exists? The answer in the case of arbitrary Hilbert spaces is given in the following theorem.

We state the following elementary result.

Lemma 2.2. *Let $A\in B(\mathcal{H}_1,\mathcal{H}_2)$, T and S be closed subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then the following statements are equivalent:*

- **(a)** *A has a $\{2\}$-inverse $X\in B(\mathcal{H}_2,\mathcal{H}_1)$ such that $\mathcal{R}(X)=T$ and $\mathcal{N}(X)=S$;*
- **(b)** *$A:T\to A(T)$ is invertible and $A(T)\oplus S=\mathcal{H}_2$.*

In the case when **(a)** *or* **(b)** *holds, X is unique and is denoted by $A^{(2)}_{T,S}$.*

Now, we generalize the result from [5].

Theorem 2.3. *Suppose that A, T and S satisfy the condition* **(a)** *or* **(b)** *from Lemma 2.2 and let $Y\in B(\mathcal{H}_2,\mathcal{H}_1)$ be such that $\mathcal{R}(Y)=T$ and $\mathcal{N}(Y)=S$. If there exists a Hilbert space $\mathcal{H}_3$ and a left invertible operator $E\in B(\mathcal{H}_3,\mathcal{H}_1)$ such that $\mathcal{R}(E)=T$, then*

$$W=E^*YAE\in B(\mathcal{H}_3)$$

is invertible in $B(\mathcal{H}_3)$ and

$$A^{(2)}_{T,S}=EW^{-1}E^*Y.$$

Proof. With the help of the proof of Theorem 2.3, notice that $E:\mathcal{H}_3\to T$ is invertible, $A:T\to AT$ is inverible and $Y:AT\to T$ is invertible.

Since $\mathcal{N}(E^*)^\perp = \mathcal{R}(E) = T$, it follows that $E^* : T \to \mathcal{H}_3$ is invertible, so W is invertible. Now, it is easy to verify that $EW^{-1}E^*Y$ is a $\{2\}$-inverse of A. Also, $\mathcal{N}(Y) = S$, $W^{-1}E^*Y : AT \to \mathcal{H}_3$ is invertible and $\mathcal{R}(EW^{-1}E^*Y) = \mathcal{R}(E) = T$, $\mathcal{N}(EW^{-1}E^*Y) = S$, so

$$A^{(2)}_{T,S} = EW^{-1}E^*Y. \quad \square$$

Theorem 2.4. *Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $X : \mathcal{H}_2 \to \mathcal{H}_1$. Then $X \in A\{2\}$ if and only if there exist Hilbert spaces $\mathcal{H}_3$, $\mathcal{H}_4$, $\mathcal{H}_5$ and operators*

$$C \in B(\mathcal{H}_4, \mathcal{H}_1),\ D \in B(\mathcal{H}_2, \mathcal{H}_3),\ W_1 \in B(\mathcal{H}_5, \mathcal{H}_4),\ W_2 \in B(\mathcal{H}_3, \mathcal{H}_5),$$

such that DAC is g-invertible and W_2DACW_1 is invertible. In this case:

$$X = CW_1(W_2DACW_1)^{-1}W_2D. \tag{2.9}$$

Proof. If X possesses the form (2.9), it is not difficult to verify $X \in A\{2\}$. On the other hand, using the method from [10], it is easy to verify that $X \in A\{2\}$ if and only if there exist operators C and D, such that

$$X = C(DAC)^{(1,2)}D,\ \ C \in B(\mathcal{H}_4, \mathcal{H}_1),\ \ D \in B(\mathcal{H}_2, \mathcal{H}_3).$$

According to part **(a)** of Theorem 2.1, $X \in A\{2\}$ if and only if there exist operators W_1 and W_2, such that W_2DACW_1 is invertible, and X possesses the form (2.9). $\square$

We introduce a general representation of the Drazin inverse based on an arbitrary full-rank factorization of A^l, $l \geq k = \operatorname{asc}(A) = \operatorname{des}(A)$. The following theorem is a natural generalization of a Cline's result from [6], introduced for complex matrices. We shall assume that A is not a nilpotent operator, i.e. $A^D \neq 0$.

Theorem 2.5. *Let $\mathcal{X}$ be a Banach space. If $A \in B(\mathcal{X})$, $l \geq k = \operatorname{asc}(A) = \operatorname{des}(A) < \infty$ and $A^l = P_{A^l}Q_{A^l}$ is the full-rank decomposition of A^l, then*

$$A^D = P_{A^l}(Q_{A^l}AP_{A^l})^{-1}Q_{A^l}.$$

Proof. If $\operatorname{asc}(A) = \operatorname{des}(A) = k < \infty$, then it is well-known that $\mathcal{N}(A^l) = \mathcal{N}(A^k)$ and $\mathcal{R}(A^l) = \mathcal{R}(A^k)$ for all $l \geq k$,

$$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2, \tag{2.10}$$

where $\mathcal{X}_1 = \mathcal{N}(A^l)$ and $\mathcal{X}_2 = \mathcal{R}(A^l)$, $A(\mathcal{X}_i) \subset \mathcal{X}_i$ for $i = 1, 2$, $A_1 = A|_{\mathcal{X}_1}$ is nilpotent and $A_2 = A|_{\mathcal{X}_2}$ is invertible (A is not nilpotent) [3], [4]. We can write

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad A^D = \begin{bmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{bmatrix}$$

with respect to the decomposition (2.10) ([3],[4]). Since $\mathcal{N}(A^l)$ and $\mathcal{R}(A^l)$ are complementary and closed subspaces of $\mathcal{X}$, it follows that A^l is g-invertible, so there exists the full-rank decomposition $A^l = P_{A^l} Q_{A^l}$, where $P_{A^l} \in B(\mathcal{Z}, \mathcal{X})$ is left invertible and $Q_{A^l} \in B(\mathcal{X}, \mathcal{Z})$ is right invertible, for some Banach space $\mathcal{Z}$. By the isomorphism theorem [3], we can take that $\mathcal{Z} = \mathcal{X}_2$. We conclude that P_{A^l} and Q_{A^l} have the following representations with respect to (2.10):

$$P_{A^l} = \begin{bmatrix} M \\ \tilde{P} \end{bmatrix} \qquad \text{and} \qquad Q_{A^l} = \begin{bmatrix} N & \tilde{Q} \end{bmatrix},$$

where $\tilde{P}, \tilde{Q} \in B(\mathcal{X}_2)$, $M \in B(\mathcal{X}_2, \mathcal{X}_1)$, $N \in B(\mathcal{X}_1, \mathcal{X}_2)$. Now, P_{A^l} is left invertible and Q_{A^l} is right invertible, so P_{A^l} and Q_{A^l} are g-invertible operators, $\mathcal{N}(P_{A^l}) = \{0\}$ and $\mathcal{R}(Q_{A^l}) = \mathcal{X}_2$. It follows that $\mathcal{R}(P_{A^l}) = \mathcal{R}(A^l) = \mathcal{X}_2$ and $\mathcal{N}(Q_{A^l}) = \mathcal{N}(A^l) = \mathcal{X}_1$, so $M = 0$, $N = 0$ and

$$P_{A^l} = \begin{bmatrix} 0 \\ \tilde{P} \end{bmatrix} \qquad \text{and} \qquad Q_{A^l} = \begin{bmatrix} 0 & \tilde{Q} \end{bmatrix}.$$

It is easy to verify that $\tilde{P}$ is left invertible and $\tilde{Q}$ is right invertible in $B(\mathcal{X}_2)$. But

$$\begin{bmatrix} 0 & 0 \\ 0 & A_2^l \end{bmatrix} = A^l = P_{A^l} Q_{A^l} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}\tilde{Q} \end{bmatrix},$$

so $A_2^l = \tilde{P}\tilde{Q}$. Since A_2^l is invertible, it follows that $\tilde{P}$ and $\tilde{Q}$ are invertible in $B(\mathcal{X}_2)$.

Now, $Q_{A^l} A P_{A^l} = \tilde{Q} A_2 \tilde{P}$ is invertible in $B(\mathcal{X}_2)$, so

$$A^D = \begin{bmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}(\tilde{Q} A_2 \tilde{P})^{-1} \tilde{Q} \end{bmatrix} = P_{A^l} (Q_{A^l} A P_{A^l})^{-1} Q. \qquad \square$$

As a corollary, we get the following result.

Corollary 2.1. *If $\mathcal{X}$ is a Banach space, $A \in B(\mathcal{X})$ and $\mathrm{asc}(A) = \mathrm{des}(A) = k < \infty$ and $A^l = P_{A^l} Q_{A^l}$ is an arbitrary full-rank decomposition of A^l, $l \geq k$, then*

(a) $(A^D)^l = P_{A^l}(Q_{A^l} A^l P_{A^l})^{-1} Q_{A^l} = P_{A^l}(Q_{A^l} P_{A^l})^{-2} P_{A^l}$;

(b) $AA^D = P_{A^l}(Q_{A^l} P_{A^l})^{-1} Q_{A^l}$;

(c) *If $\mathcal{X}$ is a Hilbert space, then* $(A^D)^\dagger = (Q_{A^l})^\dagger Q_{A^l} A P_{A^l} (P_{A^l})^\dagger$.

Proof. **(a)** Follows from $(A^D)^l = (A^l)^\#$ and Theorem 2.6.

(b) According to Theorem 2.6 it follows that $Q_{A^l} P_{A^l} = \tilde{Q}\tilde{P}$, so an easy computation shows that

$$P_{A^l}(Q_{A^l} P_{A^l})^{-1} Q_{A^l} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = AA^D.$$

(c) Follows from Theorem 2.1 **(d)** and Theorem 2.6. □

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