SPECTRAL PERMANENCE FOR THE MOORE-PENROSE INVERSE

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ABSTRACT. C* algebra "spectral permanence" extends from the ordinary inverse to the Moore-Penrose inverse.

"Spectral permanence" for C* algebras says that if $T : A \to B$ is an isometric C* homomorphism then the image spectrum $\sigma_B(Ta)$ is always the same as the original spectrum $\sigma_A(a)$: equivalently invertibility $Ta \in B^{-1}$ implies invertibility $a \in A^{-1}$. In this note we extend this to *relative regularity*, and offer a fresh proof of the Harte/Mbekhta theorem ([5] Theorem 6) which says that relatively regular C* algebra elements always have "Moore-Penrose inverse". Instead of [5] the "poor man's path" between projections, we proceed via the *Drazin inverse*.

1. Spectral permanence

Suppose $T : A \to B$ is a homomorphism of semigroups: here a "semigroup" A is assumed to have an identity 1 , and we write A^{-1} for its subgroup of invertibles; then there is inclusion

1.1 $$T(A^{-1}) \subseteq B^{-1} \subseteq B\ ,$$

and hence also

1.2 $$A^{-1} \subseteq T^{-1}(B^{-1}) \subseteq A\ ;$$

now if there is equality in (1.2) we shall say that T has *the Gelfand property*,

1.3 $$A^{-1} = T^{-1}(B^{-1})\ :$$

this famously is the case when A is a commutative Banach algebra and $T : A \to B = C(X)$ is the Gelfand mapping from A to the continuous functions on the "maximal ideal space" $X = \sigma(A)$ of A. Another familiar example is the *left regular representation*

1.4 $$T = L : a \mapsto L_a \quad (A \to A^A \equiv \mathrm{Map}(A, A))\ ,$$

where, for each $x \in A$,

$$L_a(x) = ax\ .$$

This holds also for the natural embedding

1.5 $$A = \mathrm{comm}_B(K) \subseteq B\ ,$$

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of the *commutant* of a subset $K \subseteq B$,

1.6 $$\mathrm{comm}_B(K) = \{b \in B : a \in K \Longrightarrow ba = ab\}\ .$$

The same is also true, among rings, of the quotient mapping

1.7 $$T : A \to A/\mathrm{Rad}(A)\ ,$$

where

1.8 $$\mathrm{Rad}(A) = \{a \in A : 1 - Aa \subseteq A^{-1}\}$$

is the *Jacobson radical* of the ring A. Our final example, if not the most "elementary", is the process of taking the *determinant* of a square matrix: it is precisely because of the Gelfand property that it "determines" whether or not a matrix is invertible.

Generally when A is a complex linear algebra there is a *spectrum*

1.9 $$\sigma_A(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}\}\ ,$$

for elements $a \in A$: thus complex analysis is harnessed to the theory of invertibility. Now inclusion (1.1) takes the form

1.10 $$\sigma_B(Ta) \subseteq \sigma_A(a)\ ,$$

while equality (1.3) corresponds to equality in (1.10), where it is referred to as "spectral permanence". Generally if a homomorphism $T : A \to B$ of Banach algebras is isometric, or more generally bounded below, then there is inclusion

1.11 $$\partial\sigma_A(a) \subseteq \sigma_B(Ta) \subseteq \sigma_A(a)\ ,$$

where we write ∂K for the *topological boundary* of $K \subseteq \mathbf{C}$: thus the condition, for a particular $a \in A$, that

1.12 $$\sigma_A(a) \subseteq \partial\sigma_A(a)$$

is sufficient for "spectral permanence at a" in the sense of equality in (1.10).

2. Generalized permanence

If A is a semigroup we shall write

2.1 $$A^\cap = \{a \in A : a \in aAa\}$$

for the "regular" or *relatively regular* elements of A, those $a \in A$ which have a *generalized inverse* $b \in A$ for which

2.2 $$a = aba\ \ :$$

we remark that if (2.2) holds the products

$$p = ba = p^2\ ,\ q = ab = q^2$$

are both *idempotent*. Generally if $T : A \to B$ is a homomorphism there is inclusion

2.3 $$T(A^\cap) \subseteq B^\cap \subseteq B\ ,$$

and hence also

2.4 $$A^\cap \subseteq T^{-1}(B^\cap) \subseteq A\ .$$

If there is equality in (2.4) we shall say that T has *generalized permanence*. This happens for example when

2.5 $$T^{-1}(0) \subseteq A^\cap\ ,\ T(A) = B\ \ :$$

recall the implication

$$(a - aAa) \cap A^\cap \neq \emptyset \Longrightarrow a \in A^\cap \ .$$

If in particular there is $b \in A$ for which

2.6 $$a - aba = 0 = ab - ba \ ,$$

then $a \in A$ is very special; we shall say that $a \in A$ is *simply polar*, and refer to the product bab as the *group* inverse for $a \in A$: more generally a group inverse for a power a^n gives rise to a *Drazin inverse* for a. We remark that it is necessary and sufficient for $a \in A$ to be simply polar that

2.7 $$a \in a^2 A \cap A a^2 \ :$$

indeed [9] there is implication

2.8 $$a^2 u = a = v a^2 \Longrightarrow au = va \ , \ aua = a = ava \ ,$$

giving (2.6) with $b = vau$. In particular if $A = B(X)$ for a Banach space X then relatively regular elements have closed range:

Theorem 2.1. *If $a \in A = B(X)$ for a normed space X then*

2.9 $$a \in A^\cap \Longrightarrow a(X) = \mathrm{cl}\ a(X) \ .$$

If X is complete then necessary and sufficient for $a \in A$ to be simply polar is that it has ascent ≤ 1,

2.10 $$a^{-2}(0) \subseteq a^{-1}(0) \ ; \ \textit{equivalently}\ a^{-1}(0) \cap a(X) = O \equiv \{0\} \ ,$$

and also descent ≤ 1,

2.11 $$a(X) \subseteq a^2(X) \ ; \ \textit{equivalently}\ a^{-1}(0) + a(X) = X \ .$$

Proof. If $a = aba \in A$ then the range

$$a(X) = ab(X) = (1 - ab)^{-1}(0)$$

is the null space of the complementary idempotent, therefore closed. Now the complementary subspaces $a^{-1}(0)$ and $a(X)$ determine the idempotent $q : X \to X$, defined by setting

$$q(x) \in a(X) \ ; \ x - q(x) \in a^{-1}(0)$$

for each $x \in X$, whose boundedness, together with the closedness of the range $a(X)$, follows ([4] Theorem 4.8.2) from the open mapping theorem, and finally, if $x \in X$,

$$b(x) = bq(x) \ ; \ b(ax) = q(x) \bullet$$

□

We remark, even together with the assumption $a \in A^\cap$, the conditions (2.10) and (2.11) are ([4] (7.3.6.8)) not sufficient for simple polarity (2.6) when $A = B(X)$ for an incomplete normed space X.

More generally if there is $n \in \mathbf{N}$ for which a^n is simply polar we shall also say that $a \in A$ is "polar", or *Drazin invertible*. If $a \in A$ is polar then there is $b \in A$ for which $ab = ba$ and $a - aba$ is *nilpotent*.

3. Drazin permanence

More generally if we write

3.1 $$\mathrm{QN}(A) = \{a \in A : 1 - \mathbf{C}a \subseteq A^{-1}\}$$

for the *quasinilpotents* of A then $a \in \mathrm{QN}(A)$ if and only if

$$\sigma_A(a) \subseteq \{0\} \ ,$$

while with some complex analysis we can prove that if $a \in \mathrm{QN}(A)$ then

3.2 $$\|a^n\|^{1/n} \to 0 \ (n \to \infty) \ .$$

In the ultimate generalization of "group invertibility", we shall write $\mathrm{QP}(A)$ for the *quasipolar* elements $a \in A$, those which have a *spectral projection* $p \in A$ for which

3.3 $$p = p^2 \ ; \ ap = pa \ ; \ a + p \in A^{-1} \ ; \ ap \in \mathrm{QN}(A) \ .$$

Now [7] the spectral projection and the *Koliha-Drazin inverse*

3.4 $$a^\bullet = p \ , \ a^\times = (a+p)^{-1}(1-p)$$

are uniquely determined and lie in the double commutant of $a \in A$. It is easy to see that if (3.3) is satisfied then

3.5 $$0 \notin \mathrm{acc}\ \sigma_A(a) \ :$$

the origin cannot be an accumulation point of the spectrum; conversely if (3.5) holds then we can display the spectral projection as a sort of "vector-valued winding number"

3.6 $$a^\bullet = \frac{1}{2\pi i}\oint_{\gamma(0)} (z-a)^{-1}dz \ ,$$

where we integrate counter clockwise round a small circle $\gamma = \gamma(0)$ centre the origin whose connected hull is a disc $\eta\gamma$ whose intersection with the spectrum is at most the point $\{0\}$. By the same technique we can display the Koliha-Drazin inverse in the form

3.7 $$a^\times = \frac{1}{2\pi i}\oint_{\gamma(\sigma(a)\setminus\{0\})} z^{-1}(z-a)^{-1}dz \ .$$

Now generally for a homomorphism $T : A \to B$ there is inclusion

3.8 $$T\ \mathrm{QP}(A) \subseteq \mathrm{QP}(B) \ ,$$

while if $T : A \to B$ has spectral permanence in the sense (1.3) then it is clear from (3.5) that there is also "Drazin permanence" in the sense that

3.9 $$\mathrm{QP}(A) = T^{-1}\mathrm{QP}(B) \subseteq A \ :$$

Theorem 3.1. *For Banach algebra homomomorphisms $T : A \to B$ there is implication*

spectral permanence $\Longrightarrow$ Drazin permanence.

Proof. Equality in (1.10) together with (3.5) $\bullet$ □

In general we cannot deduce "generalized permanence", equality in (2.4), from spectral permanence (1.3):

Theorem 3.2. *If $T : A \to B$ and A is commutative then there is inclusione*

3.10 $$T(A^\cap) \subseteq \mathrm{QP}(B) \cap B^\cap \ .$$

Thus if A is commutative there is implication, for arbitrary $a \in A$,

3.11 $$T(a) \in B^\cap \setminus \mathrm{QP}(B) \Longrightarrow a \notin A^\cap \ .$$

Proof. If A is commutative then everything in $A^\cap$ is simply polar • □

For a specific example take $T : A \to B$ the embedding

$$A = \mathrm{comm}_B^2(a) \subseteq B = B(X)$$

with $X = \ell_2$ and $a \in A$ either the forward or the backward shift, or alternatively the left regular representation $L : A \to B(A)$. Here of course $\mathrm{comm}^2(a)$ is $\mathrm{comm}\ \mathrm{comm}(K)$ with $K = \{a\}$. Conversely however if $A = B(X)$ for a normed space X then there is implication

3.12 $$L_a A = \mathrm{cl}\ L_a A \Longrightarrow a(X) = \mathrm{cl}\ a(X) \ :$$

indeed if $ax_n \to y$ and $\varphi \in X^*$ and $\varphi(x) = 1$ then, with $\varphi \odot y : w \mapsto \varphi(w)y$,

3.13 $$L_a(\varphi \odot y) = L_a(b) \Longrightarrow y = a(bx) \ .$$

For another example observe that generalized permanence fails for the process of factoring out the radical of a ring, unless it is semisimple:

3.14 $$\mathrm{Rad}(A) \cap A^\cap = O \ .$$

4. Moore-Penrose permanence

We recall that a "C*algebra" is a Banach algebra which also has an *involution* $a \mapsto a^*$ which is conjugate linear, reverses multiplication, respects the identity and satisfies the "B* condition"

4.1 $$\|a^* a\| = \|a\|^2 \ (a \in A) \ .$$

Historically the term "C* algebra" was reserved for closed *-subalgebras of the algebras $B(X)$ for Hilbert spaces X; however the *Gelfand/Naimark/Segal* (GNS) representation

4.2 $$\Gamma : A \to B(\Xi_A)$$

takes an arbitrary "B* algebra" A isometrically into the algebra of operators on a rather large Hilbert space Ξ_A built from its "states": a defect of (4.2) would be that if already $A = B(X)$ we do not get back $\Xi_A = X$. In the opinion of at least one writer these terms "B* algebra" and "C* algebra" could easily ([4] Chapter 8) have been *Hilbert algebra*. When in particular $A = B(X)$ for a Hilbert space X then the closed range condition is sufficient for relative regularity $a \in A^\cap$: indeed we can satisfy (2.2) by setting

4.3 $$b(x) = b(qx) \ ; \ b(ax) = p(x) \ (x \in X) \ ,$$

where $q^* = q = q^2$ and $p^* = p = p^2$ are the orthogonal projections on the range $a(X)$ and the orthogonal complement $a^{-1}(0)^\perp$ of the null space. The element $b \in A$ given by (4.3) satisfies four conditions:

4.4 $$a = aba \ ; \ b = bab \ ; \ (ba)^* = ba \ ; \ (ab)^* = ab \ ,$$

and is known as the *Moore-Penrose* inverse of $a \in B(X)$: more generally in a C*algebra A the conditions (4.4) uniquely determine at most one element

4.5 $$b = a^\dagger \in A \ ,$$

lying ([5] Theorem 5) in the double commutant of $\{a, a^*\}$, and still known as a "Moore-Penrose inverse" for $a \in A$. Now it is a result of Harte and Mbekhta ([5] Theorem 6) that generally there is equality ($A^\dagger$ denotes the set of all Moore-Penrose invertible elements of A):

4.6 $$A^\cap = A^\dagger \ :$$

in an arbitrary C*algebra, every relatively regular element has a Moore Penrose inverse. The argument, and a slight generalization, proceeds with the aid of the Drazin inverse. Generally for a C*algebra A we write

4.7 $$\mathrm{Re}(A) = \{a \in A : a^* = a\} \ ,$$

for the subspace of *hermitian* elements: now we claim

Theorem 4.1. *If A is a C*algebra then*

4.8 $$\mathrm{Re}(A) \cap A^\cap \subseteq \mathrm{QP}(A) \ ,$$

and

4.9 $$a \in A^\cap \Longrightarrow a^*a \in A^\cap \Longrightarrow a^*a \in \mathrm{QP}(A) \ .$$

Proof. When $A = B(X)$ then the "ascent" conditions (2.9) hold for hermitian elements $a \in \mathrm{Re}(A)$: this is because [6] generally, if $a \in B(X)$,

4.10 $$x \in X \Longrightarrow \|ax\|^2 \le \|x\| \ \|a^*ax\| \ ,$$

giving $a^{-1}(0) = (a^*a)^{-1}(0)$. Also if $a = a^*$ has closed range then also the "descent" conditions (2.10) hold:

4.11 $$\mathrm{cl}\ a(X) = (a^*)^{-1}(0)^\perp \Longrightarrow \mathrm{cl}\ a(X) + (a^*)^{-1}(0) = X \ ,$$

giving also if a has closed range, $a^*(X) = a^*a(X)$, which is therefore also closed. To establish (4.8) and (4.9) for general A look at isometric $T : A \to B(X)$, recalling (1.12), hence spectral permanence and then Drazin permanence • □

Theorem 4.1 says that self adjoint C* elements have [2] "property EP".

Our main result is a slight generalization, and a new proof, of the Harte/Mbekhta result (4.6), and at the same time "generalized permanence", equality in (2.4), for isometric C* homomorphisms:

Theorem 4.2. *If $T : A \to B$ is an isometric C* homomorphism then it has both a "left" and a "right" Gelfand property, and hence also Drazin permanence; there is equality*

4.12 $$A^\dagger = T^{-1}B^\cap \ .$$

Proof. If $a \in A$ then the product $a^*a \in A^+ \subseteq \mathrm{Re}(A)$ is *positive*, with

4.13 $$\partial\sigma_A(a^*a) = \sigma_A(a^*a) \subseteq [0, \infty) \subseteq \mathbf{R} \ ,$$

giving "spectral permanence at" a^*a. Since also

4.14 $$a \in A^{-1}_{left} \Longleftrightarrow a^*a \in A^{-1}$$

this converts to equality in both a left and a right version of (1.10).

Now by (4.9) the product $a^*a \in \mathrm{QP}(A)$ has a Koliha-Drazin inverse, and even a group inverse: but now (cf [5] Theorem 10)

4.15 $$a^\dagger = (a^*a)^\times a^* = (a^*a)^\dagger a^*$$

gives a Moore-Penrose inverse $a^\dagger$ for $a \in A$ • □

In the situation of (4.8) we now have more detail:

4.16 $$a = a^* \in A^\cap \Longrightarrow a^\dagger = a^\times \ ; \ 1 - a^\dagger a = a^\bullet \ .$$

We remark that it is rather easy ([5] Theorem 7) to see

4.17 $$a \in A^\dagger \Longrightarrow a^*a \in A^\cap \ :$$

for if $b = a^\dagger$ then

4.18 $$a^*a = a^*b^*a^*aba = a^*(ab)^*(ab)^*a = a^*a(bb^*)a^*a \ .$$

Theorem 4.2 has an obvious extension to homomorphisms with closed range:

Theorem 4.3. *If $T : A \to B$ has closed range then there is implication, for arbitrary $a \in A$,*

4.19 $$T(a) \in B^\cap \Longrightarrow a + T^{-1}(0) \in (A/T^{-1}(0))^\cap \ .$$

Proof. Apply Theorem 4.2 to the bounded below $T^\wedge : A/T^{-1}(0) \to B$ • □

5. Irreducible representations

If $T : A \to B(X)$ is an isometric * homomorphism, with a Hilbert space X, and if there is a closed subspace $Y \subseteq X$ for which, for arbitrary $a \in A$, there is inclusion

5.1 $$T(a)Y \subseteq Y \ ; \ T(a)Y^\perp \subseteq Y^\perp \ ,$$

then we shall say that the representation has been *reduced* to a representation $T_Y : A \to B(Y)$; if it follows from (5.1) that

5.2 $$Y = X \text{ or } Y = O \equiv \{0\}$$

then we shall describe the representation $T : A \to B(X)$ as *irreducible*. Now if $B(X)$ is the bounded operators on a Banach space we shall write $B_{00}(X)$ for the ideal of *finite rank* operators, and $B_0(X)$ for the larger closed ideal of *compact operators*; we remark that ([4] Theorem 6.8.5)

5.3 $$B_0(X) \cap B(X)^\cap = B_{00}(X) \ :$$

a compact operator is relatively regular if and only if it is finite rank. When X is a Hilbert space then $B_0(X) = \mathrm{cl}\ B_{00}(X)$ is also the norm closure of the finite rank operators, which in turn are the linear subspace generated by the *rank one* operators:

5.4 $$X^* \odot X = \{x^* \odot y : x, y \in X\} \ ,$$

where for arbitrary $x, y \in X$ we set

5.5 $$(x^* \odot y)(w) = \langle w; x\rangle y \quad (w \in X) \ .$$

Irreducible C* subalgebras $A \subseteq B(X)$ have an important property ([1] Theorem 6.3.3) relating them to the compact ideal $B_0(X)$: there is implication

5.6 $$A \cap B_0(X) \neq O \Longrightarrow B_0(X) \subseteq A \ .$$

In words if A contains even one non zero compact operator then it contains them all.

As a companion to (5.6) we observe (cf [1] Theorem 5.2.1) that if $J \subseteq A$ is a two-sided ideal there is implication, when $A = B(X)$,

5.7 $$J \neq O \Longrightarrow A \cap B_{00}(X) \subseteq J \ .$$

In words a non trivial two-sided ideal of irreducible A contains all the finite rank operators in A. If in particular the ideal is closed then it contains all the compact operators.

If $T : A \to B$ is a homomorphism of rings then it also brings ([4] Chapter 7) a *Fredholm theory* to the departure ring A: generally for $T : A \to B$ to have the Gelfand property means that "Fredholm implies invertible". Intermediate between the invertibles and the Fredholms is the semigroup of *Weyl* elements:

5.8 $$A^{-1} \subseteq A^{-1} + T^{-1}(0) \subseteq T^{-1}(B^{-1}) \subseteq A \ .$$

More generally if we replace invertibles by left and by right invertibles we get "left" and "right" Fredholm and Weyl elements. For example if, with a Banach space X,

5.9 $$T = \pi : B(X) \to D = B(X)/B_0(X)$$

is the *Calkin* quotient then (Atkinson's theorem) the Fredholm operators are those with finite dimensional null space and closed range of finite codimension, while (Schechter's theorem) the Weyl operators are the Fredholm operators "of index zero", for which those two finite dimensions are equal. If instead

5.10 $$T = \pi_0 : B(X) \to D_0 = B(X)/B_{00}(X)$$

then, since

5.11 $$B_{00}(X) \subseteq B(X)^{\cap} \ ; \ \pi_0 B_0(X) \subseteq \mathrm{Rad}(D_0)$$

the Fredholm theory is unchanged: there is, recalling (2.5), equality

5.12 $$\pi^{-1}D^{-1} = \pi_0^{-1}D_0^{-1} \subseteq B(X)^{\cap} \ .$$

Now if $A \subseteq B(X)$ is irreducible then more generally if $J \subseteq A$ is a non trivial closed two-sided ideal then the homomorphisms

$$\pi_A : A \to D = B(X)/B_0(X)$$

and

$$\rho_J : A \to D_J = A/(J \cap B_0(X))$$

generate, according to ([1] Theorem 6.3.4), the same Fredholm theory:

Theorem 5.1. *If $A \subseteq B(X)$ is irreducible and*

$$O \neq AJ + JA \subseteq J \neq A$$

then there is equality

5.13 $$\pi_A^{-1}D^{-1} = \rho_J^{-1}D_J^{-1} \ .$$

Proof. It is clear at once that if $\rho_J(a) \in A/(J \cap B_0(X))$ is left or right invertible then so is $\pi_A(a) \in D = B(X)/B_0(X)$. Conversely if $\pi_A(a) \in D_{left}^{-1}$ then two cases appear.

Case 1) $a \in A_{left}^{-1}$, then $\rho_J(a) \in (A/(J \cap B_0(X)))_{left}^{-1}$.

Case 2) $a \notin A^{-1}_{left}$, then $a^*a \notin A^{-1}$. Since $a \in A^{\cap}$, by (4.9) $a^*a \in \mathrm{QP}(A)$ with spectral projection

5.14 $$0 \neq p = (a^*a)^{\bullet} \in J \cap B_{00}(X) \ ,$$

Appealing to irreducibility and (5.7), with

$$1 = ((a^*a+p)^{-1}a^*)a + (a^*a+p)^{-1}p \in Aa + J \cap B_{00}(X) \ .$$

It follows $\rho_J(a) \in (D_J)^{-1}_{left}$.

Similarly we can start with $\pi_A(a) \in D^{-1}_{right}$ to obtain $\rho_J(a) \in (D_J)^{-1}_{right}$.

Alternatively, if $\pi_A(a) \in D^{-1}_{left}$, then $a \in A$ has a Moore-Penrose inverse $a^\dagger \in A$, and now

5.15 $$1 - a^\dagger a \in A \cap B_{00}(X) \subseteq J$$

is the orthogonal projection with finite dimensional range $a^{-1}(0)$ $\bullet$ □

We remark that Theorem 5.2.1 of [1] holds for $A = B(X)$, while if (5.7) holds and there are any non zero finite rank operators in A then they will all be in J. It seems to us that the argument of [1] requires our (5.7) rather than their Theorem 5.2.1: one might hope to derive our (5.7) by a combination of the arguments of their (5.2.1) and their (6.3.3).

References

1. S. R. Caradus, W. E. Pfaffenberg and B. Yood, *Calkin algebras and algebras of operators on Banach spaces*, Dekker, 1974.
2. D. S. Djordjević, J. J. Koliha and I. Straskraba, *Factorization of EP elements of C* algebras*, Linear Multilinear Algebra **57** (2009) 587–594.
3. D. S. Djordjević and V. Rakočević, *Lectures on generalized inverses*, University of Nis, 2008.
4. R. E. Harte, *Invertibility and singularity*, Dekker, 1988.
5. R. E. Harte and M. Mbekhta, *On generalized inverses in C* algebras*, Studia Math. 103 (1992), 71–77.
6. R. E. Harte and M. O'Searcoid, *Positive elements and the B* condition*, Math. Zeit. **193** (1986), 1–9.
7. J. J. Koliha, *A generalized Drazin inverse*, Glasgow Math. Jour. **38** (1996), 367–381.
8. M. Takesaki, *Theory of operator algebras I*, Springer, 2002.
9. M. Xavier, *On the converse of a theorem of Harte and Mbekhta*, Studia Math. **184** (2008), 149–152.

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