Partial isometries and EP elements in Banach algebras

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Abstract

New characterizations of partial isometries and EP elements in Banach algebra are presented.

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1 Introduction

Generalized inverses of matrices have important roles in theoretical and numerical methods of linear algebra. The most significant fact is that we can use generalized inverses of matrices, in the case when ordinary inverses do not exist, in order to solve some matrix equations. Similar reasoning can be applied to linear (bounded or unbounded) operators on Banach and Hilbert spaces. Then, it is interesting to consider generalized inverses of elements in Banach and C^*-algebras, or more general, in rings with or without involution.

Let $\mathcal{A}$ be a complex unital Banach algebra. An element $a \in \mathcal{A}$ is generalized (or inner) invertible, if there exists some $b \in \mathcal{A}$ such that $aba = a$ holds. In this case b is a generalized (or inner) inverse of a. If $aba = a$, then take $c = bab$ to obtain the following: $aca = a$ and $cac = c$. Such c is called a reflexive (or normalized) generalized inverse of a. Finally, if $aba = a$, then ab and ba are idempotents. In the case of the C^*-algebra, we can require that ab and ba are Hermitian. We arrive at the definition of the Moore-Penrose inverse in C^*-algebras.

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Definition 1.1. Let $\mathcal{A}$ be a unital C^*-algebra. An element $a \in \mathcal{A}$ is Moore-Penrose invertible, if there exists some $b \in \mathcal{A}$ such that

$$aba = a, \; bab = b, \; (ab)^* = ab, \; (ba)^* = ba$$

hold. In this case b is the Moore-Penrose inverse of a, usually denoted by $a^\dagger$.

If a is Moore-Penrose invertible in a C^*-algebra, then $a^\dagger$ is unique, and the notation is justified.

More general, if $\mathcal{A}$ is an unital Banach algebra, we have the following definition of Hermitian elements.

Definition 1.2. An element $a \in \mathcal{A}$ is said to be Hermitian if $\| \exp(ita) \| = 1$ for all $t \in R$.

The set of all Hermitian elements of $\mathcal{A}$ will be denoted by $\mathcal{H}(\mathcal{A})$. Now, it is natural to consider the following definition of the Moore-Penrose inverse in Banach algebras ([1], [2]).

Definition 1.3. Let $\mathcal{A}$ be a complex unital Banach algebra and $a \in \mathcal{A}$. If there exists $b \in \mathcal{A}$ such that

$$aba = a, \quad bab = b, \quad ab \text{ and } ba \text{ are Hermitian},$$

then the element b is the Moore-Penrose inverse of a, and it will be denoted by $a^\dagger$.

The Moore-Penrose inverse of a is unique in the case when it exists.

Although the Moore-Penrose inverse has many nice approximation properties, the equality $aa^\dagger = a^\dagger a$ does not hold in general. Hence, it is interesting to distinguish such elements.

Definition 1.4. An element a of a unital Banach algebra $\mathcal{A}$ is said to be EP if there exists $a^\dagger$ and $aa^\dagger = a^\dagger a$.

The name EP will be explained latter. There is another kind of a generalized inverse that commutes with the starting element.

Definition 1.5. Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. An element $b \in \mathcal{A}$ is the group inverse of a, if the following conditions are satisfied:

$$aba = a, \quad bab = b, \quad ab = ba.$$

The group inverse of a will be denoted by $a^{\#}$ which is uniquely determined (in the case when it exists).

Let X be a Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear bounded operators on X. In addition, if $T \in \mathcal{L}(X)$, then $N(T)$ and $R(T)$ stand for the null space and the range of T, respectively. The ascent of T is defined as $\mathrm{asc}(T) = \inf\{n \geq 0 : N(T^n) = N(T^{n+1})\}$, and the descent of T is defined as $\mathrm{dsc}(T) = \inf\{n \geq 0 : R(T^n) = R(T^{n+1})\}$. In both cases the infimum of the empty set is equal to ∞. If $\mathrm{asc}(T) < \infty$ and $\mathrm{dsc}(T) < \infty$, then $\mathrm{asc}(T) = \mathrm{dsc}(T)$.

Necessary and sufficient for $T^{\#}$ to exist is the fact that $\mathrm{asc}(T) = \mathrm{dsc}(T) \leq 1$. If $T \in \mathcal{L}(X)$ is a closed range operator, then $T^{\#}$ exists if and only if $X = N(T) \oplus R(T)$ (see [3]). Obviously, $R((T^{\#})^n) = R(T^{\#}) = R(T) = R(T^n)$ and $N((T^{\#})^n) = N(T^{\#}) = N(T) = N(T^n)$, for every non-negative integer n. Now the name follows: EP means "equal projections" on $R(T^k)$ parallel to $N(T^k)$ for all positive integers k.

Finally, if $a \in \mathcal{A}$ is an EP element, then clearly $a^{\#}$ exists. In fact, $a^{\#} = a^{\dagger}$. On the other hand, if a exists, then necessary and sufficient for a to be EP is that $aa^{\#}$ is a Hermitian element of $\mathcal{A}$. Furthermore, in this case $a^{\#} = a^{\dagger}$.

The left multiplication by $a \in \mathcal{A}$ is the mapping $L_a : \mathcal{A} \to \mathcal{A}$, which is defined as $L_a(x) = ax$ for all $x \in \mathcal{A}$. Observe that, for $a, b \in \mathcal{A}$, $L_{ab} = L_a L_b$ and that $L_a = L_b$ implies $a = b$. If $a \in \mathcal{A}$ is both Moore-Penrose and group invertible, then $L_{a^{\dagger}} = (L_a)^{\dagger}$ and $L_{a^{\#}} = (L_a)^{\#}$ in the Banach algebra $\mathcal{L}(\mathcal{A})$. According to [4, Remark 12], necessary and sufficient condition for $a \in \mathcal{A}$ to be EP is that $L_a \in \mathcal{L}(\mathcal{A})$ is EP.

A similar statement can be proved if we consider $R_a \in \mathcal{L}(\mathcal{A})$ instead of $L_a \in \mathcal{L}(\mathcal{A})$, where the mapping $R_a : \mathcal{A} \to \mathcal{A}$ is the right multiplication by a, and defined as $R_a(x) = xa$ for all $x \in \mathcal{A}$.

Let $\mathcal{V}(\mathcal{A}) = \mathcal{H}(\mathcal{A}) + i\mathcal{H}(\mathcal{A})$. Recall that according to [5, Hilfssatz 2(c)], for each $a \in \mathcal{V}(\mathcal{A})$ there exist necessary unique Hermitian elements $u, v \in \mathcal{H}(\mathcal{A})$ such that $a = u + iv$. As a result, the operation $a^* = u - iv$ is well defined. Note that $^* : \mathcal{V}(\mathcal{A}) \to \mathcal{V}(\mathcal{A})$ is not an involution, in particular $(ab)^*$ does not in general coincide with b^*a^*, $a, b \in \mathcal{V}(\mathcal{A})$. However, if $\mathcal{A} = \mathcal{V}(\mathcal{A})$ and for every $h \in \mathcal{H}(\mathcal{A})$, $h^2 = u + iv$ with $uv = vu$, $u, v \in \mathcal{H}(\mathcal{A})$, then $\mathcal{A}$ is a C^*-algebra whose involution is the just considered operation, see [5].

An element $a \in \mathcal{V}(\mathcal{A})$ satisfying $aa^* = a^*a$ is called *normal*. If $a = u + iv \in \mathcal{V}(\mathcal{A})$ $(u, v \in \mathcal{H}(\mathcal{A}))$, it is easy to see that a is normal if and only if $uv = vu$. An element $a \in \mathcal{V}(\mathcal{A})$ satisfying $a = aa^*a$ is called *a partial isometry* [6].

Note that necessary and sufficient for $a \in \mathcal{A}$ to belong to $\mathcal{H}(\mathcal{A})$ is that

$L_a \in \mathcal{H}(\mathcal{L}(\mathcal{A}))$. Therefore, $a \in \mathcal{V}(\mathcal{A})$ is normal if and only if $L_a \in \mathcal{V}(\mathcal{L}(\mathcal{A}))$ is normal. Observe that if $a \in \mathcal{V}(\mathcal{A})$ then $L_a \in \mathcal{V}(\mathcal{L}(\mathcal{A}))$ and $L_{a^*} = (L_a)^*$.

Theorem 1.1. [7] *Let X be a Banach space and consider $T \in \mathcal{L}(X)$ such that $T^\dagger$ exists and $T \in \mathcal{V}(\mathcal{L}(X))$. Then the following statements hold.*

(i) $R(T^*) \subseteq R(T)$ *if and only if* $T = TTT^\dagger$.

(ii) $N(T) \subseteq N(T^*)$ *if and only if* $T = T^\dagger TT$.

In addition, if the conditions of statements (i) and (ii) are satisfied, then T is an EP operator.

Notice that $R(T^*) \subseteq R(T)$ is equivalent to $T^* = TT^\dagger T^*$, by $R(T) = R(TT^\dagger) = N(I - TT^\dagger)$. The condition $N(T) \subseteq N(T^*)$ is equivalent to $T^* = T^*T^\dagger T$, because $N(T) = N(T^\dagger T) = R(I - T^\dagger T)$ [7]. Hence, by Theorem 1.1, we deduce the following.

Corollary 1.1. *Let X be a Banach space and consider $T \in \mathcal{L}(X)$ such that $T^\dagger$ exists and $T \in \mathcal{V}(\mathcal{L}(X))$. Then the following statements hold.*

(i) $T^* = TT^\dagger T^*$ *if and only if* $T = TTT^\dagger$.

(ii) $T^* = T^*T^\dagger T$ *if and only if* $T = T^\dagger TT$.

There are many papers characterizing EP elements, partial isometries, or related classes (such as normal elements). See, for example [4], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22] and [23]. Properties of the Moore-Penrose inverse in various structures can be found in [1], [2], [24], [25], [26], [27] and [28].

In [8] O.M. Baksalary, G.P.H. Styan and G. Trenkler used an elegant representation of complex matrices to explore various classes of matrices, such as partial isometries and EP. Inspired by [8], in paper [21] we use a different approach, exploiting the structure of rings with involution to investigate partial isometries and EP elements.

In this paper we characterize elements in Banach algebras which are EP and partial isometries.

2 Partial isometry and EP elements

Before the main theorem, we give some characterizations of partial isometries in Banach algebras in the following theorem.

Theorem 2.1. *Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{V}(\mathcal{A})$ such that $a^\dagger$ and $a^\#$ exist. Then the following statements are equivalent:*

(i) *a is a partial isometry;*

(ii) *$a^\# a^* a = a^\#$;*

(iii) *$aa^* a^\# = a^\#$;*

Proof. (i) $\Rightarrow$ (ii): If $aa^*a = a$, then

$$a^\# a^* a = (a^\#)^2 (aa^* a) = (a^\#)^2 a = a^\#.$$

(ii) $\Rightarrow$ (i): From $a^\# a^* a = a^\#$, it follows

$$aa^* a = a^2 (a^\# a^* a) = a^2 a^\# = a.$$

(i) $\Leftrightarrow$ (iii): This part can be proved similarly. □

In the following result we present equivalent conditions for an bounded linear operator T on Banach space X to be a partial isometry and EP. Compare with [21, Theorem 2.3] where we studied necessary and sufficient conditions for an element a of a ring with involution to be a partial isometry and EP.

Theorem 2.2. *Let X be a Banach space and consider $T \in \mathcal{L}(X)$ such that $T^\dagger$ and $T^\#$ exist and $T \in \mathcal{V}(\mathcal{L}(X))$. Then the following statements are equivalent:*

(i) *T is a partial isometry and EP;*

(ii) *T is a partial isometry and normal;*

(iii) *$T^* = T^\#$;*

(iv) *$TT^* = T^\dagger T$ and $T = TTT^\dagger$;*

(v) *$T^* T = TT^\dagger$ and $T = T^\dagger TT$;*

(vi) *$TT^* = TT^\#$ and $T = TTT^\dagger$;*

(vii) *$T^* T = TT^\#$ and $T = T^\dagger TT$;*

(viii) *$T^* T^\dagger = T^\dagger T^\#$;*

(ix) *$T^\dagger T^* = T^\# T^\dagger$;*

(x) $T^\dagger T^* = T^\dagger T^\#$ *and* $T = TTT^\dagger$;

(xi) $T^* T^\dagger = T^\# T^\dagger$ *and* $T = T^\dagger TT$;

(xii) $T^* T^\# = T^\# T^\dagger$ *and* $T = T^\dagger TT$;

(xiii) $T^* T^\dagger = T^\# T^\#$ *and* $T = T^\dagger TT$;

(xiv) $T^* T^\# = T^\# T^\#$ *and* $T = T^\dagger TT$;

(xv) $TT^* T^\# = T^\dagger$ *and* $T = T^\dagger TT$;

(xvi) $T^* T^2 = T$ *and* $T = T^\dagger TT$;

(xvii) $T^2 T^* = T$ *and* $T = TTT^\dagger$;

(xviii) $TT^\dagger T^* = T^\#$ *and* $T = TTT^\dagger$;

(xix) $T^* T^\dagger T = T^\#$ *and* $T = T^\dagger TT$.

Proof. (i) $\Rightarrow$ (ii): If T is EP, then $T = TTT^\dagger$ and, by Corollary 1.1, $T^* = TT^\dagger T^*$. Since T is a partial isometry, we have

$$TT^*T^\# = (TT^*T)(T^\#)^2 = T(T^\#)^2 = T^\#$$

and

$$T^*T^\#T = TT^\dagger T^*T^\#T = T^\dagger(TT^*T)T^\# = T^\dagger TT^\# = T^\# TT^\# = T^\#.$$

Thus, $TT^*T^\# = T^*T^\#T$ and $T = T^\dagger TT$ imply T is normal, by [7, Theorem 3.4(i)].

(ii) $\Rightarrow$ (iii): The condition T is normal and [7, Theorem 3.4(vii)] imply $T^* = TT^*T^\#$. Because T is a partial isometry, we have

$$T^* = TT^*T^\# = (TT^*T)(T^\#)^2 = T(T^\#)^2 = T^\#.$$

(iii) $\Rightarrow$ (i): Using the equality $T^* = T^\#$, we get:

$$TT^* = TT^\# = T^\#T = T^*T \text{ and } TT^*T = TT^\#T = T.$$

By [7, Theorem 3.3], T is normal gives T is EP. The condition (i) is satisfied.

(ii) $\Rightarrow$ (iv): By [7, Theorem 3.4(ii)], T is normal gives $TT^*T^\# = T^\#TT^*$ and $T = TTT^\dagger$. Now

$$TT^* = T(T^\#TT^*) = T(TT^*)T^\# = (TT^*T)T^\# = TT^\#.$$

Since T is normal implies T is EP, then $TT^* = T^{\#}T = T^\dagger T$.

(iv) $\Rightarrow$ (vi): Assume that $TT^* = T^\dagger T$ and $T = TTT^\dagger$. Then

$$T^{\#}(TT^*) = T^{\#}T^\dagger T = (T^{\#})^2 TT^\dagger T = T^{\#}$$

implying

$$TT^*T^{\#} = T(T^{\#}TT^*)T^{\#} = TT^{\#}T^{\#} = T^{\#}$$

and $T^{\#}TT^* = TT^*T^{\#}$. By [7, Theorem 3.4(ii)], T is normal and, by [7, Theorem 3.3], T is EP. Therefore, $TT^* = T^\dagger T = TT^\dagger = TT^{\#}$.

(vi) $\Rightarrow$ (ii): Let $TT^* = TT^{\#}$ and $T = TTT^\dagger$. Then

$$T(TT^*) = TTT^{\#} = T = (TT^{\#})T = TT^*T$$

which yields that T is a partial isometry and normal by [7, Theorem 3.4(x)].

(ii) $\Rightarrow$ (v) $\Rightarrow$ (vii) $\Rightarrow$ (ii): These implications can be proved in the same way as (ii) $\Rightarrow$ (iv) $\Rightarrow$ (vi) $\Rightarrow$ (ii) using [7, Theorem 3.4(i)] and [7, Theorem 3.4(ix)].

(i) $\Rightarrow$ (viii): From (i) follows (iii) $T^* = T^{\#}$ and T is EP which gives (viii).

(viii) $\Rightarrow$ (xi): Suppose that $T^*T^\dagger = T^\dagger T^{\#}$. Then

$$\begin{aligned} TT^{\#} &= T^2(T^{\#})^2 = TTT^\dagger T(T^{\#})^2 = TT(T^\dagger T^{\#}) = TTT^*T^\dagger \\ &= TT(T^*T^\dagger)TT^\dagger = TTT^\dagger T^{\#}TT^\dagger = TTT^\dagger TT^{\#}T^\dagger = TT^\dagger. \end{aligned}$$

Hence, $TT^{\#}$ is Hermitian and T is EP. Now condition (xi) is satisfied by

$$T^*T^\dagger = T^\dagger T^{\#} = T^{\#}T^\dagger \quad \text{and} \quad T^\dagger TT = TT^\dagger T = T.$$

(xi) $\Rightarrow$ (xvi): The assumptions $T^*T^\dagger = T^{\#}T^\dagger$ and $T = T^\dagger TT$ give, by Corollary 1.1,

$$T^*T^2 = (T^*T^\dagger)TT^2 = T^{\#}T^\dagger TT^2 = (T^{\#})^2TT^\dagger TT^2 = T.$$

(xvi) $\Rightarrow$ (xiv): Multiplying $T^*T^2 = T$ by $(T^{\#})^3$ from the right side, we get $T^*T^{\#} = T^{\#}T^{\#}$. Hence, T satisfies condition (xiv).

(xiv) $\Rightarrow$ (xii): If $T^*T^{\#} = T^{\#}T^{\#}$ and $T = T^\dagger TT$, then we see that $T^*T = (T^*T^{\#})T^2 = T^{\#}T^{\#}T^2 = T^{\#}T$. Thus, by (vii) $\Leftrightarrow$ (i), we get that T is EP, and

$$T^*T^{\#} = (T^*T)(T^{\#})^2 = T^{\#}T(T^{\#})^2 = (T^{\#})^2 = T^{\#}T^\dagger.$$

(xii) $\Rightarrow$ (vii): Applying $T^*T^\# = T^\# T^\dagger$ and $T = T^\dagger TT$, we obtain the condition (vii):

$$T^*T = (T^*T^\#)T^2 = T^\#(T^\dagger T^2) = T^\# T.$$

(i) $\Rightarrow$ (ix) $\Rightarrow$ (x) $\Rightarrow$ (xvii): Similarly as (i) $\Rightarrow$ (viii) $\Rightarrow$ (xi) $\Rightarrow$ (xvi).

(xvii) $\Rightarrow$ (vi): Suppose that $T^2T^* = T$ and $T = TTT^\dagger$. Then $TT^* = T^\# T^2 T^* = T^\# T$ and the condition (vi) is satisfied.

(xiii) $\Rightarrow$ (xi): Multiplying the equality $T^*T^\dagger = T^\# T^\#$ by $TT^\dagger$ from the right side, we obtain $T^*T^\dagger = T^\# T^\dagger$. So, we deduce that condition (xi) holds.

(xi) $\Rightarrow$ (xiii): By (xi), we have that T is EP and condition (xiii) is satisfied.

(xv) $\Rightarrow$ (i): Let $TT^*T^\# = T^\dagger$ and $T = T^\dagger TT$. Now, we observe that

$$TT^*T = (TT^*T^\#)T^2 = T^\dagger TT = T$$

and

$$T^\dagger = TT^*T^\# = T^\# T(TT^*T)(T^\#)^2 = T^\# TT(T^\#)^2 = T^\#.$$

Therefore, T is a partial isometry and EP.

(i) $\Rightarrow$ (xv): The hypothesis T is EP gives $T = T^\dagger TT$ and, because (i) implies (iii),

$$TT^*T^\# = TT^\# T^\# = T^\# = T^\dagger.$$

(xviii) $\Rightarrow$ (iii): By the assumption $TT^\dagger T^* = T^\#$ and $T = TTT^\dagger$, we obtain $T^* = TT^\dagger T^* = T^\#$.

(iii) $\Rightarrow$ (xviii): From $T^* = T^\#$, we get

$$TT^\dagger T^* = TT^\dagger T^\# = TT^\dagger T(T^\#)^2 = T^\#$$

and T is EP implying $T = TTT^\dagger$.

(iii) $\Leftrightarrow$ (xix): Analogy as (iii) $\Leftrightarrow$ (xviii). □

Now, we return to a general case, i.e. $\mathcal{A}$ is a complex unital Banach algebra, and $a \in \mathcal{A}$ is both Moore-Penrose and group invertible.

Corollary 2.1. *Theorem 2.2 holds if we change $\mathcal{L}(X)$ for an arbitrary complex Banach algebra $\mathcal{A}$, and we change T by an $a \in \mathcal{A}$ such that $a^\dagger$ and $a^\#$ exist.*

Proof. If $a \in \mathcal{A}$ satisfies the hypothesis of this theorem, then $L_a \in \mathcal{L}(\mathcal{A})$ satisfies the hypothesis of Theorem 2.2. Now, if any one of statements (i)-(xix) holds for a, then the same statement holds for L_a. Therefore, L_a is a partial isometry and EP in $\mathcal{L}(A)$. By [4, Remark 12], it follows that a is EP in $\mathcal{A}$. It is well-known that if $a \in \mathcal{V}(\mathcal{A})$ then $L_a \in \mathcal{V}(\mathcal{L}(\mathcal{A}))$ and $L_{a^*} = (L_a)^*$. Since L_a is a partial isometry, $L_a L_{a^*} L_a = L_a$, i.e $L_{aa^*a} = L_a$. So, we deduce that $aa^*a = a$ and a a partial isometry in $\mathcal{A}$.

A similar statement can be proved if we consider $R_a \in \mathcal{L}(\mathcal{A})$ instead of $L_a \in \mathcal{L}(\mathcal{A})$. □

The cancellation property and the identity $(ab)^* = b^*a^*$ are important when we proved the equivalent statements characterizing the condition of being a partial isometry and EP in a ring with involution $\mathcal{R}$ in [21]. Since $^* : \mathcal{V}(\mathcal{A}) \to \mathcal{V}(\mathcal{A})$ is not in general an involution, and it is not clear if the cancellation property holds for Moore-Penrose invertible elements of $\mathcal{V}(\mathcal{A})$, in most statements of Theorem 2.2 an additional condition needs to be considered.

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