NEW TYPE OF MATRIX SPLITTING AND ITS APPLICATIONS

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Abstract. One possible type of the matrix splitting is introduced. Using this matrix splitting, we introduce a few properties and representations of generalized inverses as well as iterative methods for computing various solutions of singular linear systems. This matrix splitting is a generalization of the known index splitting from [13] and a proper splitting from [4]. Using a generalization of the condition number and introduced representations of generalized inverses, we obtain several norm estimates.

1. Introduction

Let $\mathbb{C}^n$ be the set of n-dimensional complex vectors, $\mathbb{C}^{m\times n}$ be the set of $m\times n$ complex matrices, and $\mathbb{C}_r^{m\times n}=\{X\in\mathbb{C}^{m\times n}:\quad \operatorname{rank}(X)=r\}$. By I_r we denote the identity matrix of the order r. We use $\mathcal{N}(A)$ to denote the kernel, $\mathcal{R}(A)$ to denote the image of A, $\sigma(A)$ to denote the set of all eigenvalues of A and $\rho(A)$ to denote the spectral radius of A. If L and M are complementary subspaces of $\mathbb{C}^n$, then $P_{L,M}$ denotes the projector on L along M.

For any matrix $A\in\mathbb{C}^{m\times n}$ consider the following equations in X:

$$(1)\quad AXA=A,\quad (2)\quad XAX=X,\quad (3)\quad (AX)^*=AX,\quad (4)\quad (XA)^*=XA$$

and if $m=n$, also

$$(1^k)\quad A^{k+1}X=A^k,\qquad (5)\qquad AX=XA.$$

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For a sequence $\mathcal{S}$ of elements from the set $\{1,2,3,4,5\}$, the set of matrices obeying the equations represented in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$. If X satisfies (1) and (2), it is said to be a reflexive generalized inverse of A. The Moore-Penrose inverse of A is its unique $\{1,2,3,4\}$ inverse: $A^\dagger = A\{1,2,3,4\}$. A matrix $X = A^D$ is said to be the Drazin inverse of A if (1^k) (for some positive integer k), (2) and (5) are satisfied. The group inverse $A^\#$ is the unique $\{1,2,5\}$ inverse of A, and exists if and only if the index of A satisfies $\operatorname{ind}(A)=\min_k\{k: \ \operatorname{rank}(A^{k+1})=\operatorname{rank}(A^k)\}=1$ [6], [10].

We use the following known results from [2]:

Lemma 1.1. *If $B \in \mathbb{C}^{n\times n}$ and L, M are complementary subspaces of $\mathbb{C}^n$, then*

(a) $P_{L,M}B = B$ *if and only if* $\mathcal{R}(B) \subseteq L$;
(b) $BP_{L,M} = B$ *if and only if* $\mathcal{N}(B) \supseteq M$.

The idea of splitting of matrices is originated by the regular splitting theory, introduced in [11]. Several various types of matrix splittings are defined in [16] and [17]. The definitions of splittings are collected in the following definition (see [17]).

Definition 1.1. Let U, V be $n \times n$ matrices. Then the decomposition $A = U - V$ is called:

(a) a regular splitting of A if $U^{-1} \geq 0$ and $V \geq 0$;
(b) a non-negative splitting of A if $U^{-1} \geq 0$, $U^{-1}V \geq 0$ and $VU^{-1} \geq 0$;
(c) a weak non-negative splitting of A if $U^{-1} \geq 0$ and either $U^{-1}V \geq 0$ (the first type) or $VU^{-1} \geq 0$ (the second type);
(d) a weak splitting of A if U is nonsingular and either $U^{-1}V \geq 0$ (the first type) or $VU^{-1} \geq 0$ (the second type);
(e) a convergent splitting of A if $\rho(U^{-1}V) = \rho(VU^{-1}) < 1$.

The concept of a regular splitting is used in characterizations of the usual inverse and in iterative methods for solving linear systems. These results are extended to the Moore-Penrose inverse and rectangular linear systems in [3], [4]. This extension is based on the application of the proper splitting [4].

In [13] it is presented the index splitting of a singular square $n \times n$ matrix A and its relative iterations for the minimal P-norm solution of a singular linear system $Ax = b$, $x, b \in \mathbb{C}^n$. Also a few representations of the Drazin inverse are introduced in [13].

Definition 1.2. Let $A \in \mathbb{C}^{n\times n}$ with $k = \operatorname{ind}(A)$. Then the splitting $A = U - V$ is called an index splitting of A provided that

$$\mathcal{R}(U) = \mathcal{R}(A^k), \qquad \mathcal{N}(U) = \mathcal{N}(A^k).$$

In the case $\operatorname{ind}(A) = 1$ the index splitting is known as a proper splitting [4].

Our main idea in this paper is the generalization of these results to outer inverses with prescribed range and kernel (so called $A^{(2)}_{T,S}$ inverses). For this purpose we introduce a new type of a matrix splitting, which is a generalization of all so far known splittings. It is known that all important generalized inverses: the Moore-Penrose inverse, the weighted Moore-Penrose inverse, the Drazin inverse, the group inverse, the Bott-Duffin inverse and the generalized Bott-Duffin inverse are all $A^{(2)}_{T,S}$ generalized inverses.

We are also motivated by the fact that the theory of outer inverses with prescribed range and kernel is very actual (see for example [2], [5], [12], [14], [15]). Also, it is known that $\{2\}$-inverses have many applications, for example, in the iterative methods for solving the nonlinear equations and the applications to statistics [14].

Now we describe main results of this paper. In the second section we introduce one type of the matrix splitting, applicable for rectangular matrices. We also develop a few properties and representations of generalized inverses, based on this matrix splitting. Finally, using these representations, we introduce iterative methods for computing various solutions of the linear system.

This type of the matrix splitting is called the $\{T, S\}$ splitting and it is useful in the representation of the generalized inverse $A^{(2)}_{T,S}$ and the solution $A^{(2)}_{T,S}b$ of a given linear system $Ax = b$. In a partial case

$$T = \mathcal{R}(U) = \mathcal{R}(A^k), \quad S = \mathcal{N}(U) = \mathcal{N}(A^k)$$

the results arising from the $\{T, S\}$ splitting reduce to the known representations of the Drazin inverse and the minimal P-norm solution of the system $Ax = b$, introduced in [13].

From the other hand, in the case $T = \mathcal{R}(A^*) = \mathcal{R}(U^*)$ and $S = \mathcal{N}(A^*) = \mathcal{N}(U^*)$ our matrix splitting reduces to the well-known proper splitting, introduced in [4].

In the third section we introduce a generalization of the condition numbers $K(A) = \|A\|\|A^\dagger\|$ and $K_D(A) = \|A\|\|A^D\|$, which is based on the application of the generalized inverse $A^{(2)}_{T,S}$. Using this generalization of the condition number and representations of the generalized inverse $A^{(2)}_{T,S}$ we develop several norm estimates. The results of this section are generalizations of the perturbation theory for the Drazin inverse introduced in [15].

2. New matrix splitting

In the beginning we introduce a possible generalization of the index splitting from [13]. This matrix splitting is applicable in characterizations and representations of $\{2\}$ generalized inverse $A^{(2)}_{T,S}$. Also, this matrix splitting can be applied in the construction of an iterative method which produces various solutions of a given singular linear system $Ax = b$.

Definition 2.1. Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$ and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. Then the splitting $A = U - V$ is called the $\{T, S\}$ *splitting* of A if the following condition is satisfied:

$$UT \oplus S = \mathbb{C}^m \tag{2.1}$$

Remark 2.1. In the case

$$m = n, \quad T = \mathcal{R}(U) = \mathcal{R}(A^k), \quad S = \mathcal{N}(U) = \mathcal{N}(A^k), \quad k \ge \operatorname{ind}(A),$$

the notion of the $\{T, S\}$ splitting reduces to the known notion of the index splitting.

In the following theorem we introduce a characterization of the generalized inverse $A^{(2)}_{T,S}$.

Theorem 2.1. *Let $A \in \mathbb{C}^{m\times n}_r$, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$ and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$, such that*

$$AT \oplus S = \mathbb{C}^m. \tag{2.2}$$

Assume that $A = U - V$ is a $\{T, S\}$ splitting of A and $\dim(T) \le \operatorname{rank}(U)$.

Then the generalized inverse $A^{(2)}_{T,S}$ satisfies the following conditions:

$$U^{(2)}_{T,S} - A^{(2)}_{T,S} = -U^{(2)}_{T,S} V A^{(2)}_{T,S} = -A^{(2)}_{T,S} V U^{(2)}_{T,S}, \tag{2.3}$$

$$A^{(2)}_{T,S} = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S} = U^{(2)}_{T,S}(I - VU^{(2)}_{T,S})^{-1} \tag{2.4}$$

and

$$U^{(2)}_{T,S} = (I + A^{(2)}_{T,S}V)^{-1}A^{(2)}_{T,S} = A^{(2)}_{T,S}(I + VA^{(2)}_{T,S})^{-1}. \tag{2.5}$$

Proof. Using known results from [2], [5], it is not difficult to verify the existence of the generalized inverses $A^{(2)}_{T,S}$ and $U^{(2)}_{T,S}$. From Lemma 1.1 we obtain

$$\begin{aligned}
-U^{(2)}_{T,S}VA^{(2)}_{T,S} &= -U^{(2)}_{T,S}(U-A)A^{(2)}_{T,S} = -U^{(2)}_{T,S}UA^{(2)}_{T,S} + U^{(2)}_{T,S}AA^{(2)}_{T,S} \\
&= -P_{\mathcal{R}(U^{(2)}_{T,S}U),\mathcal{N}(U^{(2)}_{T,S}U)}A^{(2)}_{T,S} + U^{(2)}_{T,S}P_{\mathcal{R}(AA^{(2)}_{T,S}),\mathcal{N}(AA^{(2)}_{T,S})} \\
&= U^{(2)}_{T,S} - A^{(2)}_{T,S}.
\end{aligned}$$

Analogously we can prove $U^{(2)}_{T,S} - A^{(2)}_{T,S} = -A^{(2)}_{T,S}VU^{(2)}_{T,S}$. Thus, the proof of (2.3) is completed.

To prove (2.4) we need to prove that $I - U^{(2)}_{T,S}V$ is invertible. Consider an arbitrary $x \in \mathbb{C}^n$ satisfying

$$(I - U^{(2)}_{T,S}V)x = 0.$$

Then we get $x = U^{(2)}_{T,S}Vx$, which implies

$$x \in \mathcal{R}(U^{(2)}_{T,S}) = T$$

and

$$Ax \in AT. \tag{2.6}$$

Also, from

$$U^{(2)}_{T,S}U(I - U^{(2)}_{T,S}V)x = 0$$

we get $U^{(2)}_{T,S}Ax = 0$, which means

$$Ax \in \mathcal{N}(U^{(2)}_{T,S}) = S. \tag{2.7}$$

From (2.2), (2.6) and (2.7) we get $Ax = 0$ and

$$x \in \mathcal{N}(A) \cap T.$$

Since there exists $A^{(2)}_{T,S}$, the restriction of A to T is an one-to-one operator. Hence $x = 0$ and we get the invertibility of the matrix $I - U^{(2)}_{T,S}V$. From $\sigma(MN) \cup \{0\} = \sigma(NM) \cup \{0\}$ for arbitrary rectangular matrices M and N, it follows that $I - VU^{(2)}_{T,S}$ is invertible also.

Since $U = A-(-V)$ is the $\{T, S\}$ splitting of U, in the same way as above we can prove that $I + A^{(2)}_{T,S}V$ and $I + VA^{(2)}_{T,S}$ are invertible.

Now the equalities in (2.4) and (2.5) follow immediately from the equalities in (2.3) and the invertibility of matrices $I-U^{(2)}_{T,S}V$, $I-VU^{(2)}_{T,S}$, $I+A^{(2)}_{T,S}V$ and $I + VA^{(2)}_{T,S}$. □

Analogous characterizations and representations of the Drazin inverse and the Moore-Penrose inverse can be derived in certain cases of Theorem 2.1.

Corollary 2.1. *Assume that* $A = U - V$ *is a* $\{T, S\}$ *splitting of* $A \in \mathbb{C}^{m\times n}$. *Let the condition* (2.2) *be satisfied. Then the following statements are valid:*

(a) *In the case*

$$m = n, \quad T = \mathcal{R}(U) = \mathcal{R}(A^k), \quad S = \mathcal{N}(U) = \mathcal{N}(A^k), \quad k \geq \operatorname{ind}(A),$$

we get

$$A^D = (I - U^{\#}V)^{-1}U^{\#}.$$

(b) *In the case*

$$T = \mathcal{R}(U^*) = \mathcal{R}(A^*), \quad S = \mathcal{N}(U^*) = \mathcal{N}(A^*)$$

we have

$$A^{\dagger} = (I - U^{\dagger}V)^{-1}U^{\dagger}.$$

Proof. **(a)** From Theorem 2.1 we immediately obtain

$$A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)} = (I - U^{(2)}_{\mathcal{R}(U),\mathcal{N}(U)}V)^{-1}U^{(2)}_{\mathcal{R}(U),\mathcal{N}(U)},$$

which is equivalent to

$$A^D = (I - U^{\#}V)^{-1}U^{\#}.$$

(b) This part of the proof also follows from Theorem 2.1, since $A^{(2)}_{T,S} = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)} = A^{\dagger}$ and $U^{(2)}_{T,S} = U^{(2)}_{\mathcal{R}(U^*),\mathcal{N}(U^*)} = U^{\dagger}$. □

Remark 2.2. As it is shown in part **(a)** of Corollary 2.1, in the case $m = n$, $\mathcal{R}(U) = T = \mathcal{R}(A^k)$, $\mathcal{N}(U) = S = \mathcal{N}(A^k)$, $k \geq \operatorname{ind}(A)$, the result of Theorem 2.1 reduces to the known characterization and representation of the Drazin inverse, introduced in [13, Theorem 3.1]. Moreover, it is known that many of important generalized inverses, besides the Drazin inverse, such as: the Moore-Penrose inverse, weighted Moore-Penrose inverse, weighted Drazin inverse, Bott-Duffin inverse, can be expressed as $\{2\}$ inverse having a corresponding range T and null space S [2], [5], [14]. Theorem 2.1 is applicable to each of these inverses.

So far, we have used the introduced matrix splitting to characterize the generalized inverses $A^{(2)}_{T,S}$ and $U^{(2)}_{T,S}$. But, these matrix splittings can be used in characterizations of the generalized inverse $V^{(2)}_{T,S}$.

Corollary 2.2. *Let $A \in \mathbb{C}^{m\times n}_r$, let T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$ and S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. If the conditions in* (2.1) *are satisfied, $A = U - V$, and the following conditions hold:*

$$VT \oplus S = \mathbb{C}^m, \quad \dim(T) \leq \min\{\operatorname{rank}(U), \operatorname{rank}(V)\},$$

then

$$V^{(2)}_{T,S} = (I - U^{(2)}_{T,S}A)^{-1}U^{(2)}_{T,S} = U^{(2)}_{T,S}(I - AU^{(2)}_{T,S})^{-1}.$$

Proof. The conditions in (2.1) and $\dim(T) \leq \operatorname{rank}(U)$ ensure the existence of the generalized inverse $U^{(2)}_{T,S}$. Also, the conditions $VT \oplus S = \mathbb{C}^m$ and $\dim(T) \leq \operatorname{rank}(V)$ ensure the existence of the generalized inverse $V^{(2)}_{T,S}$. Finally, from (2.1) we conclude that $V = U - A$ is a $\{T, S\}$ splitting of V. The proof immediately follows from Theorem 2.1. □

In the following statement we derive a few iterative methods related to a consistent linear system $Ax = b$.

Corollary 2.3. *If the conditions from Theorem 2.1 are satisfied and $x \in T$, then:*

(i) *The vector $A^{(2)}_{T,S}b$ is the unique solution of the system*

$$x = U^{(2)}_{T,S}Vx + U^{(2)}_{T,S}b$$

for any $b \in \mathbb{C}^n$.

(ii) *The iteration*

$$x_{i+1} = U_{T,S}^{(2)} V x_i + U_{T,S}^{(2)} b, \quad b \in \mathbb{C}^n,$$

converges to $A_{T,S}^{(2)} b$ *for every* $x_0 \in \mathbb{C}^n$ *if and only if* $\rho(U_{T,S}^{(2)} V) < 1$.

Proof. (i) From $(I - U_{T,S}^{(2)} V)x = U_{T,S}^{(2)} b$ we have $x = (I - U_{T,S}^{(2)} V)^{-1} U_{T,S}^{(2)} b$. Applying Theorem 2.1 we get $x = A_{T,S}^{(2)} b$, which completes this part of the proof. The part (ii) follows from the part (i). □

Recall that a set $K \subset \mathbb{R}^n$ is called a full cone if the following is satisfied:

$\lambda \geq 0$ implies $\lambda K \subset K$, K is convex and closed, $K + K = \mathbb{R}^n$, $K \cap (-K) = \{0\}$ and the interior of K is nonempty.

A full cone $K \subset \mathbb{R}^n$ induces a partial ordering in $\mathbb{R}^n$ given by

$$x \overset{K}{\leq} y \quad \text{if and only if} \quad y - x \in K.$$

A sequence (x_i) in $\mathbb{R}^n$ is K-monotone nondecreasing if $x_{i-1} \overset{K}{\leq} x_i$ for all $i = 1, 2, \ldots$.

We need the following auxiliary results.

Lemma 2.1. *Let K be a full cone in $\mathbb{R}^n$ and (s_i) be a K-monotone nondecreasing sequence. If there exists some $t \in \mathbb{R}^n$ such that $t - s_i \in K$ for all $i = 1, 2, \ldots$, then the sequence (s_i) converges.*

Lemma 2.2 [9]**.** *If K is a full cone in $\mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ has the property $AK \subset K$, then K contains an eigenvector of A whose eigenvalue is $\rho(A)$.*

Now we formulate equivalent conditions for $\rho(U_{T,S}^{(2)} V) < 1$.

Theorem 2.2. *Let $L \subset \mathbb{R}^m$ and $K\mathbb{R}^n$ be full cones and $A = U - V$ be a $\{T, S\}$ splitting of $A \in \mathbb{R}^{m\times n}$, such that the conditions from Theorem 2.1 are satisfied, $U_{T,S}^{(2)} L \subset K$ and $U_{T,S}^{(2)} V K \subset K$. Then the following statements are equivalent:*

(a) $A_{T,S}^{(2)} L \subset K$.

(b) $A_{T,S}^{(2)} V K \subset K$.

(c) $\rho(U_{T,S}^{(2)} V) = \dfrac{\rho(A_{T,S}^{(2)} V)}{1 + \rho(A_{T,S}^{(2)} V)} < 1$.

Proof. This result is similar as [4, Theorem 2 and Theorem 3], but the proof needs some corrections. Thus, for the convenience of the reader, we give a complete proof.

(c) $\Longrightarrow$ **(b)** Since $\rho(U^{(2)}_{T,S}V) < 1$ we get

$$(I - U^{(2)}_{T,S}V)^{-1} = \sum_{j=0}^{\infty}(U^{(2)}_{T,S}V)^j$$

and from (2.4) we get

$$A^{(2)}_{T,S}V = \sum_{j=1}^{\infty}(U^{(2)}_{T,S}V)^j.$$

Since $U^{(2)}_{T,S}VK \subset K$ we conclude $A^{(2)}_{T,S}VK \subset K$.

(b) $\Longrightarrow$ **(a)** If $A^{(2)}_{T,S}VK \subset K$, we shall prove that $\rho(U^{(2)}_{T,S}V) < 1$.

Since $U^{(2)}_{T,S}VK \subset K$, by Lemma 2.2 we get that there exist some $x \in K$ such that $U^{(2)}_{T,S}Vx = \rho(U^{(2)}_{T,S}V)x$. From (2.4) we get

$$A^{(2)}_{T,S}Vx = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S}Vx = \frac{\rho(U^{(2)}_{T,S}V)}{1-\rho(U^{(2)}_{T,S}V)}x.$$

Since $U^{(2)}_{T,S}Vx \in K$, $A^{(2)}_{T,S}Vx \in K$ and $K \cap (-K) = \{0\}$, we conclude $\frac{\rho(U^{(2)}_{T,S}V)}{1-\rho(U^{(2)}_{T,S}V)} < 1$ and $\rho(U^{(2)}_{T,S}V) < 1$.

Now, from (2.4) we have

$$A^{(2)}_{T,S}L = \sum_{j=0}^{\infty}(U^{(2)}_{T,S}V)^j U^{(2)}_{T,S}L \subset \sum_{j=0}^{\infty}(U^{(2)}_{T,S}V)^j K \subset K.$$

(a) $\Longrightarrow$ **(c)** Denote $S_p = \sum_{n=0}^{p-1}(U^{(2)}_{T,S}V)^n$ for each positive integer p. Since

$$U^{(2)}_{T,S}AA^{(2)}_{T,S} = U^{(2)}_{T,S} \quad \text{and} \quad U^{(2)}_{T,S}UA^{(2)}_{T,S} = A^{(2)}_{T,S},$$

we get

$$\begin{aligned} S_pU^{(2)}_{T,S} &= S_pU^{(2)}_{T,S}AA^{(2)}_{T,S} = S_pU^{(2)}_{T,S}UA^{(2)}_{T,S} - S_pU^{(2)}_{T,S}VA^{(2)}_{T,S} \\ &= S_pA^{(2)}_{T,S} - \sum_{n=1}^{p}(U^{(2)}_{T,S}V)^j A^{(2)}_{T,S} = (I - (U^{(2)}_{T,S}V)^p)A^{(2)}_{T,S}. \end{aligned}$$

Now $(U_{T,S}^{(2)}V)^p A_{T,S}^{(2)}L \subset K$ and hence

$$(A_{T,S}^{(2)} - S_p U_{T,S}^{(2)})L = (U_{T,S}^{(2)}V)^p A_{T,S}^{(2)}L \subset K.$$

If $l \in L$, then $A_{T,S}^{(2)}l - S_p U_{T,S}^{(2)}l \in K$. Denote $t = A_{T,S}^{(2)}l$ and $s_p = S_p U_{T,S}^{(2)}l$. Then t and s_i $(i = 1, 2, \dots)$ satisfy the conditions from Lemma 2.1 and

$$\lim_{i\to\infty}(s_i - s_{i-1}) = \lim_{i\to\infty}(U_{T,S}^{(2)}V)^i U_{T,S}^{(2)}l = 0.$$

From Lemma 2.2 there exists $y \in K$ such that $U_{T,S}^{(2)}Vy = \rho(U_{T,S}^{(2)}V)y$. Also, there exists some x such that $y = U_{T,S}^{(2)}x$. Since $\mathbb{R}^m = L + (-L)$, we can write $x = l_1 - l_2$ for some $l_1, l_2 \in L$. Then for all i we get

$$[\rho(U_{T,S}^{(2)}V)]^i y = (U_{T,S}^{(2)}V)^i U_{T,S}^{(2)}l_1 - (U_{T,S}^{(2)}V)^i U_{T,S}^{(2)}l_2.$$

We conclude that the sequence $(\rho(U_{T,S}^{(2)}V)^i$ converges to 0 as $i \to \infty$. Now $\rho(U_{T,S}^{(2)}V) < 1$ and again we get $A_{T,S}^{(2)}VK \subset K$.

To prove that $\rho(U_{T,S}^{(2)}V) = \frac{\rho(A_{T,S}^{(2)}V)}{1+\rho(A_{T,S}^{(2)}V)}$, notice that in the implication **(b)** $\Longrightarrow$ **(a)** we have already proved $\rho(A_{T,S}^{(2)}V) \geq \frac{\rho(U_{T,S}^{(2)}V)}{1-\rho(U_{T,S}^{(2)}V)}$, or, equivalently,

$$\rho(U_{T,S}^{(2)}V) \leq \frac{\rho(A_{T,S}^{(2)}V)}{1+\rho(A_{T,S}^{(2)}V)}.$$

The reverse inequality can be proved in the same manner. Since $A_{T,S}^{(2)}VK \subset K$, there exists some $z \in K$ such that $A_{T,S}^{(2)}Vz = \rho(A_{T,S}^{(2)}V)z$. Then, from (2.5) we get

$$U_{T,S}^{(2)}Vz = (I + A_{T,S}^{(2)}V)^{-1}A_{T,S}^{(2)}Vz = \frac{\rho(A_{T,S}^{(2)}}{1+\rho(A_{T,S}^{(2)}V)}z.$$

Thus, **(c)** follows. □

3. Norm estimates and matrix splitting

It is known that the condition number of a given regular matrix A is defined by $\mathrm{cond}(A) = \|A\|\|A^{-1}\|$. In [8] the notion of the condition number of an arbitrary matrix is defined by $K(A) = \|A\|\|A^{\dagger}\|$. In [7] the following generalization of the condition number is introduced

$$\mathcal{K}(A) = \|A\| \cdot \inf\{\|A^{(1)}\| : A^{(1)} \in A\{1\}\}.$$

The condition number $K_D(A) = \|A\|\|A^D\|$, which is defined with respect to the Drazin inverse, is introduced in [15].

We introduce a generalized condition number of a matrix which is based on the application of the generalized inverse $A^{(2)}_{T,S}$.

Definition 3.1. For a given matrix $A \in \mathbb{C}^{m\times n}$ let T and S be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, such that there exists the $A^{(2)}_{T,S}$ inverse. The generalized condition number is introduced as

$$K_{T,S}(A) = \|A\|\|A^{(2)}_{T,S}\|.$$

Remark 3.1. For $T = \mathcal{N}(A)^{\perp}$ and $S = \mathcal{R}(A)^{\perp}$ the introduced generalized condition number $K_{T,S}(A)$ reduces to the well-known generalized condition number $K(A)$.

In the case $T = \mathcal{R}(A^k)$, $S = \mathcal{N}(A^k)$, where $k = \mathrm{rank}(A)$, the condition number $K_{T,S}(A)$ reduces to the condition number $K_D(A)$.

In this section we show the usefulness of the introduced generalized condition number $K_{T,S}(A)$.

Proposition 3.1. *Consider the system of linear equations*

$$Ax = b, \tag{3.1}$$

where $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. Let T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r = \mathrm{rank}(A)$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$, such that $AT \oplus S = \mathbb{C}^m$. If (3.1) *is solvable, $x \in T$ satisfies*

$$x = A^{(2)}_{T,S}b, \qquad (x + \delta x) = A^{(2)}_{T,S}(b + \delta b),$$

then

$$\frac{\|\delta x\|}{\|x\|} \leq K_{T,S}(A)\frac{\|\delta b\|}{\|b\|}.$$

Proof. This result can be proved similarly as in [1, Theorem 5]. □

In the following theorem we establish the norm estimate for $A^{(2)}_{T,S} - U^{(2)}_{T,S}$ using the generalized condition number. Here $A = U - V$ is the $\{T, S\}$ splitting of A.

Theorem 3.1. *Assume the conditions from Theorem 2.1 are satisfied. If* $\|A^{(2)}_{T,S}V\| < 1$*, then the following norm estimate can be established:*

$$\|U^{(2)}_{T,S} - A^{(2)}_{T,S}\| \le \frac{\|A^{(2)}_{T,S}V\|\|A^{(2)}_{T,S}\|}{1 - \|A^{(2)}_{T,S}V\|} \le K_{T,S}(A)\frac{\|A^{(2)}_{T,S}V\|}{\|A\|(1 - \|A^{(2)}_{T,S}V\|)}.$$

Proof. Since $\|A^{(2)}_{T,S}V\| < 1$, from Theorem 2.1 we get the following expression

$$\begin{aligned} U^{(2)}_{T,S} - A^{(2)}_{T,S} &= (I + A^{(2)}_{T,S}V)^{-1}A^{(2)}_{T,S} - A^{(2)}_{T,S} \\ &= \left(\sum_{k=0}^{\infty}(-1)^k(A^{(2)}_{T,S}V)^k - I\right)A^{(2)}_{T,S} = \sum_{k=1}^{\infty}(-1)^k(A^{(2)}_{T,S}V)^kA^{(2)}_{T,S}. \end{aligned}$$

Hence

$$\|U^{(2)}_{T,S} - A^{(2)}_{T,S}\| \le \frac{\|A^{(2)}_{T,S}V\|\|A^{(2)}_{T,S}\|}{1 - \|A^{(2)}_{T,S}V\|} \le K_{T,S}(A)\frac{\|A^{(2)}_{T,S}V\|}{\|A\|(1 - \|A^{(2)}_{T,S}V\|)}. \quad \square$$

Remark 3.2. Notice that Theorem 2.1 and Theorem 3.1 are generalizations of the main result of Wang and Wei from [15, Theorem 3.2], stated for the Drazin inverse. Moreover, in the verification of these results we do not need the additional condition $AA^{(2)}_{T,S}VA^{(2)}_{T,S}A = V$ which is essentially used in [15].

Suppose that we know the matrix $(1+\delta)A$ instead of the matrix A, where δ is a complex number. If there exist $A^{(2)}_{T,S}$ and $(A+\delta A)^{(2)}_{T,S}$ inverses for suitable chosen subspaces T and S, one could ask the following question: can we give a norm estimate for $A^{(2)}_{T,S} - (A + \delta A)^{(2)}_{T,S}$? The answer is contained in the following theorem.

Theorem 3.2. *For $A \in \mathbb{C}^{m\times n}$ let δ be a given complex number, let T and S be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, such that there exist $A^{(2)}_{T,S}$ and $(A+\delta A)^{(2)}_{T,S}$ inverses. If $\|(A+\delta A)^{(2)}_{T,S}\delta A\| < 1$ is satisfied, then the following holds:*

$$\|A^{(2)}_{T,S} - (A+\delta A)^{(2)}_{T,S}\| \leq K_{T,S}(A+\delta A)\frac{\|(A+\delta A)^{(2)}_{T,S}\delta A\|}{\|A+\delta A\|(1-\|(A+\delta A)^{(2)}_{T,S}\delta A\|)}.$$

Proof. Let $U = A + \delta A$. Since there exists $A^{(2)}_{T,S}$ and $U^{(2)}_{T,S}$ inverses, we may consider $A = U - \delta A$ as the splitting of A. Now, the conditions from Theorem 3.1 are satisfied and the result follows immediately from Theorem 3.1. □

As an application, consider a perturbation of a linear system. We assume $A = U - V$ and there exist $A^{(2)}_{T,S}$ and $U^{(2)}_{T,S}$ for given subspaces T and S. Let there be given a linear system $Ax = b$ and a perturbed system $Uy = c$. If $b \in A(T)$ and $c \in U(T)$, then the unique solutions in T are given as $x = A^{(2)}_{T,S}b$ and $y = U^{(2)}_{T,S}c$.

The following result can be proved similarly as in [15, Theorem 4.1], using our Theorem 2.1 and Theorem 3.1. For the convenience of the reader we give a complete proof.

Theorem 3.3. *If $A = U - V$ is a $\{T,S\}$-splitting of A, such that there exist $A^{(2)}_{T,S}$, $U^{(2)}_{T,S}$ and $\|A^{(2)}_{T,S}\|\|V\| < 1$, then*

$$\frac{\|y-x\|}{\|x\|} \leq \frac{K_{T,S}(A)}{1-K_{T,S}(A)\|V\|/\|A\|}\left(\frac{\|E\|}{\|A\|}+\frac{\|f\|}{\|b\|}\right).$$

Here $x = A^{(2)}_{T,S}b$, $y = U^{(2)}_{T,S}c$, $b \in A(T)$, $c \in U(T)$ and $f = c - b$.

Proof. From (2.3) we get

$$\begin{aligned} y - x &= U^{(2)}_{T,S}(b+f) - A^{(2)}_{T,S}b = (U^{(2)}_{T,S} - A^{(2)}_{T,S})b + U^{(2)}_{T,S}f \\ &= -U^{(2)}_{T,S}VA^{(2)}_{T,S}b + U^{(2)}_{T,S}f = -U^{(2)}_{T,S}Vx + U^{(2)}_{T,S}f. \end{aligned}$$

Notice that from Theorem 3.1 we easily get

$$\|U^{(2)}_{T,S}\| \leq \frac{\|A^{(2)}_{T,S}\|}{1-\|A^{(2)}_{T,S}\|\|V\|}.$$

Also $\frac{\|A\|\|x\|}{\|b\|} \geq 1$. Now we have

$$\begin{aligned}
\|y-x\| &\leq \|U_{T,S}^{(2)}\|\|V\|\|x\| + \|U_{T,S}^{(2)}\|\|f\| \\
&\leq \frac{\|A_{T,S}^{(2)}\|}{1-\|A_{T,S}^{(2)}\|\|V\|}\left(\|V\|\|x\| + \frac{\|f\|}{\|b\|}\|A\|\|x\|\right) \\
&= \frac{\|A\|\|A_{T,S}^{(2)}\|\|x\|}{1-\|A_{T,S}^{(2)}\|\|V\|}\left(\frac{\|V\|}{\|A\|} + \frac{\|f\|}{\|b\|}\right) \\
&= \frac{K_{T,S}(A)\|x\|}{1-K_{T,S}(A)\|V\|/\|A\|}\left(\frac{\|V\|}{\|A\|} + \frac{\|f\|}{\|b\|}\right). \quad \square
\end{aligned}$$

In the next statement we present a few general results for all so far known matrix splittings.

Theorem 3.4. *Let A be an arbitrary matrix and $A = U - V$ be a matrix splitting satisfying the following conditions*

$$U^g U = P_{L,M}, \quad \text{where } \mathcal{R}(A^g) \subseteq L \tag{3.2}$$

$$AA^g = P_{F,G}, \quad \text{where } \mathcal{N}(U^g) \supseteq G \tag{3.3}$$

$$x \in \mathcal{R}(A^g), \tag{3.4}$$

where A^g (respectively U^g) denotes one of the inverses A^{-1}, A^D, $A_{T,S}^{(2)}$ or $A^\dagger$ (respectively U^{-1}, U^D, $U_{T,S}^{(2)}$ or $U^\dagger$). Then the following statements are valid:

(a) $U^g - A^g = -U^g V A^g = -A^g V U^g$.

(b) $A^g = (I - U^g V)^{-1} U^g = U^g (I - V U^g)^{-1}$.

(c) $U^g = (I + A^g V)^{-1} A^g = A^g (I + V A^g)^{-1}$.

(d) *The vector $A^g b$ is the unique solution of the system $x = U^g V x + U^g b$.*

(e) *The iteration $x_{i+1} = U^g V x_i + U^g b$ converges to $A^g b$ for every $x_0 \in \mathbb{C}^n$ if and only if $\rho(U^g V) < 1$.*

(f) $\|U^g - A^g\| \leq \dfrac{\|A^g V\|\|A^g\|}{1 - \|A^g V\|} \leq K_g(A) \dfrac{\|A^g V\|}{\|A\|(1 - \|A^g V\|)}$,
where $K_g(A) = \|A\|\|A^g\|$.

Proof. All of these results follows from Theorem 2.1 and Theorem 3.1 for particular choices of T and S. Historically, we mention special cases.

The case $A^g = A^D$ is presented in [13], and it is based on the application of the index splitting.

The case $A^g = A^\dagger$ is presented in [4], using the proper splitting. □

Remark 3.3. In a view of the results of Theorem 3.4 we state a few general principles which can be used in the construction of future matrix splittings. We say that the splitting $A = U - V$ is proper for the generalized inverses A^g if the conditions (3.2), (3.3) and (3.4) are satisfied. If these conditions are satisfied, then statements similar to statements stated before in Theorem 3.4 are valid.

4. Conclusion

In this paper we introduce one possible matrix splitting, so called $\{T, S\}$ splitting, which can be applied to rectangular matrices. This matrix splitting is a generalization of the known index splitting introduced in [13] and a proper splitting introduced in [4]. It is used in representations and characterizations of generalized inverses, as well as in the construction of some iterative methods which are applied for various solutions of singular linear systems. Also, these applications of $\{T, S\}$ matrix splitting, applications of the index splitting from [13] and applications of the matrix splitting of regular matrices [17], possesses a general form.

A generalization of the condition number is introduced. Some error estimates are established by means of the introduced condition number and the introduced representations of the generalized inverse $A^{(2)}_{T,S}$.

We also stated a few principles which can be used in the construction of eventually future matrix splittings.

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