SPLITTINGS OF OPERATORS AND GENERALIZED INVERSES

DRAGAN S. DJORDJEVIĆ AND PREDRAG S. STANIMIROVIĆ

ABSTRACT. In this paper we extend the notion of the proper splitting of rectangular matrices introduced and investigated in (Berman, A. and Neumann, M., SIAM J. Appl. Math. **31** (1976), 307–312; and Berman, A. and Plemmons, R. J., SIAM J. Numer. Anal. **11** (1974), 145–154) to g-invertible operators on Banach spaces. Also, we extend and generalize the notion of the index splitting of square matrices introduced and investigated in (Wei, Y., Appl. Math. Comput., **95**, (1998), 115–124) introducing the $\{T,S\}$-splitting for arbitrary operators on Banach spaces. The index splitting is a partial case of $\{T,S\}$-splitting. The obtained results extend and generalize various well-known results for square and rectangular complex matrices.

1. Introduction

Several various types of matrix splittings can be found in [12] and [13]. The idea of splitting of matrices is originated in the regular splitting theory, introduced in [9]. The concept of a regular splitting is used in characterizations of the usual inverse and in iterative methods for solving linear systems. These results are extended to the Moore-Penrose inverse of a complex rectangular matrix and rectangular linear systems in [2], [3]. This extension is based on the application of the proper splitting [3]. In [10] the index splitting of a singular square $n \times n$ matrix A and its relative iterations for the minimal P-norm solution of a singular linear system $Ax = b$, $x, b \in \mathbb{C}^n$ are

1991 *Mathematics Subject Classification.* 47A05, 15A09, 47A50, 65F20, 65J10.
Key words and phrases. Generalized inverses, splittings of operators, iterative methods.

presented. Also a few representations of the Drazin inverse, based on the index splitting, are introduced in [10].

In this paper we investigate a general problem of splittings for bounded operators on Banach spaces. The concept of the proper splitting is extended to g-invertible operators on Banach spaces and the concept of the $\{T, S\}$-splitting is introduced. The index splitting is a special cases of the $\{T, S\}$-splitting. The introduced splittings are used in the representation of generalized inverses as well as in the construction of iterative processes for solving singular linear systems.

The paper is organized as follows. In Section 2 we introduce the proper splitting for g-invertible operators on Banach spaces. In Section 3 we introduce the $\{T, S\}$-splitting of bounded operators on Banach spaces.

2. Proper splitting on Banach spaces

Let X and Y denote arbitrary Banach spaces and let $\mathcal{L}(X, Y)$ denote the set of all bounded linear operators from X into Y. For $A \in \mathcal{L}(X, Y)$ we use $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, to denote the range and the kernel of A. If $A \in \mathcal{L}(X)$, then $\rho(A)$ denotes the spectral radius of A.

Recall that $A \in \mathcal{L}(X, Y)$ is called g-invertible, if there exists an operator $B \in \mathcal{L}(Y, X)$, satisfying $ABA = A$. In this case B is called an *inner generalized inverse* of A, and will be denoted by $A^{(1)}$. It is well-known that $A \in \mathcal{L}(X, Y)$ is *g-invertible* if and only if $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, are closed and complemented subspaces of Y and X (see [5] and [8]). If $A \in \mathcal{L}(X, Y)$ is g-invertible, then let T be a closed subspace of X satisfying $T \oplus \mathcal{N}(A) = X$. Also, let S be a closed subspace of Y satisfying $\mathcal{R}(A) \oplus S = Y$. Then A has the following matrix form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix},$$

where $A_1 = A|_T : T \to \mathcal{R}(A)$ is invertible. Now, any inner generalized inverse of A can be defined as

$$A^{(1)}_{T,S,M} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & M \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix} \to \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix},$$

where $M : S \to \mathcal{N}(A)$ is an arbitrary bounded operator. If $M = 0$, then $A^{(1)}_{T,S,0} = A^{(1,2)}_{T,S}$ becomes the unique reflexive generalized inverse of A (an inner and outer generalized inverse of A) associated with the corresponding subspaces T and S. Notice that if we have $A^{(1)}$, then we can define $T = \mathcal{R}(A^{(1)}A)$, $S = \mathcal{R}(I - AA^{(1)})$ and $M = A^{(1)}|_S : S \to \mathcal{N}(A)$, i.e.

$$\begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} = (I - A^{(1)}A)A^{(1)}(I - AA^{(1)}) : \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix} \to \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix}.$$

Hence, the unique correspondence between $A^{(1)}$ and T, S, M is established.

The existence of $A^{(1)}_{T,S,M}$ enables us to define the following generalization of the condition number:

$$\kappa_{T,S,M}(A) = \|A\|\|A^{(1)}_{T,S,M}\|.$$

The proper splitting for g-invertible operators on Banach spaces is introduced by the following definition.

Definition 2.1. Let $A \in \mathcal{L}(X, Y)$ be a g-invertible operator and let $U, V \in \mathcal{L}(X, Y)$. Then the splitting $A = U - V$ is called a *proper splitting* of A, if $\mathcal{R}(A) = \mathcal{R}(U)$ and $\mathcal{N}(A) = \mathcal{N}(U)$.

Notice that the original definition of the proper splitting leads to the Moore-Penrose inverse of a complex matrix. Since we consider operators on Banach spaces, the usual concepts of the scalar product, orthogonality and the Moore-Penrose inverse are not available. Hence, we use inner inverses of a given operator.

Theorem 2.1. *Let $A \in \mathcal{L}(X, Y)$ be g-invertible and let $A = U - V$ be a proper splitting of A.*

(a) *If $A^{(1)} = A^{(1)}_{T,S,M}$ is an inner generalized inverse of A for some $M : S \to \mathcal{N}(A)$, then any inner generalized inverse of U has the form*

$U^{(1)}_{T,S,N}$ for some $N : S \to \mathcal{N}(A)$. In particular, there exists the inner generalized inverse $U^{(1)}_{T,S,M}$ of U.

(b) *$A^{(1)}_{T,S,K} - U^{(1)}_{T,S,K} = U^{(1)}_{T,S,N} V A^{(1)}_{T,S,M} = A^{(1)}_{T,S,M} V U^{(1)}_{T,S,N}$ for arbitrary $K, M, N : S \to \mathcal{N}(A)$.*

(c) *$A^{(1)}_{T,S,M} = (I - U^{(1)}_{T,S,M} V)^{-1} U^{(1)}_{T,S,M} = U^{(1)}_{T,S,M}(I - V U^{(1)}_{T,S,M})^{-1}$ for arbitrary $M : S \to \mathcal{N}(A)$.*

(d) *$U^{(1)}_{T,S,M} = (I + A^{(1)}_{T,S,M} V)^{-1} A^{(1)}_{T,S,M} = A^{(1)}_{T,S,M}(I + V A^{(1)}_{T,S,M})^{-1}$ for arbitrary $M : S \to \mathcal{N}(A)$.*

(e) *If $x \in \mathcal{R}(U^{(1)}_{T,S,M})$ for some $M : S \to \mathcal{N}(A)$, then $x_0 = A^{(1)}_{T,S,M} b$ is the unique solution of the equation $x = U^{(1)}_{T,S,M} V x + U^{(1)}_{T,S,M} b$ in the subspace $\mathcal{R}(A^{(1)}_{T,S,M})$.*

(f) *The iteration $x_{i+1} = U^{(1)}_{T,S,M} V x_i + U^{(1)}_{T,S,M} b$ converges to $A^{(1)}_{T,S,M} b$ for every $x_0 \in X$ and arbitrary $M : S \to \mathcal{N}(A)$, if and only if $\rho(U^{(1)}_{T,S,M} V) < 1$.*

(g) *If $\|A^{(1)}_{T,S,M} V\| < 1$ for some $M : S \to \mathcal{N}(A)$, then*

$$\begin{aligned}\|U^{(1)}_{T,S,M} - A^{(1)}_{T,S,M}\| &\le \frac{\|A^{(1)}_{T,S,M} V\| \|A^{(1)}_{T,S,M}\|}{1 - \|A^{(1)}_{T,S,M} V\|} \\ &\le \kappa_{T,S,M}(A) \frac{\|A^{(1)}_{T,S,M} V\|}{\|A\|(1 - \|A^{(1)}_{T,S,M} V\|)}.\end{aligned}$$

Proof. Let us take $T = \mathcal{R}(A^{(1)}A)$ and $S = \mathcal{R}(I - AA^{(1)})$. Then, with respect to the previous consideration, we have

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A^{(1)} = A^{(1)}_{T,S,M} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & M \end{bmatrix},$$

where $M \in \mathcal{L}(S, \mathcal{N}(A))$ is arbitrary.

(a) Since $\mathcal{N}(U) = \mathcal{N}(A)$ and $\mathcal{R}(U) = \mathcal{R}(A)$, we conclude that U must have the following form

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix},$$

where $U_1 = U|_T : T \to \mathcal{R}(A)$ is invertible. Hence, an arbitrary inner generalized inverse of U has the form

$$U^{(1)} = U^{(1)}_{T,S,N} = \begin{bmatrix} U_1^{-1} & 0 \\ 0 & N \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix} \to \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix},$$

where $N \in \mathcal{L}(S, \mathcal{N}(A))$ is arbitrary. In particular, $U^{(1)}_{T,S,M}$ exists.

(b) In this case V has the form $V = \begin{bmatrix} U_1 - A_1 & 0 \\ 0 & 0 \end{bmatrix}$ and hence

$$\begin{aligned} U^{(1)}_{T,S,N} V A^{(1)}_{T,S,M} &= \begin{bmatrix} U_1^{-1} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} U_1 - A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & M \end{bmatrix} \\ &= \begin{bmatrix} A_1^{-1} - U_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = A^{(1)}_{T,S,K} - U^{(1)}_{T,S,K}. \end{aligned}$$

The second equality can be verified in the same way.

(c) Since the operator

$$I - U^{(1)}_{T,S,M} V = \begin{bmatrix} U_1^{-1} A_1 & 0 \\ 0 & I \end{bmatrix}$$

is invertible, we have

$$(I - U^{(1)}_{T,S,M} V)^{-1} U^{(1)}_{T,S,M} = \begin{bmatrix} A_1^{-1} U_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} U_1^{-1} & 0 \\ 0 & M \end{bmatrix} = A^{(1)}_{T,S,M}.$$

The second equality can be obtained in the same manner.

(d) This part follows from the part (c), since $U = A - (-V)$ is the proper splitting of U.

(e) To prove this part, it is enough to notice that $\mathcal{R}(A^{(1)}_{T,S,M}) = \mathcal{R}(U^{(1)}_{T,S,M})$ and use part (c).

(f) This part follows immediately from the part (e), knowing that for any $B \in \mathcal{L}(X)$ the following equivalence holds: $B^n \to 0$ if and only if $\rho(B) < 1$.

(g) Since $\|A^{(1)}_{T,S,M} V\| < 1$, from part (d) we get

$$\begin{aligned} U^{(1)}_{T,S,M} - A^{(1)}_{T,S,M} &= (I + A^{(1)}_{T,S,M} V)^{-1} A^{(1)}_{T,S,M} - A^{(1)}_{T,S,M} \\ &= \left(\sum_{k=0}^{\infty} (-1)^k (A^{(1)}_{T,S,M} V)^k - I \right) A^{(1)}_{T,S,M} \\ &= \sum_{k=1}^{\infty} (-1)^k (A^{(1)}_{T,S,M} V)^k A^{(1)}_{T,S,M}. \end{aligned}$$

Hence

$$\|U^{(1)}_{T,S,M} - A^{(1)}_{T,S,M}\| \le \frac{\|A^{(1)}_{T,S,M}V\|\|A^{(1)}_{T,S,M}\|}{1-\|A^{(1)}_{T,S,M}V\|}$$

$$\le \kappa_{T,S,M}(A)\frac{\|A^{(1)}_{T,S,M}V\|}{\|A\|(1-\|A^{(1)}_{T,S,M}V\|)}. \quad \square$$

The operator $W = U^{(1)}_{T,S,M}V$ is called the *iteration operator corresponding to a proper splitting* $A = U - V$. Notice that V has the matrix form $V = \begin{bmatrix} U_1 - A_1 & 0 \\ 0 & 0 \end{bmatrix}$. Hence,

$$W = U^{(1)}_{T,S,M}V = \begin{bmatrix} I - U_1^{-1}A_1 & 0 \\ 0 & 0 \end{bmatrix},$$

$\mathcal{R}(W) \subset T$ and $\mathcal{N}(W) \supset \mathcal{N}(A)$. Also, $I - W$ is invertible. In the next theorem we show how we can reconstruct a proper splitting of a given operator, if the iteration operator is already known.

Theorem 2.2. *Let $A \in \mathcal{L}(X,Y)$ be g-invertible, let T be a closed subspace of X satisfying $X = T \oplus \mathcal{N}(A)$, and let $W \in \mathcal{L}(X)$ be an operator, such that $I - W$ is invertible, $\mathcal{R}(W) \subset T$ and $\mathcal{N}(W) \supset \mathcal{N}(A)$. Then there exists the unique proper splitting $A = U - V$ of A, such that W is the corresponding iteration operator, i.e. $W = U^{(1)}_{T,S,M}V$ for any $M \in \mathcal{L}(S, \mathcal{N}(A))$. Moreover, this splitting can be reconstructed as $U = A(I-W)^{-1}$ and $V = U - A$.*

Proof. From $\mathcal{R}(W) \subset T$ and $\mathcal{N}(W) \supset \mathcal{N}(A)$ we conclude that W has the matrix form

$$W = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix}.$$

Since $I - W = \begin{bmatrix} I - W_1 & 0 \\ 0 & I \end{bmatrix}$ is invertible, we get that $I - W_1$ is invertible. First we show that the proper splitting can be reconstructed. Let us take

$$U = A(I-W)^{-1} = \begin{bmatrix} A_1(I-W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix}.$$

Then $U^{(1)}_{T,S,M}$ has the form

$$U^{(1)}_{T,S,M} = \begin{bmatrix} (I - W_1)A_1^{-1} & 0 \\ 0 & M \end{bmatrix}$$

for an arbitrary $M \in \mathcal{L}(S, \mathcal{N}(A))$. Taking $V = U - A$ we easily verify that $W = U^{(1)}_{T,S,M}V$ is the corresponding iteration operator.

To prove the uniqueness, suppose that $A = K - L$ is another proper splitting of A such that $W = K^{(1)}_{T,S,M}L$ for any $M \in \mathcal{L}(S, \mathcal{N}(A))$. Then K has the matrix form

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix},$$

where K_1 is invertible, and $L = \left[\begin{smallmatrix} K_1 - A_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. From $W = K^{(1)}_{T,S,M}L = U^{(1)}_{T,S,M}V$ we get $I - K_1^{-1}A_1 = I - U_1^{-1}A_1$, hence $U_1 = K_1$, $U = K$ and $V = L$. Thus, the uniqueness is proved. $\square$

If the iteration operator W is given and a proper splitting $A = U - V$ is constructed such that $W = U^{(1)}_{T,S,M}V$, then $A = U - V$ is called a *proper splitting induced by the iteration operator* W. The previous result is a generalization of the corresponding result from [11], stated for complex matrices. This result is used in [4] for constructing iterative methods for solving certain systems of equations.

3. $\{\mathbf{T}, \mathbf{S}\}$-splitting on Banach spaces

In this section we introduce the $\{T, S\}$-splitting for operators on Banach spaces, which is induced a particular outer generalized inverse. As a special case of the $\{T, S\}$-splitting we get the index splitting.

Basic references for the following results are [5] and [8]. If $A \in \mathcal{L}(X, Y)$ is a non-zero operator, then there always exists an non-zero operator $B \in \mathcal{L}(Y, X)$ satisfying $BAB = B$. This operator B is an *outer generalized inverse* of A. In this case let $T = \mathcal{R}(B)$ and $S = \mathcal{N}(B)$. Then B is denoted by $A^{(2)}_{T,S}$. If T and S, respectively, are given subspaces of X and Y, then for $A \in \mathcal{L}(X,Y)$ there exists $A^{(2)}_{T,S} \in \mathcal{L}(Y,X)$ if and only if the following is satisfied: T, $A(T)$ and S, respectively, are closed and complemented subspaces of X, Y and Y, the restriction $A|_T : T \to A(T)$ is invertible and $A(T) \oplus S = Y$. In this case $A^{(2)}_{T,S}$ is the unique outer generalized inverse of A satisfying $\mathcal{R}(A) = T$ and $\mathcal{N}(A) = S$. Also, the matrix form of A is given by

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \to \begin{bmatrix} A(T) \\ S \end{bmatrix}.$$

Here T_1 is an arbitrary closed subspace of X complementary to T and $A_1 = A|_T : T \to A(T)$ is invertible. Then $A^{(2)}_{T,S}$ has the matrix form

$$A^{(2)}_{T,S} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \to \begin{bmatrix} T \\ T_1 \end{bmatrix}.$$

If T can be chosen such that $T \oplus \mathcal{N}(A) = X$, then $A^{(2)}_{T,S} = A^{(1,2)}_{T,S}$ is the unique reflexive generalized inverse of A corresponding to subspaces T and S. The applications of the generalized inverse $A^{(2)}_{T,S}$ can be found in [1], [6], [11].

The existence of $A^{(2)}_{T,S}$ enables us to define the following generalization of the condition number:

$$\kappa_{T,S}(A) = \|A\| \|A^{(2)}_{T,S}\|.$$

We associate a $\{T,S\}$-splitting to an outer generalized inverse of a given operator.

Definition 3.1. Let $A \in \mathcal{L}(X,Y)$ and T, S be subspaces of X and Y, such that there exists the generalized inverse $A^{(2)}_{T,S}$. Then $A = U - V$ is called a $\{T,S\}$-*splitting* of A if $U^{(2)}_{T,S}$ exists.

In the proof of the main result of this section, we shall use the following well-known statement (see, for example [1, Exercise 23, p. 55]).

Lemma 3.1. *If $B \in \mathcal{L}(X)$, L and M are closed subspaces of X such that $X = L \oplus M$ and $P_{L,M}$ is the projection from X onto L parallel to L, then*

(a) $P_{L,M}B = B$ *if and only if* $\mathcal{R}(B) \subseteq L$;

(b) $BP_{L,M} = B$ *if and only if* $\mathcal{N}(B) \supseteq M$.

Theorem 3.1. *Let $A \in \mathcal{L}(X,Y)$ be given, and closed subspaces T and S, respectively, such that there exists the generalized inverse $A^{(2)}_{T,S}$. If $A = U - V$ is a $\{T,S\}$-splitting of A, then the following results hold:*

(a) $A^{(2)}_{T,S} - U^{(2)}_{T,S} = U^{(2)}_{T,S}VA^{(2)}_{T,S} = A^{(2)}_{T,S}VU^{(2)}_{T,S}$.

(b) $A^{(2)}_{T,S} = (I - U^{(2)}_{T,S}V)^{-1}U^{(2)}_{T,S} = U^{(2)}_{T,S}(I - VU^{(2)}_{T,S})^{-1}$.

(c) $U^{(2)}_{T,S} = (I + A^{(2)}_{T,S}V)^{-1}A^{(2)}_{T,S} = A^{(2)}_{T,S}(I + VA^{(2)}_{T,S})^{-1}$.

(d) *If $x \in T$, then $x_0 = A^{(2)}_{T,S}b$ is the unique solution of the equation $x = U^{(2)}_{T,S}Vx + U^{(2)}_{T,S}b$ in the subspace T.*

(e) *The iteration $x_{i+1} = U^{(2)}_{T,S}Vx_i + U^{(2)}_{T,S}b$ converges to $A^{(2)}_{T,S}b$ for every $x_0 \in X$ if and only if $\rho(U^{(2)}_{T,S}V) < 1$.*

(f) *If $\|A^{(2)}_{T,S}V\| < 1$, then*

$$\|U^{(2)}_{T,S} - A^{(2)}_{T,S}\| \leq \frac{\|A^{(2)}_{T,S}V\|\|A^{(2)}_{T,S}\|}{1 - \|A^{(2)}_{T,S}V\|} \leq \kappa_{T,S}(A)\frac{\|A^{(2)}_{T,S}V\|}{\|A\|(1 - \|A^{(2)}_{T,S}V\|)}.$$

Proof. (a) By Definition 3.1 the generalized inverse $U^{(2)}_{T,S}$ exists. From the matrix form of A (and, similarly, the matrix form of U), we can verify that $A^{(2)}_{T,S}A$ and $U^{(2)}_{T,S}U$ are projections from X onto T, but $AA^{(2)}_{T,S} = P_{A(T),S}$ and $UU^{(2)}_{T,S} = P_{U(T),S}$. Using Lemma 3.1 we compute

$$U^{(2)}_{T,S}VA = U^{(2)}_{T,S}(U - A)A^{(2)}_{T,S} = U^{(2)}_{T,S}UA^{(2)}_{T,S} - U^{(2)}_{T,S}AA^{(2)}_{T,S} = A^{(2)}_{T,S} - U^{(2)}_{T,S}.$$

The second equality of (a) can be proved in the same way.

(b) From (a) we have $A_{T,S}^{(2)}(I - VU_{T,S}^{(2)}) = U_{T,S}^{(2)}$. Notice that

$$I - VU_{T,S}^{(2)} = I - UU_{T,S}^{(2)} + AU_{T,S}^{(2)}.$$

We see that $I - UU_{T,S}^{(2)}$ is the projection from Y onto S parallel to $U(T)$, $AU_{T,S}^{(2)}$ is an invertible operator from $U(T)$ to $A(T)$ and $\mathcal{N}(AU_{T,S}^{(2)}) = S$. Since $Y = U(T) \oplus S = A(T) \oplus S$, we easily prove that $I - UU_{T,S}^{(2)} + AU_{T,S}^{(2)}$ is invertible. Hence, it follows $A_{T,S}^{(2)} = U_{T,S}^{(2)}(I - VU_{T,S}^{(2)})^{-1}$. The first equality in (b) can be proved similarly.

(c) Since $U = A - (-V)$ is a $\{T, S\}$-splitting of U, the statement from (c) follows from (b).

Part (d) follows from the part (b), and the part (e) follows from part (d). Finally, the part (f) can be proved in the same way as part (g) of Theorem 2.1. $\square$

If $A = U - V$ is a $\{T, S\}$-splitting of $A \in \mathcal{L}(X, Y)$, then the operator $W = U_{T,S}^{(2)} V$ is called the corresponding *iteration operator*. This operator obviously satisfies $\mathcal{R}(W) \subset T$ and $I - W$ is invertible. In the next theorem we show how to reconstruct a $\{T, S\}$-splitting if the corresponding iteration operator is given.

Theorem 3.2. *Let $A \in \mathcal{L}(X, Y)$, T and S be given such that there exists the generalized inverse $A_{T,S}^{(2)}$. If $W \in \mathcal{L}(X)$ satisfies $\mathcal{R}(W) \subset T$ and $I - W$ is invertible, then the operators $U = A(I - W)^{-1}$ and $V = U - A$ induce a $\{T, S\}$-splitting $A = U - V$ of A and $W = U_{T,S}^{(2)} V$ is the corresponding iteration operator.*

Proof. Taking $T_1 = \mathcal{N}(A_{T,S}^{(2)} A)$ we obtain $A_{T,S}^{(2)} A = P_{T,T_1}$. Using the matrix forms presented above we get

$$A_{T,S}^{(2)} A = \begin{bmatrix} I & A_1^{-1} A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \to \begin{bmatrix} T \\ T_1 \end{bmatrix}.$$

Hence $A_2 = 0$ and $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_3 \end{bmatrix}$. Since $\mathcal{R}(W) \subset T$ we conclude that W has the matrix form

$$W = \begin{bmatrix} W_1 & W_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ T_1 \end{bmatrix} \to \begin{bmatrix} T \\ T_1 \end{bmatrix}.$$

The operator

$$I - W = \begin{bmatrix} I - W_1 & -W_2 \\ 0 & I \end{bmatrix}$$

is invertible, so $I - W_1$ must be invertible,

$$(I - W)^{-1} = \begin{bmatrix} (I - W_1)^{-1} & (I - W_1)^{-1} W_2 \\ 0 & I \end{bmatrix}$$

and

$$U = A(I - W)^{-1} = \begin{bmatrix} A_1(I - W_1)^{-1} & A_1(I - W_1)^{-1} W_2 \\ 0 & A_3 \end{bmatrix}.$$

Since the restriction $U|_T = A(I - W_1)^{-1} : T \to U(T) = A(T)$ is invertible and $U(T) \oplus S = Y$, we conclude that there exists the unique $U^{(2)}_{T,S}$ inverse given as

$$U^{(2)}_{T,S} = \begin{bmatrix} (I - W_1)A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \to \begin{bmatrix} T \\ T_1 \end{bmatrix}.$$

Now we take

$$V = U - A = \begin{bmatrix} A_1((I - W_1)^{-1} - I) & A_1(I - W_1)^{-1} W_2 \\ 0 & -A_3 \end{bmatrix}$$

and a simple matrix calculation shows that $W = U^{(2)}_{T,S} V$ holds. $\square$

If the iteration operator W is given and a $\{T, S\}$-splitting $A = U - V$ is constructed such that $W = U^{(2)}_{T,S} V$, then $A = U - V$ is called a $\{T, S\}$-*splitting induced by the iteration operator* W.

The previous result is also a generalization of the corresponding result from [11].

4. Particular cases

As corollaries, many representations and properties of the well-known generalized inverses can be obtained. In [7] the generalized Drazin inverse is introduced in the following way. Let $\operatorname{acc} G$ denote the set of all accumulation point of the set G, $G \subset \mathbb{C}$. Let $A \in \mathcal{L}(X)$, $0 \notin \operatorname{acc} \sigma(A)$ and $P = P(A, 0)$ be the spectral idempotent of A corresponding to $\{0\}$. In this case $X = \mathcal{N}(P) \oplus \mathcal{R}(P)$ and

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix},$$

where $A_1 = A|_{\mathcal{N}(P)} : \mathcal{N}(P) \to \mathcal{N}(P)$ is invertible and $A_2 : \mathcal{R}(P) \to \mathcal{R}(P)$ is quasinilpotent. In this case the generalized Drazin inverse of A has the form

$$A^d = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix}.$$

Obviously the generalized Drazin inverse can be chosen as a particular outer generalized inverse with prescribed range and kernel: $A^d = A^{(2)}_{\mathcal{N}(P),\mathcal{R}(P)}$.

If A_2 is nilpotent, i.e. there exists the least non-negative integer k such that $A_2^k = 0$, then $\operatorname{ind}(A) = k$ and A^d reduces to the well-known Drazin inverse of A, usually denoted by A^D. If $\operatorname{ind}(A) = 0$, then A is invertible and $A^D = A^{-1}$. If $\operatorname{ind}(A) = 1$, then $A^D = A^\#$ is well-known as the group inverse of A. Notice that $A^\#$ is also an inner generalized inverse of A.

One special case of $\{T, S\}$ splittings can be used for the representation of the generalized Drazin inverse. The following result is a generalization of the well-known representation for the Drazin inverse of a complex square matrix [10].

Corollary 4.1. *Let $A \in \mathcal{L}(X)$, $0 \notin \operatorname{acc} \sigma(A)$ and P be the spectral projection of A corresponding to $\{0\}$. If $U \in \mathcal{L}(X)$ satisfies $\mathcal{R}(U) = \mathcal{N}(P)$ and $\mathcal{N}(U) = \mathcal{R}(P)$, then $A = U - V$ is a $\{\mathcal{N}(P), \mathcal{R}(P)\}$-splitting of A and*

$$A^d = (I - U^\# V)^{-1} U^\#.$$

Proof. From $\mathcal{R}(U) = \mathcal{N}(P)$ and $\mathcal{N}(U) = \mathcal{R}(P)$, we get $\operatorname{ind}(U) \leq 1$ and $U^{(2)}_{\mathcal{N}(P),\mathcal{R}(P)} = U^{\#}$. The rest of the proof follows from Theorem 3.1. $\square$

Now, suppose that X and Y are Hilbert spaces. If $A \in \mathcal{L}(X,Y)$ has closed range, then $A^\dagger$ denotes the Moore-Penrose inverse of A. The Moore-Penrose inverse of A is the unique operator $A^\dagger \in \mathcal{L}(Y,X)$ satisfying $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$ and $(A^\dagger A)^* = A^\dagger A$. The Moore-Penrose inverse can also be obtained as an outer inverse with prescribed range and kernel. Let A^* denote the conjugate operator of A. If $\mathcal{R}(A)$ is closed, then $A^\dagger = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}$. Thus the connection with Section 3 is established. Notice that the same result can be obtained from Section 1: $A^\dagger = A^{(1)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}$. As a corollary we get the following result.

Corollary 4.2. *Suppose that X, Y are Hilbert spaces and $A \in \mathcal{L}(X,Y)$ has closed range. If $A = U - V$ is the proper splitting of A, then*

$$A^\dagger = (I - U^\dagger V)^{-1} U^\dagger.$$

Proof. Obviously, $U^\dagger$ exists and $U^\dagger = U^{(1)}_{\mathcal{R}(A^*),\mathcal{N}(A^*),0} = U^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}$. Now the result can be deduced from Theorem 2.1 or Theorem 3.1. $\square$

References

[1] Ben-Israel, A. and Greville, T.N.E., *Generalized Inverses: Theory and Applications*, Wiley-Interscience, New York, 1974.

[2] Berman, A. and Neumann, M., *Proper splittings of rectangular matrices*, SIAM J. Appl. Math. **31** (1976), 307–312.

[3] Berman, A. and Plemmons, R. J., *Cones and iterative methods for best least squares solutions of linear systems*, SIAM J. Numer. Anal. **11** (1974), 145–154.

[4] Zhi-Hao Cao, He-Bing Wu and Zhong-Yun Liu, *Two stage iterative methods for consistent Hermitian positive semidefinite systems*, Linear Algebra Appl. **294** (1999), 217–238.

[5] Caradus, S.R., *Generalized Inverses and Operator Theory*, Queen's paper in pure and applied mathematics, Queen's University, Kingston, Ontario, 1978.

[6] Chen, Y., *Finite algorithms for* (2)*-generalized inverse* $A^{(2)}_{T,S}$, Linear and Multilinear Algebra **40** (1995), 61–68.

[7] Koliha, J.J., *A generalized Drazin inverse*, Glasgow Math. J. **38** (1996), 367–381.

[8] Nashed, M. Z., *Inner, outer and generalized inverses in Banach and Hilbert spaces*, Numer. Funct. Anal. Optim. **9** (1987), 261–325.

[9] Varga, R.S., *Matrix Iterative analysis*, Prentice Hall, Englewood Cliffs, N.J., 1962.

[10] Wei, Y., *Index splitting for the Drazin inverse and the singular linear system*, Appl. Math. Comput. **95** (1998), 115–124.

[11] Wei, Y., *A characterization and representation of the generalized inverse* $A^{(2)}_{T,S}$ *and its applications*, Linear Algebra Appl. **280** (1998), 87–96.

[12] Woźnicki, Z.I., *Nonnegative splitting theory*, Japan J. Industr. Appl. Math. **11** (1994), 289–342.

[13] Woźnicki, Z.I., *Matrix splitting principles*, Novi Sad J. Math. **28** (1998), 197–209.

Department of Mathematics, Faculty of Science, University of Niš
Ćirila i Metodija 2, P.O. Box 224, 18000 Niš, Yugoslavia

E-mail: `dragan@pmf.pmf.ni.ac.yu` `ganedj@EUnet.yu`
`pecko@pmf.pmf.ni.ac.yu`