Additive results for the Wg-Drazin inverse

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Abstract

In this paper we prove the formula for the expression $(A+B)^{d,W}$ in terms of $A, B, W, A^{d,W}, B^{d,W}$, assuming some conditions for A, B and W. Here $S^{d,W}$ denotes the generalized W-weighted Drazin inverse of a linear bounded operator S on a Banach space.

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1 Introduction

Let X and Y denote arbitrary Banach spaces. We use $\mathcal{B}(X,Y)$ to denote the set of all linear bounded operators from X to Y. Set $\mathcal{B}(X) = \mathcal{B}(X,X)$. Let $A \in \mathcal{B}(X,Y)$ and $W \in \mathcal{B}(Y,X)$ be nonzero operators. If there exists some $S \in \mathcal{B}(X,Y)$ satisfying

$$(AW)^{k+1}SW = (AW)^k, \ SWAWS = S, \ AWS = SWA,$$

for some nonnegative integer k, then S is called the W-weighted Drazin inverse of A and denoted by $S = A^{D,W}$ [12], [13], [15]. If there exists $A^{D,W}$, then we say that A is W-Drazin invertible and $A^{D,W}$ must be unique [12]. If $X = Y$, $A \in \mathcal{B}(X)$ and $W = I$, then $S = A^D$, the ordinary Drazin inverse of A. Further related results can also be found in [3, 4, 7, 11, 14, 16, 17].

Let $\mathcal{B}_W(X,Y)$ be the space $\mathcal{B}(X,Y)$ equipped with the multiplication $A * B = AWB$ and the norm $\|A\|_W = \|A\|\|W\|$. Then $\mathcal{B}_W(X,Y)$ becomes a Banach algebra [6]. $\mathcal{B}_W(X,Y)$ has the unit if and only if W is invertible, in which case W^{-1} is that unit.

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Let $\mathcal{A}$ be a Banach algebra. Then $a \in \mathcal{A}$ is quasipolar if and only if there exists $b \in \mathcal{A}$ such that

$$ab = ba, \quad bab = b, \quad a - aba \text{ is quasinilpotent.}$$

The element b, if exists, is unique [9] (Theorem 7.5.3), [10]. Such b is the generalized Drazin inverse, or Koliha-Drazin inverse of a, and it is denoted by a^d.

Let $W \in \mathcal{B}(Y, X)$ be a fixed nonzero operator. An operator $A \in \mathcal{B}(X, Y)$ is called Wg–Drazin invertible if A is quasipolar in the Banach algebra $\mathcal{B}_W(X, Y)$. The Wg–Drazin inverse $A^{d,W}$ of A is defined as the g–Drazin inverse of A in the Banach algebra $\mathcal{B}_W(X, Y)$ [6].

Let us recall that if $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X)$ then the following conditions are equivalent [6]:

(1) A is Wg-Drazin invertible,

(2) AW is quasipolar in $\mathcal{B}(Y)$ with $(AW)^d = A^{d,W}W$,

(3) WA is quasipolar in $\mathcal{B}(X)$ with $(WA)^d = WA^{d,W}$.

Then, the Wg-Drazin inverse $A^{d,W}$ of A satisfies

$$A^{d,W} = ((AW)^d)^2 A = A((WA)^d)^2.$$

Lemma 1.1 [6] *Let $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X)\backslash\{0\}$. Then A is Wg-Drazin invertible if and only if there exist topological direct sums $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ such that*

$$A = A_1 \oplus A_2, \qquad W = W_1 \oplus W_2,$$

where $A_i \in \mathcal{B}(X_i, Y_i)$, $W_i \in \mathcal{B}(Y_i, X_i)$, with A_1, W_1 invertibe, and W_2A_2 and A_2W_2 quasinilpotent in $\mathcal{B}(X_2)$ and $\mathcal{B}(Y_2)$, respectively. The Wg-Drazin inverse of A is given by

$$A^{d,W} = (W_1A_1W_1)^{-1} \oplus 0$$

with $(W_1A_1W_1)^{-1} \in \mathcal{B}(X_1, Y_1)$ and $0 \in \mathcal{B}(X_2, Y_2)$.

Recall that if A^D and B^D exist, it is possible that $(A+B)^D$ does not exist. Moreover, if $(A + B)^D$ exists, then we do not always know how to calculate $(A + B)^D$ in terms of A, B, A^D, B^D. In this paper we investigate some special cases of this phenomenon. In [5] Hartwig, Wang and Wei obtained a formula for the Drazin inverse of a sum of two matrices, when one of the products of these matrices vanishes. Djordjević and Wei generalized their results to bounded linear operators on Banach spaces [8]. In [1], Castro Gonzalez extended these additive Drazin inverse results to complex matrices using weaker conditions. Finally, Castro-Gonzalez and Koliha extended the results for the generalized Drazin inverse of Banach algebra elements [2]. In this paper we extend previous results to linear bounbed operators on Banach spaces, and give a formula for computing the Wg-Drazin inverse of a sum of two operators.

We state one lemma concerning g–Drazin inverse of a partitioned matrix that will be needed later (see Djordjević and Wei [8]).

Lemma 1.2 *If $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ are g–Drazin invertible, $C \in \mathcal{B}(Y, X)$ and $D \in \mathcal{B}(X, Y)$, then*

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} A & 0 \\ D & B \end{bmatrix}$$

are also g–Drazin invertible and

$$M^d = \begin{bmatrix} A^d & S \\ 0 & B^d \end{bmatrix}, \quad N^d = \begin{bmatrix} A^d & 0 \\ R & B^d \end{bmatrix},$$

where

$$\begin{aligned} S &= (A^d)^2 \left[\sum_{n=0}^{\infty} (A^d)^n C B^n \right] (I - BB^d) + \\ &+ (I - AA^d) \left[\sum_{n=0}^{\infty} A^n C (B^d)^n \right] (B^d)^2 - A^d C B^d \end{aligned}$$

and

$$\begin{aligned} R &= (B^d)^2 \left[\sum_{n=0}^{\infty} (B^d)^n D A^n \right] (I - AA^d) + \\ &+ (I - BB^d) \left[\sum_{n=0}^{\infty} B^n D (A^d)^n \right] (A^d)^2 - B^d D A^d. \end{aligned}$$

We also need the following important results from [8].

Lemma 1.3 *If $P, Q \in \mathcal{B}(X)$ are quasinilpotent and $PQ = 0$ or $PQ = QP$, then $P + Q$ is also quasinilpotent. Hence, $(P + Q)^d = 0$.*

Lemma 1.4 *If $P \in \mathcal{B}(X)$ is g–Drazin invertible, $Q \in \mathcal{B}(X)$ is quasinilpotent and $PQ = 0$, then $P + Q$ is g–Drazin invertible and*

$$(P + Q)^d = \sum_{i=0}^{\infty} Q^i (P^d)^{i+1}.$$

We also state the following useful result.

Lemma 1.5 *Let $\mathcal{A}$ be a complex Banach algebra with the unit 1, and let p be an idempotent of $\mathcal{A}$. If $x \in p\mathcal{A}p$, then $\sigma_{p\mathcal{A}p}(x) = \sigma_{\mathcal{A}}(x)$, where $\sigma_{\mathcal{A}}(x)$ denotes the spectrum of x in the algebra $\mathcal{A}$, and $\sigma_{p\mathcal{A}p}(x)$ denotes the spectrum of x in the algebra $p\mathcal{A}p$.*

2 Wg–Drazin inverse of a sum of two operators

First we state one particular case of our main result.

Theorem 2.1 *Let $W \in \mathcal{B}(Y, X)$, and let $B \in \mathcal{B}(X, Y)$ be Wg–Drazin invertible and $N \in \mathcal{B}(X, Y)$ such that $WN \in \mathcal{B}(X)$ is quasinilpotent. If $NWB^{d,W} = 0$ and $(I - WBWB^{d,W})WNWB = 0$, then*

$$(1) \qquad (WN + WB)^d = (WB)^d + ((WB)^d)^2 \left(\sum_{i=0}^{\infty} ((WB)^d)^i WNS(i) \right),$$

where

$$(2) \qquad \begin{aligned} S(i) &= (I - WBWB^{d,W})(WN + WB)^i \\ &= (I - WBWB^{d,W}) \left(\sum_{j=0}^{i} (WB)^{i-j} (WN)^j \right). \end{aligned}$$

Moreover, for all $i \geq l \geq 1$, we have

$$S(i) = (WB)^{i-l+1} S(l - 1) = S(l - 1)(WN)^{i-l+1}.$$

Proof. Since B is Wg–Drazin invertible, by Lemma 1.1, we conclude that B and W have the matrix forms

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix},$$

where B_1, W_1 are invertible, and W_2B_2 is quasinilpotent. From $NWB^{d,W} = 0$ it follows that N has the matrix form

$$N = \begin{bmatrix} 0 & N_1 \\ 0 & N_2 \end{bmatrix}.$$

Since $WN = \begin{bmatrix} 0 & W_1N_1 \\ 0 & W_2N_2 \end{bmatrix}$ is quasinilpotent, from Lemma 1.5 we conclude that W_2N_2 is quasinilpotent. From $(I - WBWB^{d,W})WNWB = 0$ it follows that $W_2N_2W_2B_2 = 0$. Thus, for any $i \geq 0$,

$$(W_2N_2 + W_2B_2)^i = \sum_{j=0}^{i} (W_2B_2)^{i-j}(W_2N_2)^j = \sum_{j=0}^{i} (W_2B_2)^j (W_2N_2)^{i-j}.$$

From Lemma 1.4, we see that $W_2N_2 + W_2B_2$ is quasinilpotent. Now, from Lemma 1.2, we get

$$\begin{aligned}
(WN + WB)^d &= \left(\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \begin{bmatrix} 0 & N_1 \\ 0 & N_2 \end{bmatrix} + \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \right)^d \\
&= \begin{bmatrix} W_1B_1 & W_1N_1 \\ 0 & W_2N_2 + W_2B_2 \end{bmatrix}^d = \begin{bmatrix} (W_1B_1)^{-1} & X \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

where

$$\begin{aligned}
X &= (W_1B_1)^{-2} \left[\sum_{i=0}^{\infty} (W_1B_1)^{-i} W_1N_1 (W_2N_2 + W_2B_2)^i \right] \\
&= (W_1B_1)^{-2} \left[\sum_{i=0}^{\infty} (W_1B_1)^{-i} W_1N_1 \left(\sum_{j=0}^{i} (W_2B_2)^{i-j} (W_2N_2)^j \right) \right].
\end{aligned}$$

Write $S(i) = (I - WBWB^{d,W}) \left(\sum_{j=0}^{i} (WB)^{i-j}(WN)^j \right)$, for all $i \geq 0$. Now, for all $i \geq 1$, we have

$$\begin{aligned}
S(i) &= \begin{bmatrix} 0 & 0 \\ 0 & (W_2B_2)^i \end{bmatrix} + \sum_{j=1}^{i} \begin{bmatrix} 0 & 0 \\ 0 & (W_2B_2)^{i-j} \end{bmatrix} \begin{bmatrix} 0 & W_1N_1(W_2N_2)^{j-1} \\ 0 & (W_2N_2)^j \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & \sum_{j=0}^{i} (W_2B_2)^{i-j}(W_2N_2)^j \end{bmatrix}.
\end{aligned}$$

Hence,

$$(WB)^d + ((WB)^d)^2 \left(\sum_{i=0}^{\infty} ((WB)^d)^i WNS(i) \right) =$$

$$= \begin{bmatrix} (W_1B_1)^{-1} & \sum_{i=0}^{\infty} (W_1B_1)^{-(i+2)} W_1N_1 \left(\sum_{j=0}^{i} (W_2B_2)^{i-j} (W_2N_2)^j \right) \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (W_1B_1)^{-1} & X \\ 0 & 0 \end{bmatrix} = (WN + WB)^d.$$

The second statement of the theorem are easily verified. □

As corollaries we obtain the following results.

Corollary 2.1 *Let $B, N \in \mathcal{B}(X, Y)$ satisfy conditions of Theorem* 2.1*. Then we have*

$$(WN + WB)^d (WN + WB) = (WB)^d WB + \left(\sum_{i=0}^{\infty} ((WB)^d)^{i+1} WNS(i) \right),$$

where $S(i)$ is defined in (2)*.*

Corollary 2.2 *Let $B, N \in \mathcal{B}(X, Y)$ satisfy conditions of Theorem* 2.1*.*

(i) *If $(WN)^2 = 0$, then*

$$\begin{aligned} (WN + WB)^d &= (WB)^d + ((WB)^d)^2 \left(\sum_{i=0}^{\infty} ((WB)^d)^i WN(WB)^i \right) \\ &+ ((WB)^d)^3 \left(\sum_{i=1}^{\infty} ((WB)^d)^i WN(WB)^i \right) WN. \end{aligned}$$

(ii) *If $WNWR = 0$, for all $R \in \mathcal{B}(X, Y)$, then*

$$\begin{aligned} &(WN + WB)^d WR \\ &= (WB)^d WR + ((WB)^d)^2 \left(\sum_{i=1}^{\infty} ((WB)^d)^i WN(WB)^i \right) WR. \end{aligned}$$

(iii) *If $(WB)^2 = WB$, then*

$$(WN + WB)^d = (I - WN)^{-1} WB.$$

Proof. Each of these cases follows directly from Theorem 2.1 and the following simplification.

Write $S(i) = (I - WBWB^{d,W}) \left(\sum_{j=0}^{i} (WB)^{i-j}(WN)^j \right)$, for all $i \geq 0$.

(i) Since $(WN)^2 = 0$, $WNS(i) = WN(WB)^i + WN(WB)^{i-1}WN$ for all $i \geq 1$.

(ii) Since $WNWR = 0$, $WNS(i)WR = WN(WB)^i WR$.

(iii) Since $(WB)^2 = WB$, $(WB)^d = WB$ and then the hypothesis $NWB^{d,W} = 0$ implies $NWB = N(WB)^d = NWB^{d,W} = 0$. Then from Lemma 1.4 it follows

$$\begin{aligned}
(WN + WB)^d &= \sum_{i=0}^{\infty} (WN)^i ((WB)^d)^{i+1} \\
&= \sum_{i=0}^{\infty} (WN)^i (WB)^{i+1} \\
&= \left(\sum_{i=0}^{\infty} (WN)^i \right) WB \\
&= (I - WN)^{-1} WB.
\end{aligned}$$

□

Now, we state and prove the main result.

Theorem 2.2 *Let $W \in \mathcal{B}(Y, X)$, and let $A, B \in \mathcal{B}(X, Y)$ be Wg–Drazin invertible. If $A^{d,W}WB = 0$, $AWB^{d,W} = 0$ and $(I - WBWB^{d,W})WAWB(I - WAWA^{d,W}) = 0$, then $A + B$ is Wg-Drazin invertible and*

$$\begin{aligned}
&(A + B)^{d,W} = \\
&= (A + B) \left[(WB)^d \left(I + \sum_{i=0}^{\infty} \left((WB)^d \right)^{i+1} WAZ(i) \right) (I - WAWA^{d,W}) \right]^2 \\
&+ (A + B)(I - WBWB^{d,W}) \left(I + \sum_{i=0}^{\infty} Z(i)WB \left((WA)^d \right)^{i+1} \right) \left((WA)^d \right)^2 \\
&- (A + B) \left((WB)^d \right)^2 \left(\sum_{i=0}^{\infty} \left((WB)^d \right)^i WAZ(i)WB \right) \left((WA)^d \right)^2
\end{aligned}$$

$$
\begin{aligned}
&- (A+B)(WB)^d \left(\sum_{i=0}^{\infty} WAZ(i)WB \left((WA)^d \right)^i \right) \left((WA)^d \right)^3 \\
&- (A+B) \left((WB)^d \right)^2 \times \\
&\times \left(\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \left((WB)^d \right)^i WAZ(i+k+1)WB \left((WA)^d \right)^k \right) \left((WA)^d \right)^3 \\
&- (A+B) \times \\
&\times \left[(WB)^d \left(I + \sum_{i=0}^{\infty} \left((WB)^d \right)^{i+1} WAZ(i) \right) (I - WAWA^{d,W}) \right]^2 \times \\
&\times WB(WA)^d,
\end{aligned}
$$

(3)

where

$$(4) \quad Z(i) = (I - WBWB^{d,W}) \left(\sum_{j=0}^{i} (WB)^{i-j} (WA)^j \right) (I - WAWA^{d,W}).$$

Moreover, for all $i \geq l \geq 1$, we have

$$Z(i) = (WB)^{i-l+1} Z(l-1) = Z(l-1)(WA)^{i-l+1}.$$

Proof. Since A is Wg–Drazin invertible, by Lemma 1.1, we conclude that A and W have the matrix forms

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix},$$

where A_1, W_1 are invertible and $W_2 A_2$ is quasinilpotent. From $A^{d,W} WB = 0$ it follows that B can be written as

$$B = \begin{bmatrix} 0 & 0 \\ B_1 & B_2 \end{bmatrix}.$$

We use Lemma 1.2 to compute $(WB)^d$ which in turn equals $WB^{d,W}$. From the assumptions $AWB^{d,W} = 0$ and $(I - WBWB^{d,W})WAWB(I - WAWA^{d,W}) = 0$, we get that $A_2 W_2 B_2^{d,W_2} = 0$ and $(I - W_2 B_2 W_2 B_2^{d,W_2})$ $W_2 A_2 W_2 B_2 = 0$. We see that the conditions of Theorem 2.1 are satisfied with: B_2, W_2, A_2, respectively, instead of B, W, N.

From Lemma 1.2 we have that

$$\begin{aligned}
&(A+B)^{d,W} = (A+B)((W(A+B))^d)^2 = (A+B)((WA+WB)^d)^2 \\
&= (A+B)\left(\begin{bmatrix} W_1A_1 & 0 \\ W_2B_1 & W_2A_2+W_2B_2 \end{bmatrix}^d\right)^2 \\
&= (A+B)\begin{bmatrix} (W_1A_1)^{-1} & 0 \\ X & (W_2A_2+W_2B_2)^d \end{bmatrix}^2 \\
&= \begin{bmatrix} A_1 & 0 \\ B_1 & A_2+B_2 \end{bmatrix} \times \\
&\times \begin{bmatrix} (W_1A_1)^{-2} & 0 \\ X(W_1A_1)^{-1} + (W_2A_2+W_2B_2)^d X & ((W_2A_2+W_2B_2)^d)^2 \end{bmatrix} \\
&= \begin{bmatrix} A_1(W_1A_1)^{-2} & 0 \\ X' & (A_2+B_2)((W_2A_2+W_2B_2)^d)^2 \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
X &= (I-(W_2A_2+W_2B_2)(W_2A_2+W_2B_2)^d) \times \\
&\times \left(\sum_{i=0}^{\infty}(W_2A_2+W_2B_2)^i W_2B_1(W_1A_1)^{-i}\right)(W_1A_1)^{-2} \\
&- (W_2A_2+W_2B_2)^d W_2B_1(W_1A_1)^{-1}
\end{aligned}$$

and

$$X' = B_1(W_1A_1)^{-2} + (A_2+B_2)[X(W_1A_1)^{-1} + (W_2A_2+W_2B_2)^d X].$$

Using Theorem 2.1 we get

$$(W_2A_2+W_2B_2)^d = (W_2B_2)^d + ((W_2B_2)^d)^2\left(\sum_{i=0}^{\infty}((W_2B_2)^d)^i W_2A_2 S(i)\right),$$

where $S(i) = (I - W_2B_2W_2B_2^{d,W_2})\left(\sum_{j=0}^{i}(W_2B_2)^j(W_2A_2)^{i-j}\right)$ for all $i \geq 0$.

Now, we have

$$\begin{aligned}
&I-(W_2A_2+W_2B_2)(W_2A_2+W_2B_2)^d \\
&= I - W_2B_2(W_2B_2)^d - (W_2B_2)^d\left(\sum_{i=0}^{\infty}((W_2B_2)^d)^i W_2A_2 S(i)\right).
\end{aligned}$$

Since

$$(W_2A_2 + W_2B_2)^d X = -\left((W_2A_2 + W_2B_2)^d\right)^2 W_2B_1(W_1A_1)^{-1},$$

we get

$$\begin{aligned}
X' &= B_1(W_1A_1)^{-2} + (A_2 + B_2)\Bigg[\Bigg(I - W_2B_2(W_2B_2)^d \\
&- (W_2B_2)^d \sum_{i=0}^{\infty}((W_2B_2)^d)^i W_2A_2S(i)\Bigg) \times \\
&\times \left(\sum_{i=0}^{\infty}(W_2A_2 + W_2B_2)^i W_2B_1(W_1A_1)^{-(i+3)}\right) \\
&- (W_2A_2 + W_2B_2)^d W_2B_1(W_1A_1)^{-2} \\
&- \left((W_2A_2 + W_2B_2)^d\right)^2 W_2B_1(W_1A_1)^{-1}\Bigg] \\
&= B_1\left((W_1A_1)^{-1}\right)^2 + X_1 + X_2 + X_3 + X_4,
\end{aligned}$$

where X_1, X_2, X_3 and X_4 are the following terms:

$$\begin{aligned}
X_1 &= (A_2 + B_2)(I - W_2B_2(W_2B_2)^d) \times \\
&\times \left(\sum_{i=0}^{\infty}(W_2A_2 + W_2B_2)^i W_2B_1(W_1A_1)^{-i}\right)(W_1A_1)^{-3} \\
&= (A_2 + B_2)(I - W_2B_2(W_2B_2)^d) \times \\
&\times \left(\sum_{i=0}^{\infty} S(i) W_2B_1(W_1A_1)^{-i}\right)(W_1A_1)^{-3}
\end{aligned}$$

and the last equality follows by using (2) in Theorem 2.1. Moreover,

$$\begin{aligned}
X_2 &= -(A_2 + B_2)(W_2B_2)^d \left(\sum_{i=0}^{\infty}\left((W_2B_2)^d\right)^i W_2A_2S(i)\right) \times \\
&\times \left(\sum_{i=0}^{\infty}(W_2A_2 + W_2B_2)^i W_2B_1(W_1A_1)^{-i}\right)(W_1A_1)^{-3} \\
&= -(A_2 + B_2)(W_2B_2)^d \left(\sum_{k=0}^{\infty} W_2A_2S(k)W_2B_1(W_1A_1)^{-(k+3)}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \quad (A_2+B_2)(W_2B_2)^d \times \\
&\times \quad \left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left((W_2B_2)^d\right)^{i+1} W_2A_2S(i+k+1)W_2B_1(W_1A_1)^{-(k+3)}\right)
\end{aligned}
$$

and the last equality follows by using (2) to obtain that $S(i)(W_2A_2 + W_2B_2)^k = (I - W_2B_2W_2B_2^{d,W})(W_2A_2 + W_2B_2)^{i+k} = S(i+k)$ and after we change i by $i-1$ in the last sum. Also

$$
\begin{aligned}
X_3 &= -(A_2+B_2)(W_2A_2+W_2B_2)^dW_2B_1(W_1A_1)^{-2} \\
&= -(A_2+B_2)(W_2B_2)^dW_2B_1(W_1A_1)^{-2} \\
&- \quad (A_2+B_2)\left((W_2B_2)^d\right)^2 \times \\
&\times \quad \left(\sum_{i=0}^{\infty}\left((W_2B_2)^d\right)^{i} W_2A_2S(i)W_2B_1\right)(W_1A_1)^{-2}.
\end{aligned}
$$

Finally,

$$
X_4 \quad = \quad -(A_2+B_2)\left((W_2A_2+W_2B_2)^d\right)^2 W_2B_1(W_1A_1)^{-1}.
$$

Write $Z(i) = (I - WBWB^{d,W})\left(\sum_{j=0}^{i}(WB)^{i-j}(WA)^j\right)(I - WAWA^{d,W})$. By direct computations, for all $i \geq 1$ we have,

$$
\begin{aligned}
Z(i) &= \begin{bmatrix} I & 0 \\ -(W_2B_2)^dW_2B_1 & I - W_2B_2(W_2B_2)^d \end{bmatrix} \times \\
&\times \left\{\sum_{j=0}^{i-1}\begin{bmatrix} 0 & 0 \\ (W_2B_2)^{i-j-1}W_2B_1 & (W_2B_2)^{i-j} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & (W_2A_2)^j \end{bmatrix}\right. \\
&+ \left.\begin{bmatrix} 0 & 0 \\ 0 & (W_2A_2)^i \end{bmatrix}\right\} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & (I - W_2B_2(W_2B_2)^d)\sum_{j=0}^{i}(W_2B_2)^{i-j}(W_2A_2)^j \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & S(i) \end{bmatrix},
\end{aligned}
$$

and

$$WAZ(i)WB\left((WA)^d\right)^q = \begin{bmatrix} 0 & 0 \\ W_2A_2S(i)W_2B_1(W_1A_1)^{-q} & 0 \end{bmatrix}, \quad \text{for all } q \geq 1.$$

Now, we compute the terms of the expressions (3) for $(A+B)^{d,W}$ using the block decomposition:

$$\begin{aligned}
\Sigma_1 &= (A+B)\left[(WB)^d\left(I+\sum_{i=0}^{\infty}\left((WB)^d\right)^{i+1}WAZ(i)\right)(I-WAWA^{d,W})\right]^2 \\
&= (A+B)\left\{\begin{bmatrix} 0 & 0 \\ 0 & (W_2B_2)^d \end{bmatrix}\right. \\
&+ \left.\sum_{i=0}^{\infty}\begin{bmatrix} 0 & 0 \\ \left((W_2B_2)^d\right)^{i+3}W_2B_1 & \left((W_2B_2)^d\right)^{i+2} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & W_2A_2S(i) \end{bmatrix}\right\}^2 \\
&= (A+B)\begin{bmatrix} 0 & 0 \\ 0 & (W_2B_2)^d + \sum_{i=0}^{\infty}\left((W_2B_2)^d\right)^{i+2}W_2A_2S(i) \end{bmatrix}^2 \\
&= \begin{bmatrix} A_1 & 0 \\ B_1 & A_2+B_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \left((W_2A_2+W_2B_2)^d\right)^2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & (A_2+B_2)\left((W_2A_2+W_2B_2)^d\right)^2 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\Sigma_2 &= (A+B)(I-WBWB^{d,W})\left(I+\sum_{k=0}^{\infty}Z(k)WB\left((WA)^d\right)^{k+1}\right)\left((WA)^d\right)^2 \\
&= \begin{bmatrix} A_1 & 0 \\ B_1 & A_2+B_2 \end{bmatrix}\begin{bmatrix} I & 0 \\ -(W_2B_2)^dW_2B_1 & I-W_2B_2(W_2B_2)^d \end{bmatrix}\times \\
&\times \begin{bmatrix} (W_1A_1)^{-2} & 0 \\ \sum_{k=0}^{\infty}S(k)W_2B_1(W_1A_1)^{-(k+3)} & 0 \end{bmatrix} \\
&= \begin{bmatrix} A_1(W_1A_1)^{-2} & 0 \\ X'' & 0 \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
X'' &= B_1(W_1A_1)^{-2} \\
&- (A_2+B_2)\Big[(W_2B_2)^d W_2B_1(W_1A_1)^{-2} \\
&+ (I-W_2B_2(W_2B_2)^d)\left(\sum_{k=0}^{\infty} S(k)W_2B_1(W_1A_1)^{-(k+3)}\right)\Big],
\end{aligned}$$

$$\begin{aligned}
\Sigma_3 &= -(A+B)\left((WB)^d\right)^2\left(\sum_{i=0}^{\infty}\left((WB)^d\right)^i WAZ(i)WB\right)\left((WA)^d\right)^2 \\
&= -(A+B)\begin{bmatrix} 0 & 0 \\ \sum_{i=0}^{\infty}\left((W_2B_2)^d\right)^{i+2} W_2A_2S(i)W_2B_1(W_1A_1)^{-2} & 0 \end{bmatrix} \\
&= -\begin{bmatrix} 0 & 0 \\ (A_2+B_2)\sum_{i=0}^{\infty}\left((W_2B_2)^d\right)^{i+2} W_2A_2S(i)W_2B_1(W_1A_1)^{-2} & 0 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\Sigma_4 &= -(A+B)(WB)^d\left(\sum_{i=0}^{\infty} WAZ(i)WB\left((WA)^d\right)^i\right)\left((WA)^d\right)^3 \\
&= -(A+B)\begin{bmatrix} 0 & 0 \\ (W_2B_2)^d\sum_{i=0}^{\infty} W_2A_2S(i)W_2B_1(W_1A_1)^{-(i+3)} & 0 \end{bmatrix} \\
&= -\begin{bmatrix} 0 & 0 \\ (A_2+B_2)(W_2B_2)^d\sum_{i=0}^{\infty} W_2A_2S(i)W_2B_1(W_1A_1)^{-(i+3)} & 0 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\Sigma_5 &= -(A+B)\left((WB)^d\right)^2 \times \\
&\times \left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left((WB)^d\right)^i WAZ(i+k+1)WB\left((WA)^d\right)^k\right)\left((WA)^d\right)^3 \\
&= -\begin{bmatrix} A_1 & 0 \\ B_1 & A_2+B_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ X''' & 0 \end{bmatrix} \\
&= -\begin{bmatrix} 0 & 0 \\ (A_2+B_2)X''' & 0 \end{bmatrix},
\end{aligned}$$

where

$$X''' = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left((W_2B_2)^d\right)^{i+2} W_2A_2S(i+k+1)W_2B_1(W_1A_1)^{-(k+3)},$$

$$\begin{aligned}
\Sigma_6 &= -(A+B)\times \\
&\times \left[(WB)^d\left(I+\sum_{i=0}^{\infty}\left((WB)^d\right)^{i+1} WAZ(i)\right)(I-WAWA^{d,W})\right]^2 \times \\
&\times WB(WA)^d \\
&= -(A+B)\begin{bmatrix} 0 & 0 \\ 0 & \left((W_2A_2+W_2B_2)^d\right)^2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ W_2B_1(W_1A_1)^{-1} & 0 \end{bmatrix} \\
&= -\begin{bmatrix} 0 & 0 \\ (A_2+B_2)\left((W_2A_2+W_2B_2)^d\right)^2 W_2B_1(W_1A_1)^{-1} & 0 \end{bmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5+\Sigma_6 \\
&= \begin{bmatrix} A_1(W_1A_1)^{-2} & 0 \\ X' & (A_2+B_2)((W_2A_2+W_2B_2)^d)^2 \end{bmatrix}
\end{aligned}$$

completing the proof of (3). The second statement of the theorem can easily be verified. □

We obtain some corollaries as follows.

Corollary 2.3 *Let $W \in \mathcal{B}(Y,X)$, and let $A,B \in \mathcal{B}(X,Y)$ be Wg–Drazin invertible. If $A^{d,W}WB = 0$ and $AWB(I-WAWA^{d,W}) = 0$, then*

$$\begin{aligned}
&(A+B)^{d,W} \\
&= (A+B)\left[\left(\sum_{i=0}^{\infty}\left((WB)^d\right)^{i+1}(WA)^i\right)(I-WAWA^{d,W})\right]^2 \\
&+ (A+B)(I-WBWB^{d,W})\left(\sum_{i=0}^{\infty}(WB)^i\left((WA)^d\right)^{i+2}\right. \\
&+ \left.\sum_{i=1}^{\infty}\sum_{j=1}^{i}(WB)^{i-j}(WA)^jWB\left((WA)^d\right)^{i+3}\right)
\end{aligned}$$

$$
\begin{aligned}
&- (A+B)\left((WB)^d\right)^2\left(\sum_{i=0}^{\infty}\left((WB)^d\right)^i (WA)^{i+1}WB\right)\left((WA)^d\right)^2 \\
&- (A+B)(WB)^d\left(\sum_{i=0}^{\infty}(WA)^{i+1}WB\left((WA)^d\right)^i\right)\left((WA)^d\right)^3 \\
&- (A+B)\left((WB)^d\right)^2 \times \\
&\times\left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left((WB)^d\right)^i (WA)^{i+k+2}WB\left((WA)^d\right)^k\right)\left((WA)^d\right)^3 \\
&- (A+B)\left[\left(\sum_{i=0}^{\infty}\left((WB)^d\right)^{i+1}(WA)^i\right)\right)(I-WAWA^{d,W})\Big]^2 WB(WA)^d.
\end{aligned}
$$

Proof. From $A^{d,W}WB = 0$ and $AWB(I - WAWA^{d,W}) = 0$ it follows that

$$
\begin{aligned}
A(WB)^2 &= AWB(I - WAWA^{d,W})WB + AWBWAWA^{d,W}WB \\
&= AWBWAWA^{d,W}WB \\
&= 0
\end{aligned}
$$

and thus

$$
AWB^{d,W} = A(WB)^d = AWB\left((WB)^d\right)^2 = A(WB)^2\left((WB)^d\right)^3 = 0.
$$

Then we apply Theorem 2.2, together with the simplification $WAZ(i) = (WA)^{i+1}(I - WAWA^{d,W})$ for all $i \geq 0$, to get the statement of this corollary. $\square$

Corollary 2.4 *Let $W \in \mathcal{B}(Y,X)$, and let $A, B \in \mathcal{B}(X,Y)$ be Wg–Drazin invertible. Suppose that $A^{d,W}WB = 0$ and $AWB(I - WAWA^{d,W}) = 0$.*

(i) *If $(WB)^2 = WB$, then*

$$
\begin{aligned}
&(A+B)^{d,W} \\
&= (A+B)\left[\left(WB\sum_{i=0}^{\infty}(WA)^i\right)(I - WAWA^{d,W})\right]^2 \\
&+ (A+B)(I - WB)\left(\left((WA)^d\right)^2 + \sum_{i=1}^{\infty}(WA)^i WB\left((WA)^d\right)^{i+3}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- (A+B)WB\left(\sum_{i=0}^{\infty}(WA)^{i+1}WB\right)\left((WA)^d\right)^2\\
&- (A+B)WB\left(\sum_{i=0}^{\infty}(WA)^{i+1}WB(\left((WA)^d\right)^i\right)\left((WA)^d\right)^3\\
&- (A+B)WB\left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(WA)^{i+k+2}WB(\left((WA)^d\right)^k\right)\left((WA)^d\right)^3\\
&- (A+B)\left[\left(WB\sum_{i=0}^{\infty}(WA)^i\right)(I-WAWA^{d,W})\right]^2 WB(WA)^d.
\end{aligned}
$$

(ii) *If WB is quasinilpotent, then*

$$
\begin{aligned}
(A+B)^{d,W} &= (A+B)\Bigg[\left((WA)^d\right)^2\\
&+ \left(\sum_{i=0}^{\infty}\sum_{j=0}^{i}(WB)^{i-j}(WA)^jWB\left((WA)^d\right)^i\right)\left((WA)^d\right)^3\Bigg].
\end{aligned}
$$

(iii) *If $(WB)^2=0$, then*

$$
\begin{aligned}
(A+B)^{d,W} &= (A+B)\Bigg[\left((WA)^d\right)^2\\
&+ WB\left(\sum_{i=0}^{\infty}(WA)^iWB\left((WA)^d\right)^i\right)\left((WA)^d\right)^4\\
&+ \left(\sum_{i=0}^{\infty}(WA)^iWB\left((WA)^d\right)^i\right)\left((WA)^d\right)^3\Bigg].
\end{aligned}
$$

Proof. Each of these cases follows directly from Corollary 2.3 and the following simplifications:

(i) Since $(WB)^2 = WB$, we have $WB^{d,W} = (WB)^d = WB$ and $(I - WBWB^{d,W})WB = 0$.

(ii) Since WB is quasinilpotent, we get $(WB)^d = 0$.

(iii) Since $(WB)^2 = 0$, it follows that

$$(WB)^d = WB\left((WB)^d\right)^2 = (WB)^2\left((WB)^d\right)^3 = 0. \quad \square$$

Corollary 2.5 *Let $W \in \mathcal{B}(Y,X)$, and let $A, B \in \mathcal{B}(X,Y)$ be Wg–Drazin invertible. If $AWB^{d,W} = 0$ and $(I - WBWB^{d,W})WAWB = 0$, then*

$$
\begin{aligned}
&(A+B)^{d,W} \\
&= (A+B)\left[\left(\sum_{i=0}^{\infty}\left((WB)^d\right)^{i+1}(WA)^i\right.\right. \\
&\left.\left.+\sum_{i=1}^{\infty}\sum_{j=1}^{i}\left((WB)^d\right)^{i+2}WA(WB)^j(WA)^{i-j}\right)(I-WAWA^{d,W})\right]^2 \\
&+(A+B)(I-WBWB^{d,W})\left(\sum_{i=0}^{\infty}(WB)^i\left((WA)^d\right)^{i+2}\right) \\
&-(A+B)\left((WB)^d\right)^2\left(\sum_{i=0}^{\infty}\left((WB)^d\right)^iWA(WB)^{i+1}\right)\left((WA)^d\right)^2 \\
&-(A+B)(WB)^d\left(\sum_{i=0}^{\infty}WA(WB)^{i+1}\left((WA)^d\right)^i\right)\left((WA)^d\right)^3 \\
&-(A+B)\left((WB)^d\right)^2\left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left((WB)^d\right)^iWA(WB)^{i+k+2}\left((WA)^d\right)^{k+3}\right) \\
&-(A+B)\left[\left(\sum_{i=0}^{\infty}\left((WB)^d\right)^{i+1}(WA)^i\right.\right. \\
&\left.\left.+\sum_{i=1}^{\infty}\sum_{j=1}^{i}\left((WB)^d\right)^{i+2}WA(WB)^j(WA)^{i-j}\right)(I-WAWA^{d,W})\right]^2 \\
&\times WB(WA)^d.
\end{aligned}
$$

Proof. From $AWB^{d,W} = 0$ and $(I - WBWB^{d,W})WAWB = 0$ it follows that

$$
\begin{aligned}
(AW)^2B &= A(I-WBWB^{d,W})WAWB + AWBWB^{d,W}WAWB \\
&= AWB^{d,W}WBWAWB \\
&= 0
\end{aligned}
$$

and thus

$$A^{d,W}WB = (AW)^dB = \left((AW)^d\right)^2AWB = \left((AW)^d\right)^3(AW)^2B = 0.$$

Then we apply Theorem 2.2, together with the simplification $Z(i)WB = (I - WBWB^{d,W})(WB)^{i+1}$ for all $i \geq 0$, to get the result of this corollary. □

Corollary 2.6 *Let $W \in \mathcal{B}(Y,X)$, and let $A, B \in \mathcal{B}(X,Y)$ be Wg–Drazin invertible. Suppose that $AWB^{d,W} = 0$ and $(I - WBWB^{d,W})WAWB = 0$.*

(i) *If $(WA)^2 = WA$, then*

$$
\begin{aligned}
&(A+B)^{d,W} \\
&= (A+B)\left[\left((WB)^d + \sum_{i=1}^{\infty}\left((WB)^d\right)^{i+2} WA(WB)^i\right)(I - WA)\right]^2 \\
&+ (A+B)(I - WBWB^{d,W})\left(\sum_{i=0}^{\infty}(WB)^i\right) WA \\
&- (A+B)\left((WB)^d\right)^2\left(\sum_{i=0}^{\infty}\left((WB)^d\right)^i WA(WB)^{i+1}\right) WA \\
&- (A+B)(WB)^d\left(\sum_{i=0}^{\infty} WA(WB)^{i+1}\right) WA \\
&- (A+B)\left((WB)^d\right)^2\left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left((WB)^d\right)^i WA(WB)^{i+k+2}\right) WA \\
&- (A+B)\left[\left((WB)^d + \sum_{i=1}^{\infty}\left((WB)^d\right)^{i+2} WA(WB)^i\right)(I - WA)\right]^2 \\
&\times WB(WA)^d.
\end{aligned}
$$

(ii) *If WA is quasinilpotent, then*

$$
\begin{aligned}
&(A+B)^{d,W} \\
&= (A+B)\left[(WB)^d + \sum_{i=0}^{\infty}\sum_{j=0}^{i}\left((WB)^d\right)^{i+2} WA(WB)^j(WA)^{i-j}\right]^2.
\end{aligned}
$$

Proof. We apply Corollary 2.5 and the following simplifications:

(i) Since $(WA)^2 = WA$, we have $WA^{d,W} = (WA)^d = WA$ and $(WA)^j(I - WAWA^{d,W}) = 0$ for all $j \geq 1$.

(ii) Since WA is quasinilpotent, we get $(WA)^d = 0$. $\square$

Corollary 2.7 *Let $A, B \in \mathcal{B}(X,Y)$ be Wg–Drazin invertible. If $AWB = 0$, then*

$$(A+B)^{d,W}$$

$$= (A+B)\left[(WB)^d\left(\sum_{i=0}^{\infty}\left((WB)^d\right)^i(WA)^i\right)(I-WAWA^{d,W})\right]^2$$
$$+ (A+B)(I-WBWB^{d,W})\left(\sum_{i=0}^{\infty}(WB)^i\left((WA)^d\right)^i\right)\left((WA)^d\right)^2$$
$$-(A+B)\left[(WB)^d\left(\sum_{i=0}^{\infty}\left((WB)^d\right)^i(WA)^i\right)(I-WAWA^{d,W})\right]^2$$
$$\times WB(WA)^d.$$

Proof. Since $AWB = 0$, then it follows that

$$A^{d,W}WB = A^{d,W}WAWA^{d,W}WB = (A^{d,W}W)^2AWB = 0,$$

$(I - WBWB^{d,W})WAWB = 0$, $AWB(I - WAWA^{d,W}) = 0$ and then $A^{d,W}WB = 0$. Thus, we apply Corollary 2.3, or Corollary 2.5, to get the above result. □

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