Additive results for the generalized Drazin inverse in a Banach algebra

Dragana S. Cvetković-Ilić, Dragan S. Djordjević and Yimin Wei*

Abstract

In this paper we investigate additive properties of the generalized Drazin inverse in a Banach algebra. We find some new conditions under which the generalized Drazin inverse of the sum $a+b$ could be explicitly expressed in terms of $a, a^{\mathsf{d}}, b, b^{\mathsf{d}}$. Also, some recent results of Castro and Koliha (Proc. Roy. Soc. Edinburgh Sect. A 134 (2004), 1085-1097) are extended.

2000 *Mathematics Subject Classification*: 15A09, 46H30,46H05, 15A33.
Key words: Banach algebra; generalized Drazin inverse; Drazin inverse; additive result;

1 Introduction

Let $\mathcal{A}$ be a complex Banach algebra with the unit 1. By $\mathcal{A}^{-1}$, $\mathcal{A}^{\mathsf{nil}}$, $\mathcal{A}^{\mathsf{qnil}}$ we denote the sets of all invertible, nilpotent and quasinilpotent elements in $\mathcal{A}$, respectively. Let us recall that the Drazin inverse of $a \in \mathcal{A}$ [1] is the element $x \in \mathcal{A}$ (denoted by a^{D}) which satisfies

$$xax = x, \quad ax = xa, \quad a^{k+1}x = a^k, \tag{1}$$

* Corresponding author.

Supported by Grant No. 1232 of the Ministry of Science, Technology and Development, Republic of Serbia and Natural National Science Foundation of China under grant 10471027

for some nonnegative integer k. The least such k is the index of a, denoted by $ind(a)$. When $ind(a) = 1$ then the Drazin inverse a^{D} is called the group inverse and it is denoted by $a^{\#}$. The conditions (1) are equivalent to

$$xax = x, \quad ax = xa, \quad a - a^2x \in \mathcal{A}^{\mathsf{nil}}. \tag{2}$$

The concept of the generalized Drazin inverse in a Banach algebra was introduced by Koliha [2]. The condition $a - a^2x \in \mathcal{A}^{\mathsf{nil}}$ was replaced by $a - a^2x \in \mathcal{A}^{\mathsf{qnil}}$. Hence, the generalized Drazin inverse of a is the element $x \in \mathcal{A}$ (written a^{d}) which satisfies

$$xax = x, \quad ax = xa, \quad a - a^2x \in \mathcal{A}^{\mathsf{qnil}}. \tag{3}$$

We mention that an alternative definition of the generalized Drazin inverse in a ring is also given in [3, 4, 5]. These two concepts of generalized Drazin inverse are equivalent in the case when the ring is actually a complex Banach algebra with a unit. It is well-known that a^{d} is unique whenever it exists [2]. The set $\mathcal{A}^{\mathsf{d}}$ consists of all $a \in \mathcal{A}$ such that a^{d} exists. For interesting properties of Drazin inverse see [6, 7, 8].

Let $a \in \mathcal{A}$ and let $p \in \mathcal{A}$ be a idempotent ($p = p^2$). Then we write

$$a = pap + pa(1-p) + (1-p)ap + (1-p)a(1-p)$$

and use the notations

$$a_{11} = pap, \quad a_{12} = pa(1-p), \quad a_{21} = (1-p)ap, \quad a_{22} = (1-p)a(1-p).$$

Every idempotent $p \in \mathcal{A}$ induces a representation of an arbitrary element $a \in \mathcal{A}$ given by the following matrix

$$a = \begin{bmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{bmatrix}_p = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p. \tag{4}$$

Let a^{π} be the spectral idempotent of a corresponding to $\{0\}$. It is well-known that $a \in \mathcal{A}^{\mathsf{d}}$ can be represented in the following matrix form:

$$a = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}_p,$$

relative to $p = aa^{\mathsf{d}} = 1 - a^{\pi}$, where a_{11} is invertible in the algebra $p\mathcal{A}p$ and a_{22} is quasinilpotent in the algebra $(1-p)\mathcal{A}(1-p)$. Then the generalized Drazin inverse is given by

$$a^{\mathsf{d}} = \begin{bmatrix} a_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p.$$

The motivation for this paper was the paper of Djordjević and Wei [9] and the paper of Castro and Koliha [10]. In both of these papers the conditions under which the generalized Drazin inverse $(a+b)^{\mathsf{d}}$ could be expressed in terms of $a, a^{\mathsf{d}}, b, b^{\mathsf{d}}$ were considered. In [9] this problem is investigated for a bounded linear operator on an arbitrary complex Banach space under assumption that $AB = 0$ and these results are the generalizations of the results from [11] where the same problem was considered for matrices. Castro and Koliha [10] considered the same problem for the elements of the Banach algebra with unit under some weaker conditions. They generalized the results from [9].

In the present paper we investigate additive properties of the generalized Drazin inverse in a Banach algebra and find an explicit expression for the generalized Drazin inverse of the sum $a + b$ under various conditions.

In the first part of the paper we find some new conditions, which are nonequivalent to the conditions from [10], allowing for the generalized Drazin inverse of $a + b$ to be expressed in terms of $a, a^{\mathsf{d}}, b, b^{\mathsf{d}}$. It is interesting to note that in some cases we obtain the same expression for $(a+b)^{\mathsf{d}}$ as in [10]. In the rest of the paper we generalize recent results from [10].

2 Results

First we state the following result which is proved in [12] for matrices, extended in [13] for a bounded linear operator and in [10] for arbitrary elements in a Banach algebra.

Theorem 2.1 *Let $x \in \mathcal{A}$ and let*

$$x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p ,$$

relative to the idempotent $p \in \mathcal{A}$.

(1) If $a \in (p\mathcal{A}p)^{\mathsf{d}}$ and $b \in ((1-p)\mathcal{A}(1-p))^{\mathsf{d}}$, then x is generalized Drazin invertible and

$$x^{\mathsf{d}} = \begin{bmatrix} a^{\mathsf{d}} & u \\ 0 & b^{\mathsf{d}} \end{bmatrix}_p , \qquad (5)$$

where $u = \sum_{n=0}^{\infty} (a^{\mathsf{d}})^{n+2} c b^n b^{\pi} + \sum_{n=0}^{\infty} a^{\pi} a^n c (b^{\mathsf{d}})^{n+2} - a^{\mathsf{d}} c b^{\mathsf{d}}$.

(2) If $x \in \mathcal{A}^{\mathsf{d}}$ and $a \in (p\mathcal{A}p)^{\mathsf{d}}$, then $b \in ((1-p)\mathcal{A}(1-p))^{\mathsf{d}}$ and x^{d} is given by (5).

Now, we state an auxiliary result.

Lemma 2.1 *Let $a, b \in \mathcal{A}^{\mathsf{qnil}}$ and let $ab = ba$ or $ab = 0$, then $a + b \in \mathcal{A}^{\mathsf{qnil}}$.*

Proof. If $ab = ba$ we have that

$$r(a+b) \leq r(a) + r(b),$$

which gives that $a + b \in \mathcal{A}^{\mathsf{qnil}}$. The case when $ab = 0$ follows from the equation

$$(\lambda - a)(\lambda - b) = \lambda(\lambda - (a+b)) \ \Box$$

Considering the previous lemma, the first idea was to replace the basic condition $ab = 0$ which was used in the papers [11], [9] by the condition $ab = ba$. As we expected, this condition wasn't enough to derive a formula for $(a+b)^{\mathsf{d}}$. Hence, to this aim we assume the following three conditions for $a, b \in \mathcal{A}^{\mathsf{d}}$:

$$a = ab^{\pi}, \quad b^{\pi}ba^{\pi} = b^{\pi}b \quad \text{and} \quad b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab. \tag{6}$$

Instead of the condition $ab = ba$ we assume the weaker condition $b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab$. Notice that

$$a = ab^{\pi} \Leftrightarrow ab^{\mathsf{d}} = 0 \Leftrightarrow \mathcal{A}a \subseteq Ab^{\pi}, \tag{7}$$

$$b^{\pi}ba^{\pi} = b^{\pi}b \Leftrightarrow b^{\pi}ba^{\mathsf{d}} = 0 \Leftrightarrow \mathcal{A}b^{\pi}b \subseteq \mathcal{A}a^{\pi}, \tag{8}$$

$$b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab \Leftrightarrow (ba - ab)\mathcal{A} \subseteq (b^{\pi}a^{\pi})^{\circ}, \tag{9}$$

where for $u \in \mathcal{A}$, $u^{\circ} = \{x \in \mathcal{A} \ : \ ux = 0\}$.

For matrices and bounded linear operators on a Banach space the conditions (7), (8), (9) are equivalent to

$$\mathcal{N}(b^{\pi}) \subseteq \mathcal{N}(a), \quad N(a^{\pi}) \subseteq \mathcal{N}(b^{\pi}b), \quad \mathcal{R}(ba - ab) \subseteq \mathcal{N}(b^{\pi}a^{\pi}).$$

Remark that conditions (6) are not symmetric in a, b like the conditions (3.1) from [10], so our expression for $(a+b)^{\mathsf{d}}$ is not symmetric in a, b at all.

In the next theorem under the assumption that for $a, b \in \mathcal{A}^{\mathsf{d}}$ the conditions (6) hold, we offer the following expression for $(a+b)^{\mathsf{d}}$.

Theorem 2.2 *Let $a, b \in \mathcal{A}^{\mathsf{d}}$ be such that* (6) *is satisfied. Then $a+b \in \mathcal{A}^{\mathsf{d}}$ and*

$$
\begin{aligned}
(a+b)^{\mathsf{d}} = & (b^{\mathsf{d}} + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n)a^{\pi} \\
& - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n(a^{\mathsf{d}})^{k+2}b(a+b)^{k+1} \\
& + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n a^{\mathsf{d}} b - \sum_{n=0}^{\infty} b^{\mathsf{d}} a (a^{\mathsf{d}})^{n+2} b(a+b)^n
\end{aligned}
\tag{10}
$$

Before proving Theorem 2.2 we have to prove the following result which is a special case of this theorem:

Theorem 2.3 *Let $a \in \mathcal{A}^{\mathsf{qnil}}$, $b \in \mathcal{A}^{\mathsf{d}}$ are such that $b^{\pi}ab = b^{\pi}ba$ and $a = ab^{\pi}$. Then* (6) *is satisfied, $a+b \in \mathcal{A}^{\mathsf{d}}$ and*

$$
(a+b)^{\mathsf{d}} = b^{\mathsf{d}} + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n. \tag{11}
$$

Proof. First, suppose that $b \in \mathcal{A}^{\mathsf{qnil}}$. Then $b^{\pi} = 1$ and from $b^{\pi}ab = b^{\pi}ba$ we obtain that $ab = ba$. Using Lemma 2.1, $a+b \in \mathcal{A}^{\mathsf{qnil}}$ and (11) holds. Now, we assume that b is not quasinilpotent and we consider the matrix representation of a and b relative to the $p = 1 - b^{\pi}$. We have

$$
b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p, \quad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,
$$

where $b_1 \in (p\mathcal{A}p)^{-1}$ and $b_2 \in ((1-p)\mathcal{A}(1-p))^{\mathsf{qnil}} \subset \mathcal{A}^{\mathsf{qnil}}$. From $a = ab^{\pi}$, it follows that $a_{11} = 0$ and $a_{21} = 0$. We denote $a_1 = a_{12}$ and $a_2 = a_{22}$. Hence,

$$
a + b = \begin{bmatrix} b_1 & a_1 \\ 0 & a_2 + b_2 \end{bmatrix}_p.
$$

The condition $b^{\pi}ab = b^{\pi}ba$ implies that $a_2b_2 = b_2a_2$. Hence, using Lemma 2.1, we get $a_2 + b_2 \in ((1-p)\mathcal{A}(1-p))^{\mathsf{qnil}}$. Now, by Theorem 2.1, we obtain that $a+b \in \mathcal{A}^{\mathsf{d}}$ and

$$
\begin{aligned}
(a+b)^{\mathsf{d}} &= \begin{bmatrix} b_1^{-1} & \sum_{n=0}^{\infty} b_1^{-(n+2)} a_1 (a_2+b_2)^n \\ 0 & 0 \end{bmatrix}_p \\
&= b^{\mathsf{d}} + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n \ \square
\end{aligned}
$$

Let us observe that the expressions for $(a+b)^{\mathsf{d}}$ in (11) and in (3.6), Theorem 3.3 [10] are exactly the same. If we assume that $ab = ba$ instead of $b^{\pi}ab = b^{\pi}ba$, we will get a much simpler expression for $(a+b)^{\mathsf{d}}$.

Corollary 2.1 *Let $a \in \mathcal{A}^{\mathsf{qnil}}$, $b \in \mathcal{A}^{\mathsf{d}}$ are such that $ab = ba$ and $a = ab^{\pi}$, then $a + b \in \mathcal{A}^{\mathsf{d}}$ and*

$$(a+b)^{\mathsf{d}} = b^{\mathsf{d}}.$$

Proof. From the condition $a = ab^{\pi}$, as we mentioned before, it follows that $ab^{\mathsf{d}} = 0$. Now, because the Drazin inverse b^{d} is double commutant of a, we have that

$$(b^{\mathsf{d}})^{n+2}a(a+b)^n = a(b^{\mathsf{d}})^{n+2}(a+b)^n = 0\square$$

Proof of the Theorem 2.2: If b is quasinilpotent we can apply Theorem 2.3. Hence, we assume that b is neither invertible nor quasinilpotent and consider the following matrix representation of a and b relative to the $p = 1 - b^{\pi}$:

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p, \quad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $b_1 \in (p\mathcal{A}p)^{-1}$ and $b_2 \in ((1-p)\mathcal{A}(1-p))^{\mathsf{qnil}}$. As in the proof of Theorem 2.3, from $a = ab^{\pi}$ it follows that $a = \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_p$ and

$$a + b = \begin{bmatrix} b_1 & a_1 \\ 0 & a_2 + b_2 \end{bmatrix}_p.$$

¿From the conditions $b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab$ and $b^{\pi}ba^{\pi} = b^{\pi}b$, we obtain that $a_2^{\pi}b_2a_2 = a_2^{\pi}a_2b_2$ and $b_2 = b_2a_2^{\pi}$. Now, by Theorem 2.3 it follows that $(a_2 + b_2) \in ((1-p)\mathcal{A}(1-p))^{\mathsf{d}}$ and

$$(a_2+b_2)^{\mathsf{d}} = a_2^{\mathsf{d}} + \sum_{n=0}^{\infty}(a_2^{\mathsf{d}})^{n+2}b_2(a_2+b_2)^n. \tag{12}$$

By Theorem 2.1 we get

$$(a+b)^{\mathsf{d}} = \begin{bmatrix} b_1^{-1} & u \\ 0 & (a_2+b_2)^{\mathsf{d}} \end{bmatrix}_p,$$

where $u = \sum_{n=0}^{\infty} b_1^{-(n+2)} a_1(a_2+b_2)^n (a_2+b_2)^{\pi} - b_1^{-1} a_1 (a_2+b_2)^{\mathsf{d}}$ and by b_1^{-1} we denote the inverse of b_1 in the algebra $p\mathcal{A}p$. Using (12), we have that

$$u = \sum_{n=0}^{\infty} b_1^{-(n+2)} a_1(a_2+b_2)^n = a_2^{\pi} - \sum_{n=0}^{\infty} b_1^{-(n+2)} a_1(a_2+b_2)^n a_2^{\mathsf{d}} b_2$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (b_1)^{-(n+2)} a_1 (a_2+b_2)^n (a_2^{\mathsf{d}})^{k+2} b_2 (a_2+b_2)^{k+1} - b_1^{-1} a_1 a_2^{\mathsf{d}}$$
$$- \sum_{n=0}^{\infty} b_1^{-1} a_1 (a_2^{\mathsf{d}})^{n+2} b_2 (a_2+b_2)^n$$

By a straightforward computation we obtain that (10) holds. □

Corollary 2.2 *Let $a, b \in \mathcal{A}^{\mathsf{d}}$ are such that $ab = ba$, $a = ab^{\pi}$ and $b^{\pi} = ba^{\pi} = b^{\pi} b$, then $a + b \in \mathcal{A}^{\mathsf{d}}$ and*

$$(a+b)^{\mathsf{d}} = b^{\mathsf{d}}.$$

Let us also observe that if a, b are such that a is invertible and b is group invertible than the conditions (8) and (9) are satisfied, so we have to assume just that $a = ab^{\pi}$. In the opposite case when b is invertible we get $a = 0$.

As we mentioned before, Hartwig et al. in [11] for matrices and Djordjević and Wei [9] for operators used the condition $AB = 0$ to derive the formula $(a + b)^{\mathsf{d}}$. Castro and Koliha [10] relaxed this hypothesis by assuming the following three conditions symmetric in $a, b \in \mathcal{A}^{\mathsf{d}}$,

$$a^{\pi} b = b, \quad ab^{\pi} = a, \quad b^{\pi} a b a^{\pi} = 0. \tag{13}$$

It is easy to see that $ab = 0$ implies (13), but the converse is not true (see Ex. 3.1, [10]).

It is interesting to remark that the conditions (13) and (6) are independent, neither of them implies the other one, but in some cases we obtain the same expressions for $(a + b)^{\mathsf{d}}$.

If we consider the algebra $\mathcal{A}$ of all complex 3×3 matrices and $a, b \in \mathcal{A}$ which are given in the Example 3.1 [10], we can see that the conditions (13) are satisfied, but the conditions (6) are not satisfied. In the following example we have the opposite case. We construct matrices a, b in the algebra $\mathcal{A}$ of all complex 3×3 matrices such that (6) is satisfied but (13) is not satisfied. If we assume that $ab = ba$ in Theorem 2.2 the expression for

$(a+b)^{\mathsf{d}}$ will be exactly the same as in the Theorem 3.5 [10] (in this paper Corollary 2.4).

Example. Let

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$a^{\pi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $b^{\pi} = 1$. Now, we can see that $a = ab^{\pi}$, $a^{\pi}ab = a^{\pi} = ba$ and $ba^{\pi} = b$ i.e., (6) is satisfied. Also, $a^{\pi}b = 0 \neq b$, so (13) is not satisfied.

In the rest of the paper we will present a generalization of the results from [10]. We will use some weaker conditions than in [10]. For example in the next theorem which is the generalization of Theorem 3.3 [10] we will assume that $e = (1-b^{\pi})(a+b)(1-b^{\pi}) \in \mathcal{A}^{\mathsf{d}}$ instead of $ab^{\pi} = a$. If $ab^{\pi} = a$ then $e = (1-b^{\pi})b = \begin{bmatrix} b_1 & 0 \\ 0 & 0 \end{bmatrix}_p$ for $p = 1 - b^{\pi}$ and $e^{\mathsf{d}} = b^{\mathsf{d}}$.

Theorem 2.4 *Let $b \in \mathcal{A}^{\mathsf{d}}$, $a \in \mathcal{A}^{\mathsf{qnil}}$ be such that*

$$e = (1-b^{\pi})(a+b)(1-b^{\pi}) \in \mathcal{A}^{\mathsf{d}} \quad \text{and} \quad b^{\pi}ab = 0,$$

then $a + b \in \mathcal{A}^{\mathsf{d}}$ and

$$(a+b)^{\mathsf{d}} = e^{\mathsf{d}} + \sum_{n=0}^{\infty} (e^{\mathsf{d}})^{n+2} ab^{\pi}(a+b)^n.$$

Proof. The case when $b \in \mathcal{A}^{\mathsf{qnil}}$ follows from Lemma 2.1. Hence, we assume that b is not quasinilpotent,

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p \quad \text{and} \quad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $p = 1 - b^{\pi}$. From $b^{\pi}ab = 0$ we have that $b^{\pi}a(1-b^{\pi}) = 0$, i.e., $a_{21} = 0$. Denote $a_1 = a_{11}$, $a_{22} = a_2$ and $a_{12} = a_3$. Then,

$$a + b = \begin{bmatrix} a_1 + b_1 & a_3 \\ 0 & a_2 + b_2 \end{bmatrix}_p.$$

Also, $b^\pi ab = 0$ implies that $a_2b_2 = 0$, so $a_2 + b_2 \in ((1-p)\mathcal{A}(1-p))^{\mathsf{qnil}}$, according to Lemma 2.1. Now, applying Theorem 2.1, we obtain that

$$(a+b)^{\mathsf{d}} = \begin{bmatrix} (a_1+b_1)^{\mathsf{d}} & u \\ 0 & 0 \end{bmatrix}_p,$$

where $u = \sum_{n=0}^{\infty}((a_1+b_1)^{\mathsf{d}})^{(n+2)}a_3(a_2+b_2)^n$. By a direct computation we verify that

$$(a+b)^{\mathsf{d}} = e^{\mathsf{d}} + \sum_{n=0}^{\infty}(e^{\mathsf{d}})^{n+2}ab^\pi(a+b)^n. \ \square$$

Now, as a corollary we obtain Theorem 3.3 from [10].

Corollary 2.3 *Let $b \in \mathcal{A}^{\mathsf{d}}$, $a \in \mathcal{A}^{\mathsf{qnil}}$ and let $ab^\pi = a$, $b^\pi ab = 0$. Then $a+b \in \mathcal{A}^{\mathsf{d}}$ and*

$$(a+b)^{\mathsf{d}} = b^{\mathsf{d}} + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n.$$

The next result is a generalization of Theorem 3.5 in [10]. For simplicity we use the following notation:

$$\begin{aligned} e &= (1-b^\pi)(a+b)(1-b^\pi) \in \mathcal{A}^{\mathsf{d}}, \\ f &= (1-a^\pi)(a+b)(1-a^\pi), \\ \mathcal{A}_1 &= (1-a^\pi)\mathcal{A}(1-a^\pi), \\ \mathcal{A}_2 &= (1-b^\pi)\mathcal{A}(1-b^\pi), \end{aligned}$$

where $a, b \in A^{\mathsf{d}}$ are given.

We also prove the next result which is the generalization of Theorem 3.5 [10].

Theorem 2.5 *Let $a, b \in \mathcal{A}^{\mathsf{d}}$ be such that $(1-a^\pi)b(1-a^\pi) \in \mathcal{A}^{\mathsf{d}}$, $f \in \mathcal{A}_1^{-1}$ and $e \in \mathcal{A}_2^{\mathsf{d}}$. If*

$$(1-a^\pi)ba^\pi = 0, \quad b^\pi aba^\pi = 0, \quad a^\pi = a(1-b^\pi)a^\pi = 0$$

then $a+b \in \mathcal{A}^{\mathsf{d}}$ and

$$\begin{aligned} (a+b)^{\mathsf{d}} = {} & (b^{\mathsf{d}} + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n)a^\pi + \sum_{n=0}^{\infty} b^\pi(a+b)^n a^\pi b(f)_{\mathcal{A}_1}^{-(n+2)} \\ & - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^{\mathsf{d}})^{k+1}a(a+b)^{n+k}a^\pi b(f)_{\mathcal{A}_1}^{-(n+2)} - b^{\mathsf{d}}a^\pi b(f)_{\mathcal{A}_1}^{-1} \\ & - \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n a^\pi b(f)_{\mathcal{A}_1}^{-1} + (f)_{\mathcal{A}_1}^{-1}, \end{aligned}$$

where by $(f)^{-1}_{\mathcal{A}_1}$ *we denote the inverse of* f *in* $\mathcal{A}_1$.

Proof. Obviously, if a is invertible, then the statement of the theorem holds. If a is quasinilpotent, then the result follows from Theorem 2.4. Hence, we assume that a is neither invertible nor quasinilpotent. As in the proof of Theorem 2.2, we have that

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p, \quad b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_p,$$

where $p = 1 - a^\pi$, $a_1 \in (p\mathcal{A}p)^{-1}$ and $a_2 \in ((1-p)\mathcal{A}(1-p))^{\mathsf{qnil}}$. From $(1-a^\pi)ba^\pi = 0$, we have that $b_{12} = 0$. Denote $b_1 = b_{11}$, $b_{22} = b_2$ and $b_{21} = b_3$. Then,

$$a + b = \begin{bmatrix} a_1 + b_1 & 0 \\ b_3 & a_2 + b_2 \end{bmatrix}_p.$$

The condition $a^\pi b^\pi aba^\pi = 0$ expressed in the matrix form yields

$$\begin{aligned} a^\pi b^\pi aba^\pi &= \begin{bmatrix} 0 & 0 \\ 0 & b_2^\pi \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & b_2^\pi a_2 b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Similarly, $a^\pi a(1-b^\pi) = 0$ implies that $a_2 b_2^\pi = a_2$. ¿From Corollary 2.3 we get that $a_2 + b_2 \in \mathcal{A}^{\mathsf{d}}$ and

$$(a_2 + b_2)^{\mathsf{d}} = b_2^{\mathsf{d}} + \sum_{n=0}^{\infty} (b_2^{\mathsf{d}})^{n+2} a_2 (a_2 + b_2)^n.$$

Now, using Theorem 2.1 we obtain that $a + b \in \mathcal{A}^{\mathsf{d}}$ and

$$(a+b)^{\mathsf{d}} = \begin{bmatrix} (a_1 + b_1)^{\mathsf{d}} & 0 \\ u & (a_2 + b_2)^{\mathsf{d}} \end{bmatrix}_p,$$

where

$$\begin{aligned} u = &\sum_{n=0}^{\infty} b_2^\pi (a_2 + b_2)^n b_3 (f)^{-(n+2)}_{\mathcal{A}_1} \\ &- \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (b_2^{\mathsf{d}})^{k+1} a_2 (a_2 + b_2)^{n+k} b_3 (f)^{-(n+2)}_{\mathcal{A}_1} - b_2^{\mathsf{d}} b_3 (f)^{-1}_{\mathcal{A}_1} \\ &- \sum_{n=0}^{\infty} (b_2^{\mathsf{d}})^{n+2} a_2 (a_2 + b_2)^n b_3 (f)^{-1}_{\mathcal{A}_1}. \end{aligned}$$

By a straightforward computation we obtain that the result holds.□

Corollary 2.4 *Let* $a, b \in \mathcal{A}^{\mathsf{d}}$ *satisfy the conditions* (13)*. Then* $a + b \in \mathcal{A}^{\mathsf{d}}$ *and*

$$\begin{aligned}(a+b)^{\mathsf{d}} = {} & (b^{\mathsf{d}} + \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} a(a+b)^n) a^{\pi} + \sum_{n=0}^{\infty} b^{\pi} (a+b)^n b (a^{\mathsf{d}})^{(n+2)} \\ & - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (b^{\mathsf{d}})^{k+1} a(a+b)^{n+k} b (a^{\mathsf{d}})^{(n+2)} + b^{\pi} a^{\mathsf{d}} \\ & - \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} a(a+b)^n b a^{\mathsf{d}}\end{aligned}$$

Proof. We have that $f = (1 - a^{\pi})a$, so $(f)_{\mathcal{A}_1}^{-1} = a^{\mathsf{d}}$.

The authors would like to thank the referees for their helpful comments and suggestions that help to improve this paper.

References

[1] M.P. Drazin, Pseudoinverse in associative rings and semigroups, Amer. Math. Monthly 65 (1958) 506–514.

[2] J.J. Koliha, A generalized Drazin inverse, Glasgow Math. J. 38 (1996) 367-381.

[3] R.E. Harte, Spectral projections, Irish Math. Soc. Newsletter 11 (1984) 10–15.

[4] R.E. Harte, Invertibility and Singularity for Bounded Linear Operators, New York, Marcel Dekker, 1988.

[5] R.E. Harte, On quasinilpotents in rings, PanAm. Math. J. 1 (1991) 10–16.

[6] N. Castro González, J.J. Koliha, V. Rakocevic, Continuity and general perturbation of the Drazin inverse for closed linear operators, Abstract and App. Anal. 7 (2002) 335–347.

[7] J.J. Koliha, V. Rakocevic, Holomorphic and meromorphic properties of the g-Drazin inverse, Demonstratio Mathematica 38 (2005) 657-666.

[8] J.J. Koliha, V. Rakocevic, Differentiability of the g-Drazin inverse, Studia Mathematica 168 (2005) 193-201.

[9] D.S. Djordjević, Y. Wei Additive results for the generalized Drazin inverse, J. Austral. Math. Soc. 73 (2002) 115–125.

[10] N. Castro González, J.J. Koliha, New additive results for the g-Drazin inverse, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004) 1085-1097.

[11] R.E. Hartwig, G. Wang, Y. Wei, Some additive results on Drazin inverse, Linear Algebra Appl. 322 (2001) 207–217.

[12] C.D. Meyer, Jr., N.J. Rose, The index and the Drazin inverse of block triangular matrices, SIAM J. Appl. Math. 33(1) (1977) 1–7.

[13] D.S. Djordjević, P.S. Stanimirović, On the generalized Drazin inverse and generalized resolvent, Czechoslovak Math. J. 51(126) (2001) 617–634.

Address:

Dragana S. Cvetković-Ilić:

Department of Mathematics, Faculty of Sciences, University of Niš, P.O. Box 224, Višegradska 33, 18000 Niš, Serbia

E-mail: `dragana@pmf.ni.ac.yu` `gagamaka@ptt.yu`

Dragan S. Djordjević:

Department of Mathematics, Faculty of Sciences, University of Niš, P.O. Box 224, Višegradska 33, 18000 Niš, Serbia

E-mail: `dragan@pmf.ni.ac.yu` `ganedj@EUnet.yu`

Yimin Wei:

School of Mathematical Sciences, Fudan University, Shanghai, 200433, P.R. China and Key Laboratory of Mathematics for Nonlinear Sciences (Fudan University), Ministry of Education.

E-mail: `ymwei@fudan.edu.cn`