Left-right consistency in rings III

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Abstract *A bounded linear operator T is said to be "left-right consistent" if the spectra of all the products ST and TS coincide. In this note we relate the associated "consistency spectrum" to Fredholm theory, and to the "fat boundary".*

Suppose A is a semigroup, with identity 1 and invertible group $A^{-1} = A^{-1}_{left} \cap A^{-1}_{right}$, or more generally an abstract category. Elements $a \in A$ induce left and right multiplications on A,

$$L_a : x \mapsto ax \; ; \; R_a : x \mapsto xa \; .$$

It is the relationship between these operators which gives rise to "left-right consistency". We recall [5],[3],[4] that an element $a \in A$ is said to be "left-right consistent" if for arbitrary $b \in A$ the two products ba and ab are invertible or not together:

1. Definition *If $K \subseteq A$ is arbitrary write*

1.1 $$\varpi(K) = \{a \in A : L_a^{-1}(K) = R_a^{-1}(K)\}$$

for the set of "left-right K consistent" $a \in A$. When $K = A^{-1}$ we get the "consistently invertible elements" of A.

$\varpi(K)$ is always [3],[4] a sub-semigroup of A:

1.2 $$\varpi(K)\varpi(K) \subseteq \varpi(K) \; .$$

We recall specifically [4]

1.3 $$\varpi(A^{-1}) = A^{-1} \cup \big(A \setminus (A^{-1}_{left} \cup A^{-1}_{right})\big) \; .$$

When the semigroup A is a complex linear algebra then associated with the "consistency" or otherwise of $a \in A$ and its scalar perturbations $a - \lambda$ are ([8] Definition 2) related modifications of the spectrum:

2. Definition *If ω is a spectral mapping on an algebra A then the associated consistent spectral mapping is given, for $a \in A$, by*

2.1 $$\omega_{CI}(a) = \{\lambda \in \mathbf{C} : R^{-1}_{a-\lambda} H_\omega \neq L^{-1}_{a-\lambda} H_\omega\} \; ,$$

where

2.2 $$H_\omega = \{a \in A : 0 \notin \omega(a)\} \; .$$

Thus if ω is the usual spectrum σ then H_ω is the group A^{-1} of invertibles. We remark that, in a change from [8], we are following the notation of Cao/Zhang/Zhang [1]. The characterization [4] of left-right consistent elements can ([7] Theorem 4) be expressed spectrally:

2.3 $$\sigma_{CI}(a) = \sigma(a) \setminus (\sigma^{left}(a) \cap \sigma^{right}(a)) \; .$$

For the bounded operators on a Banach space Cao/Zhang/Zhang ([1] Theorem 1) have looked at the spectral mapping theorem for this spectrum:

3. Theorem *If $a \in A$ is a Banach algebra element then there is inclusion*

3.1 $$\sigma_{CI}(a) \subseteq \text{int } \sigma(a) \ ,$$

and if $p \in$ Poly is a polynomial, inclusion

3.2 $$\sigma_{CI} p(a) \subseteq p\sigma_{CI}(a) \ .$$

Necessary and sufficient for equality in (3.2) for all polynomials is

3.3 $$\sigma_{CI}(a) = \emptyset \ .$$

If also $b \in B$ then there is inclusion

3.4 $$\sigma_{CI}(a \oplus b) \subseteq \sigma_{CI}(a)_\cup \sigma_{CI}(b) \ ;$$

the condition (3.3) is necessary for equality in (3.4).
Proof. Inclusion (3.1) follows [8] from the familiar inclusion

3.5 $$\partial\sigma(a) \subseteq \sigma^{left}(a)_\cap \sigma^{right}(a) \ .$$

Inclusion (3.2) is a consequence of the semigroup properties (1.4), and then the reverse inclusion is clear if (3.3) holds. Conversely if (3.3) fails look [1] at the polynomial

$$p = (z-\lambda)(z-\mu) \text{ with } \lambda \in \sigma_{CI}(a), \mu \in \sigma^{left}(a)_\cap \sigma^{right}(a) \ .$$

The inclusion (3.4) is clear; conversely if equality fails in (3.3) for $a \in A$ then simply choose B and $b \in B$ in such a way that

$$\sigma_B^{left}(b) = \sigma_A^{right}(a) \text{ and } \sigma_B^{right}(b) = \sigma_A^{left}(a) \ .$$

Specifically take either $b = R_a \in B = \{R_c : c \in A\}$, or $b = a \in B = A^{op}$ •

We might remark that an equivalent form for the condition (3.3) is the equality

3.6 $$\sigma^{left}(a) = \sigma^{right}(a) \ .$$

Theorem 3, and its resemblance to Theorem 5 of [7], points up a certain analogy between the consistent spectrum σ_{CI} and the Weyl spectrum ω_{ess}: for example there is inclusion, for $a = T \in B(X)$,

3.7 $$\sigma_{CI}(T) \subseteq \omega_{ess}(T) \ .$$

This of course follows from Theorem 4 of [4], since Weyl operators have invertible generalized inverses. It is also clear, since A^{-1}_{left} and A^{-1}_{right} both lie among what Müller ([11] Definition 13.11) calls the "Saphar" elements of A,

3.8 $$\{a \in aAa : L_a^{-1}(0) \subseteq L_a^\infty(A) \equiv \bigcap_{n=1}^\infty L_a^n(A)\} \ ,$$

that $\sigma_{CI}(a)$ is always disjoint from a significant portion of what might be called the "Saphar spectrum" of $a \in A$. It is useful to determine the consistent spectrum for the forward and backward shifts u, v, and the standard weight w:

4. Theorem *If $A = B(X)$ with $X = \ell_2$ and we set, for each $n \in \mathbf{N}$ and each $x \in X$,*

4.1 $$(ux)_1 = 0 \ , \ (ux)_{n+1} = x_n \ ; \ (vx)_n = x_{n+1} \ ; \ (wx)_n = (1/n)x_n$$

then $vu = 1 \neq uv$ and

4.2 $$\sigma_{CI}(u) = \text{int } \mathbf{D} = \sigma_{CI}(v) \ .$$

Also

4.3 $$\sigma_{CI}(u \oplus v) = \emptyset = \sigma_{CI}(w) = \sigma_{CI}(wu) = \sigma_{CI}(vw) \ .$$

Proof. We need only recall

4.4 $$\sigma^{left}(u) = \mathbf{S} \subseteq \mathbf{D} = \sigma^{right}(u) = \sigma(u)$$

and

4.5 $$\sigma^{right}(v) = \mathbf{S} \subseteq \mathbf{D} = \sigma^{left}(v) = \sigma(v)\ .$$

The spectrum of w has no interior, while both products wu and vw are quasinilpotent •

We also recall the disc algebra $A \subseteq C(\mathbf{D})$, and its isometric embedding $T : A \to B = C(\mathbf{S})$:

4.6 $$a = z \in A \Longrightarrow \sigma_{CI}(a) = \sigma_{CI}(a \oplus u) = \emptyset \neq \sigma_{CI}(u)\ .$$

This example shows that the condition (3.3) is not sufficient for equality in (3.4).

Theorem 5 *If $a \in A$ there is inclusion*

5.1 $$\sigma_{CI}(a) \subseteq \sigma_{CI}(L_a)_\cap \sigma_{CI}(R_a)\ .$$

Proof. Generally, recalling left and right approximate point spectrum, there is inclusion

5.2 $$\tau^{left}(a) = \tau^{left}(L_a) \subseteq \sigma^{left}(L_a) \subseteq \sigma^{left}(a) = \tau^{right}(R_a) = \sigma^{right}(R_a) = \sigma^{left}(a)$$

and

5.3 $$\tau^{right}(a) = \tau^{left}(R_a) \subseteq \sigma^{left}(R_a) \subseteq \sigma^{right}(a) = \tau^{right}(L_a) = \sigma^{right}(L_a) = \sigma^{right}(a)\ ;$$

we notice also

5.4 $$\partial\sigma(a) \subseteq \partial(a) \subseteq \tau^{left}(a)_\cap \tau^{right}(a)\ ,$$

where

5.5 $$\partial(a) = \{\lambda \in \mathbf{C} : a - \lambda \in \partial A^{-1}\}$$

is the "fat boundary" of Sonja Mouton [9],[10] •

Equality may fail in (5.1): as an example if $a = z \in A$ for the disc algebra then ([10] Example 2.3), with $T : A \to B = C(\mathbf{S})$,

5.6 $$\partial(a) = \partial(Ta) = \mathbf{S}\ ,$$

and hence

5.7 $$\sigma^{left}(L_a) = \mathbf{S} \subseteq \mathbf{D} = \sigma^{right}(L_a) = \sigma^{left}(a) = \sigma^{right}(a)\ .$$

It follows, since A is commutative, that for $a = z$

5.8 $$\sigma_{CI}(a) = \emptyset \neq \mathrm{int}\ \mathbf{D} = \sigma_{CI}(L_a)_\cap \sigma_{CI}(R_a)\ .$$

We remark finally that the consistent sdpectrum and the fat boundary are disjoint:

5.9 $$\sigma_{CI}(a)_\cap \partial(a) = \emptyset\ .$$

This is the analogue of (3.1) obtained by beefing up (3.5) to (5.4); nalternatively it is a simple observation [4] that everything in the closure of the invertibles is left-right consistent. Here the closure is of course the norm closure, but the proof is routed through what we call [2] the *spectral* closure.

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