SCHUR COMPLEMENTS IN C*– ALGEBRAS

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In this paper we introduce and study Schur complement of positive elements in a C^*-algebra and prove results on their extremal characterizations.

1 Introduction

Given a matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with A nonsingular, the classical Schur complement of A in M is the matrix

$$S = D - CA^{-1}B. \tag{1}$$

The formula (1) was first used by Schur [22], but the idea of the Schur complement goes back to Sylvester (1851), and the term Schur complement was introduced by E. Haynsworth [16].

In the beginning Schur complements were used in the theory of matrices. M.G. Krein [19] and W.N. Anderson and G.E. Trapp [4] extended the notion of Schur complements of matrices to shorted operators in Hilbert space operators, and Trapp defined the generalized Schur complement by replacing the ordinary inverse with the generalized inverse. Schur complements and generalized Schur complements were studied by a number of authors, have applications in statistics, matrix theory, electrical network theory, discrete-time regulator problem, sophisticated techniques and some other fields (see [20], [11], [10], [5], [6]).

In this paper we introduce and study the Schur complement of positive elements in a C^*-algebra $\mathcal{A}$ and among other things, we embark study the extremal characterizations of Schur complement.

Let $\mathcal{A}$ be a complex C^*-algebra with the unit 1. The Moore-Penrose inverse of an element a of $\mathcal{A}$ is the unique element $a^\dagger$ of $\mathcal{A}$ satisfying the equations

$$aa^\dagger a = a,\ a^\dagger aa^\dagger = a^\dagger,\ (aa^\dagger)^* = aa^\dagger,\ (a^\dagger a)^* = a^\dagger a$$

(see [14], [15], [17], [21]). The set of all $a \in \mathcal{A}$ that possess the *Moore-Penrose inverse* will be denoted by $\mathcal{A}^\dagger$. It is shown in ([14], [18]) that $a \in \mathcal{A}^\dagger$ if and only if $a \in a\mathcal{A}a$. We also write $\mathcal{A}^{-1}$ for the set of all invertible elements in $\mathcal{A}$. The word 'projection' will be reserved for an element q of $\mathcal{A}$ which is self-adjoint and idempotent, that is, $q^* = q = q^2$. In this paper $\mathcal{A}_h$ stands for the set of all selfadjoint elements of $\mathcal{A}$. The symbols $\mathcal{A}^\bullet{}_h$, $\mathcal{A}^\bullet{}_h$ and $\mathcal{A}_+$ denote the sets of all idempotent, projection and positive elements of $\mathcal{A}$, respectively. If $a, b \in \mathcal{A}_h$ and $a - b \in \mathcal{A}_+$, we write $a \geq b$ (or $b \leq a$). We say that $a \in \mathcal{A}$ is relatively regular, provided that there exists some $b \in \mathcal{A}$ such that $aba = a$. In this case b is called an *inner generalized inverse* of a. We use a^- to denote an arbitrary inner generalized inverse of a.

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Let $a \in \mathcal{A}$ and $s \in \mathcal{A}^{\bullet}{}_h$. Then we write

$$a = sas + sa(1-s) + (1-s)as + (1-s)a(1-s)$$

and use the notations

$$a_{11} = sas, \quad a_{12} = sa(1-s), \quad a_{21} = (1-s)as, \quad a_{22} = (1-s)a(1-s).$$

Every $s \in \mathcal{A}^{\bullet}{}_h$ induces a representation of arbitrary element $a \in \mathcal{A}$ given by the following matrix

$$a = \begin{pmatrix} sas & sa(1-s) \\ (1-s)as & (1-s)a(1-s) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Given an element $a \in \mathcal{A}$, let $\sigma(a)$ denote the *spectrum* of a and let L_a denote the *left regular representation* of a, i.e., $L_a(x) = ax, x \in \mathcal{A}$.

Let $B(X)$ denote the set of all bounded linear operators on a Banach space X. For an element T in $B(X)$ let $N(T)$ and $R(T)$ denote, respectively, the null space and the range of T. Recall that the *reduced minimum modulus* of T, $\gamma(T)$, is defined by

$$\gamma(T) = \inf\{\|Tz\|/ \operatorname{dist}(z, N(T)) : \ \operatorname{dist}(z, N(T)) > 0\}$$

and that $R(T)$ is closed if and only if $\gamma(T) > 0$. If there is an S in $B(X)$ such that $TST = T$, then $R(T)$ is closed and $\gamma(T) \geq 1/\|S\|$ ([13]). Let us recall that if $a \in \mathcal{A}^\dagger$, then it is known that $\|a^\dagger\| = 1/\gamma(L_a)$ ([21], [14]). Furthermore, (see [15]) if $0 \neq a \in \mathcal{A}_+$ then $\gamma(L_a) = \inf(\sigma(a) \setminus \{0\})$.

2 Preliminary results

We start with the following auxiliary result.

Lemma 2.1 *If $s \in \mathcal{A}^{\bullet}{}_h$ and $a \in \mathcal{A}^\dagger \cap s\mathcal{A}s$, then $a^\dagger \in s\mathcal{A}s$.*

P r o o f. Clearly, $a \in s\mathcal{A}s$ implies $a = sa = as = sas$. Thus,

$$\begin{aligned} &a(sa^\dagger s)a = aa^\dagger a = a, \ (sa^\dagger s)a(sa^\dagger s) = sa^\dagger aa^\dagger s = sa^\dagger s, \\ &(a(sa^\dagger s))^* = (aa^\dagger s)^* = (saa^\dagger s)^* = saa^\dagger s = a(sa^\dagger s), \\ &((sa^\dagger s)a)^* = (sa^\dagger a)^* = (sa^\dagger as)^* = sa^\dagger as = (sa^\dagger s)a, \end{aligned}$$

that is, $a^\dagger = sa^\dagger s \in s\mathcal{A}s$. □

Now we continue with the following extension of of Albert's results [2]. Let us remark that our methods of proof are new.

Theorem 2.2 *Let $a \in \mathcal{A}_h$, $s \in \mathcal{A}^{\bullet}{}_h$ and $a_{11} \in \mathcal{A}^\dagger$. Then $a \geq 0$ if and only if the following conditions are satisfied:*

(i) $a_{11} \geq 0$,

(ii) $a_{11}{a_{11}}^\dagger a_{12} = a_{12}$,

(iii) $a_{22} - {a_{12}}^* {a_{11}}^\dagger a_{12} \geq 0$.

P r o o f. Suppose that $a \geq 0$. Then there exist $h \in \mathcal{A}$ such that $a = hh^*$. Obviously, $a_{11} = sh(sh)^* \geq 0$. By [14, Theorem 7] and [17, Theorem 2.4], it follows that sh is relatively regular and

$$\begin{aligned} a_{11}{a_{11}}^\dagger &= (sh)(sh)^*((sh)(sh)^*)^\dagger \\ &= (sh)(sh)^*((sh)^*)^\dagger (sh)^\dagger \\ &= (sh)((sh)^\dagger (sh))^* (sh)^\dagger \\ &= (sh)(sh)^\dagger. \end{aligned}$$

Hence,

$$a_{11}{a_{11}}^{\dagger}a_{12}=(sh)(sh)^{\dagger}shh^{*}(1-s)=shh^{*}(1-s)=a_{12}.$$

Finally,

$$\begin{aligned} a_{22}-{a_{12}}^{*}a_{11}^{\dagger}a_{12} &= a_{22}-{a_{12}}^{*}((sh)(sh)^{*})^{\dagger}a_{12} \\ &= a_{22}-(1-s)hh^{*}s((sh)^{\dagger})^{*}(sh)^{\dagger}shh^{*}(1-s) \\ &= a_{22}-(1-s)h(sh)^{\dagger}shh^{*}(1-s) \\ &= (1-s)h(1-(sh)^{\dagger}(sh))((1-s)h)^{*} \\ &= [(1-s)h(1-(sh)^{\dagger}(sh))][(1-s)h(1-(sh)^{\dagger}(sh))]^{*}\geq 0. \end{aligned}$$

On the contrary, suppose that the conditions (1), (2) and (3) hold. It is easy to see that

$$(1-{a_{12}}^{*}{a_{11}}^{\dagger})a(1-{a_{12}}^{*}{a_{11}}^{\dagger})^{*}=a_{11}+(a_{22}-{a_{12}}^{*}a_{11}^{\dagger}a_{12})\geq 0.$$

Let us remark that $1-{a_{12}}^{*}{a_{11}}^{\dagger}$ is invertible, and that $(1-{a_{12}}^{*}{a_{11}}^{\dagger})^{-1}=1+{a_{12}}^{*}{a_{11}}^{\dagger}$. Thus,

$$a=(1+{a_{12}}^{*}{a_{11}}^{\dagger})(a_{11}+(a_{22}-{a_{12}}^{*}a_{11}^{\dagger}a_{12}))(1+{a_{12}}^{*}{a_{11}}^{\dagger})^{*}\geq 0,$$

and the proof is complete. □

As a corollary, we obtain the following

Corollary 2.3 *Let $a\in\mathcal{A}_h$, $s\in{\mathcal{A}^{\bullet}}_h$ and $a_{22}\in\mathcal{A}^{\dagger}$. Then $a\geq 0$ if and only if the following conditions are satisfied:*

(i) $a_{22}\geq 0$,

(ii) $a_{22}{a_{22}}^{\dagger}{a_{12}}^{*}={a_{12}}^{*}$,

(iii) $a_{11}-a_{12}{a_{22}}^{\dagger}{a_{12}}^{*}\geq 0$.

P r o o f. This follows by Theorem 2.2 with s replaced by $1-s$. □

We continue with a C^{*}-algebra type theorem of Krein [19] (see also [9]).

Theorem 2.4 *Suppose that $a\in\mathcal{A}_+$, $s\in{\mathcal{A}^{\bullet}}_h$, a_{22} is relatively regular, and set $\mathcal{M}(a,s)=\{x{\in}\mathcal{A}:0{\leq}x{\leq}a,\ sx=x\}$. Then*

$$a_{11}-a_{12}{a_{22}}^{\dagger}a_{21}=\max\mathcal{M}(a,s).$$

P r o o f. Set $b=a_{11}-a_{12}{a_{22}}^{\dagger}a_{21}$. By Corollary 2.3 we have

$$\begin{aligned} b &= a_{11}+a_{22}{a_{22}}^{\dagger}a_{21}-a(1-s){a_{22}}^{\dagger}a_{21} \\ &= a_{11}+a_{21}+a(1-s){a_{22}}^{\dagger}a_{22}-a(1-s){a_{22}}^{\dagger}(1-s)a \\ &= a-a(1-s){a_{22}}^{\dagger}(1-s)a. \end{aligned}$$

Hence,

$$a-b=a(1-s){a_{22}}^{\dagger}(1-s)a\geq 0,$$

that is, $b\leq a$. Again by Theorem 2.2, it follows that $b=a_{11}-a_{12}{a_{22}}^{\dagger}a_{21}\geq 0$. Obviously, $sb=b$, so $b{\in}\mathcal{M}(a,s)$. Let us prove that $x\in\mathcal{M}(a,s)$ implies $x\leq b$. Suppose that $x\in\mathcal{M}(a,s)$. Then $0\leq x\leq a,\ sx=x$, and it is easy to prove that $x\in s\mathcal{A}s$. Now $a-x\geq 0$ implies $x\leq b$. □

Finally, following Albert [2], Carlson, Haynsworth, and Markham [8], if $a\in\mathcal{A}_+$, $s\in{\mathcal{A}^{\bullet}}_h$ and $a_{11}\in\mathcal{A}^{\dagger}$, we define the Schur complement of a with respect to s by

$$s(a)=a_{22}-a_{21}{a_{11}}^{\dagger}a_{12}. \tag{2}$$

Let us remark that, by Theorem 2.4,

$$s(a)=\max\mathcal{M}(a,1-s).$$

3 Extremal Characterizations

In this section, we give short proofs for the extremal characterizations of the generalized Schur complement $s(a)$. Among other things, our results generalize some results for matrices [9].

Lemma 3.1 *Suppose that $s \in \mathcal{A}^{\bullet}{}_h$, $a, b \in \mathcal{A}_+$ and that $a_{11}, b_{11}, a_{11} + b_{11}$ are relatively regular. Then*

$$(a_{12} + b_{12})^*(a_{11} + b_{11})^\dagger(a_{12} + b_{12}) \le a_{12}{}^* a_{11}{}^\dagger a_{12} + b_{12}^* b_{11}^\dagger b_{12}. \tag{3}$$

P r o o f. Set $c = as(sas)^\dagger sa$. Clearly, $(sas)^\dagger \ge 0$ implies $c \ge 0$. Also, we have that $c_{11} = scs = sas = a_{11}$. By Theorem 2.2, it follows that $c_{12} = sc(1-s) = sas(sas)^\dagger sa(1-s) = a_{11}a_{11}{}^\dagger a_{12} = a_{12}$. Obviously, $c = c^*$, and $c_{21} = c_{12}^* = a_{12}{}^*$. Also, $c_{22} = (1-s)c(1-s) = (1-s)as(sas)^\dagger sa(1-s) = a_{12}{}^* a_{11}{}^\dagger a_{12}$. Hence, c has the matrix representation

$$c = \begin{pmatrix} a_{11} & a_{12} \\ a_{12}{}^* & a_{12}{}^* a_{11}{}^\dagger a_{12} \end{pmatrix}.$$

Now, set $d = bs(sbs)^\dagger sb$. From the proof for c, we conclude that $d \ge 0$ and that d has the matrix representation

$$d = \begin{pmatrix} b_{11} & b_{12} \\ b_{12}^* & b_{12}^* b_{11}^\dagger b_{12} \end{pmatrix}.$$

Thus,

$$c + d = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ (a_{12} + b_{12})^* & a_{12}{}^* a_{11}{}^\dagger a_{12} + b_{12}^* b_{11}^\dagger b_{12} \end{pmatrix} \ge 0. \tag{4}$$

Now, Theorem 2.2 (3) and (4) imply (3). $\square$

Theorem 3.2 *Suppose that $s \in \mathcal{A}^{\bullet}{}_h$, $a \in \mathcal{A}_+$, a_{11} is relatively regular, $z \in (1-s)\mathcal{A}s$ and $q = z + 1 - s$. Then*

$$qaq^* \ge s(a) \tag{5}$$

and

$$qaq^* = s(a) \tag{6}$$

if and only if

$$(z + a_{12}{}^* a_{11}{}^\dagger) a_{11} = 0. \tag{7}$$

P r o o f. Because $z = (1-s)zs$, we have

$$\begin{aligned} qaq^* &= ((1-s)zs + (1-s))a(sz^*(1-s) + (1-s)) \\ &= s(a) + (z + a_{12}{}^* a_{11}{}^\dagger) a_{11} (a_{11}{}^\dagger a_{12} + z^*). \end{aligned}$$

By Theorem 2.2, $s(a) \ge 0$ and $a_{11} \ge 0$. Furthermore,

$$\begin{aligned} &(z + a_{12}{}^* a_{11}{}^\dagger) a_{11} (a_{11}{}^\dagger a_{12} + z^*) \\ &= ((z + a_{12}{}^* a_{11}{}^\dagger) a_{11}^{1/2})((z + a_{12}{}^* a_{11}{}^\dagger) a_{11}^{1/2})^* \ge 0, \end{aligned}$$

and we obtain (5). Now, by (8), clearly we have (6) if and only if

$$(z + a_{12}{}^* a_{11}{}^\dagger) a_{11} (a_{11}{}^\dagger a_{12} + z^*) = 0,$$

which is equivalent to (7). $\square$

Corollary 3.3 *Suppose that $s \in \mathcal{A}^{\bullet}{}_h$, $a \in \mathcal{A}_+$, a_{11} is relatively regular. Then*

$$\begin{aligned} s(a) &= \min\{qaq^* : q = z + 1 - s,\ z \in (1-s)\mathcal{A}s\} \\ &= (1 - s - {a_{12}}^* {a_{11}}^{\dagger}) a (1 - s - {a_{12}}^* {a_{11}}^{\dagger})^*. \end{aligned}$$

P r o o f. By (5), (6) and (7), we can choose $z = -{a_{12}}^* {a_{11}}^{\dagger}$. □

Corollary 3.4 *If $a \in \mathcal{A}_+$, a and a_{11} are relatively regular, then $s(a)$ is relatively regular and $a^{\dagger}$ is an inner inverse of $s(a)$, that is,*

$$s(a) = s(a) a^{\dagger} s(a). \tag{8}$$

P r o o f. By Corollary 3.3, it follows that $s(a) = uau^*$, where $u = 1 - s - {a_{12}}^* {a_{11}}^{\dagger}$. Now,

$$s(a) = (ua) a^{\dagger} (au^*) = (ua) a^{\dagger} (ua)^*. \tag{9}$$

By Theorem 2.2, we know that $a_{11} \geq 0$, so ${a_{11}}^{\dagger} a_{11} = a_{11} {a_{11}}^{\dagger}$. Now, by Lemma 2.1 and Theorem 2.2,

$$\begin{aligned} ua &= (1 - s - {a_{12}}^* {a_{11}}^{\dagger}) a \\ &= {a_{12}}^* + a_{22} - {a_{12}}^* a_{11}^{\dagger} a_{11} - {a_{12}}^* a_{11}^{\dagger} a_{12} \\ &= ({a_{12}}^* {a_{12}}^* a_{11}^{\dagger} a_{11}) + (a_{22} - {a_{12}}^* a_{11}^{\dagger} a_{12}) \\ &= s(a). \end{aligned}$$

Again by Theorem 2.2 (3), $s(a) \geq 0$, and from (9) and (10) we obtain (8). □

Now as a corollary we obtain an estimation for the spectrum of a and the spectrum of $s(a)$.

Corollary 3.5 *If $a \in \mathcal{A}_+$, a and a_{11} are relatively regular, then*

$$\inf(\sigma(a) \setminus \{0\}) \leq \inf(\sigma(s(a)) \setminus \{0\}). \tag{10}$$

P r o o f. By (8) we have

$$\gamma(L_{s(a)}) \geq \frac{1}{\|a^{\dagger}\|} = \gamma(L_a),$$

and then by [15] (see (1.3)) we obtain (10). □

Theorem 3.6 *Suppose that $s \in \mathcal{A}^{\bullet}{}_h$, $a, b \in \mathcal{A}_+$, and that $a_{11},\ b_{11}$ and $a_{11} + b_{11}$ are relatively regular.*

(i) *We have*

$$s(a + b) \geq s(a) + s(b).$$

Furthermore,

$$s(a + b) = s(a) + s(b) \tag{11}$$

if and only if there exist $z \in (1-s)\mathcal{A}s$ such that

$$(z + {a_{12}}^* {a_{11}}^{\dagger}) a_{11} = (z + b_{12}^* b_{11}^{\dagger}) b_{11} = 0. \tag{12}$$

(ii) *If $a \geq b$, then*

$$s(a) \geq s(b),$$

and the equality

$$s(a) = s(b) \tag{13}$$

holds if and only if there exist $z \in (1-s)\mathcal{A}s$ satisfying (12) and

$$(z - 1 - s)(a - b) = 0. \tag{14}$$

Proof. (1) By Lemma 3.1 we have

$$\begin{aligned} s(a+b) &= a_{22}+b_{22}-(a_{12}+b_{12})^*(a_{11}+b_{11})^\dagger(a_{12}+b_{12}) \\ &\geq a_{22}+b_{22}-{a_{12}}^*{a_{11}}^\dagger a_{12}-b_{12}^*b_{11}^\dagger b_{12} \\ &= s(a)+s(b). \end{aligned}$$

If there exist $z \in (1-s)\mathcal{A}s$ such that (12) holds, then by Theorem 3.2 and (8) we have $s(a) = qaq^*$ and $s(b) = qbq^*$, where $q = z+1-s$. Thus,

$$\begin{aligned} s(a)+s(b) &= q(a+b)q^* \\ &\geq \min\{q(a+b)q^* : q = z+1-s,\ z \in (1-s)\mathcal{A}s\} \\ &= s(a+b). \end{aligned}$$

Now suppose that (11) holds and let us show (12). By Theorem 3.2, there exist $z \in (1-s)\mathcal{A}s$ such that for $q = z+1-s$ we have

$$s(a+b) = q(a+b)q^* = qaq^* + qbq^* = s(a)+p_1+s(b)+p_2, \tag{15}$$

where $p_1 = (z+{a_{12}}^*{a_{11}}^\dagger)a_{11}({a_{11}}^\dagger a_{12}+z^*)$ and $p_2 = (z+b_{12}^*b_{11}^\dagger)b_{11}(b_{11}^\dagger b_{12}+z^*)$. Clearly, $p_1, p_2 \geq 0$, and by our assumption (11), we see that (15) implies $p_1+p_2 = 0$. Thus $p_1 = p_2 = 0$, which is equivalent to (12).

To prove (2), suppose that $a \geq b$. Let $z \in (1-s)\mathcal{A}s$ be such that $(z+{a_{12}}^*{a_{11}}^\dagger)a_{11} = 0$. By Theorem 3.2 we have

$$s(a) = (z-1-s)a(z-1-s)^* \geq (z-1-s)b(z-1-s)^* \geq s(b). \tag{16}$$

To prove (13), let us remark that the second inequality holds in (16) if and only if z satisfies $(z+b_{12}^*b_{11}^\dagger)b_{11} = 0$, and the first inequality holds in (16) if and only if

$$(z-1-s)(a-b)(z-1-s)^* = 0.$$

Since $a-b \geq 0$, the last condition is equivalent to (14). $\square$

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