EXPLICIT SOLUTION OF THE OPERATOR EQUATION $A^*X + X^*A = B$

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November 15, 2005

Abstract

In this paper we find the explicit solution of the equation

$$A^*X + X^*A = B$$

for linear bounded operators on Hilbert spaces, where X is the unknown operator. This solution is expressed in terms of the Moore-Penrose inverse of the operator A. Thus, results of J. H. Hodges (Ann. Mat. Pura Appl. **44** (1957) 245–550) are extended to the infinite dimensional settings.

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1 Introduction

In this paper H and K denote arbitrary Hilbert spaces. We use $\mathcal{L}(\mathcal{H},\mathcal{K})$ to denote the set of all linear bounded operators from H to K. Also, $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H},\mathcal{H})$.

Key words and phrases: Operator equations, Moore-Penrose inverse.
2000 *Mathematics Subject Classification*: 47A62, 47A52, 15A24.
Supported by Grant No. 144003 of the Ministry of Science, Republic of Serbia.

For given operators $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{L}(\mathcal{H})$, we are interested in finding the solution $X \in \mathcal{L}(\mathcal{H},\mathcal{K})$ of the equation

$$A^*X + X^*A = B \tag{1}$$

This equation is considered for matrices over a finite field (see [7]).

We mention similar matrix equations, which have applications in control theory. These equations are investigated for matrices over fields, mostly $\mathbf{R}$ or $\mathbf{C}$. The equation $CX - XA^\top = B$ is the Sylvester equation [8]. More general equation $AX - XF = BY$ is considered in [10]. One special and important case is the Lyapunov equation $AX + XA^\top = B$ [9]. Also, the generalized Sylvester equation $AV+BW = EVJ+R$ with unknown matrices V and W, has many applications in linear systems theory (see [4]).

Present paper deals with the extension of results from [7] to infinite dimensional settings.

For $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ we use $\mathcal{R}(\mathcal{A})$ and $\mathcal{N}(A)$, respectively, to denote the range and the null-space of A. The Moore-Penrose inverse of A, denoted by $A^\dagger$, is the unique operator $A \in \mathcal{L}(\mathcal{K},\mathcal{H})$ satisfying the following conditions:

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A.$$

It is well-known that $A^\dagger$ exists if and only if $\mathcal{R}(\mathcal{A})$ is closed. For properties and applications of the Moore-Penrose inverse see [1, 3, 2, 5].

Let $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ have a closed range. Then $AA^\dagger$ is the orthogonal projection from K onto $\mathcal{R}(\mathcal{A})$ (parallel to $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$) and $A^\dagger A$ is the orthogonal projection from H onto $\mathcal{R}(\mathcal{A}^\dagger) = \mathcal{R}(\mathcal{A}^*)$ (parallel to $\mathcal{N}(A)$). It follows that A has the following matrix form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(\mathcal{A}^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(\mathcal{A}) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible. Now, the operator $A^\dagger$ has the following form:

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(\mathcal{A}) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(\mathcal{A}^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

Using these matrix forms of operators with closed ranges and properties of the Moore-Penrose inverse, we solve the equation (1).

2 Results

First, we solve the equation (1) in the case when A is invertible. It can easily be seen that the proof of the following Theorem 2.1 is valid in rings with involution.

Theorem 2.1 *Let $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ be invertible and $B \in \mathcal{L}(\mathcal{H})$. Then the following statements are equivalent:*

(a) *There exists a solution $X \in \mathcal{L}(\mathcal{H},\mathcal{K})$ of the equation* (1)*.*
(b) *$B = B^*$.*

If (a) *or* (b) *is satisfied, then any solution of the equation* (1) *has the form*

$$X = \frac{1}{2}(A^*)^{-1}B + ZA \tag{2}$$

where $Z \in \mathcal{L}(\mathcal{K})$ satisfy $Z^* = -Z$.

Proof. (a)$\to$(b): Obvious.

(b)$\to$(a): It is easy to see that any operator X of the form (2) is a solution of the equation (1). On the other hand, let X be any solution of (1). Then $X = (A^*)^{-1}B - (A^*)^{-1}X^*A$ and $(A^*)^{-1}X^* = (A^*)^{-1}BA^{-1} - XA^{-1}$. We have

$$\begin{aligned} X &= \frac{1}{2}(A^*)^{-1}B + \left(\frac{1}{2}(A^*)^{-1}BA^{-1} - (A^*)^{-1}X^*\right)A \\ &= \frac{1}{2}(A^*)^{-1}B + \left(\frac{1}{2}\left[(A^*)^{-1}X^* + XA^{-1}\right] - (A^*)^{-1}X^*\right)A \\ &= \frac{1}{2}(A^*)^{-1}B + \frac{1}{2}(XA^{-1} - (A^*)^{-1}X^*)A. \end{aligned}$$

Taking $Z = \frac{1}{2}\left(XA^{-1} - (A^*)^{-1}X^*\right)$, we get $Z^* = -Z$. $\square$

Now, we solve the equation (1) in the case when A has a closed range.

Theorem 2.2 *Let $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ have a closed range and $B \in \mathcal{L}(\mathcal{H})$. Then the following statements are equivalent:*

(a) *There exists a solution $X \in \mathcal{L}(\mathcal{H},\mathcal{K})$ of the equation* (1)*.*
(b) *$B = B^*$ and $(I - A^\dagger A)B(I - A^\dagger A) = 0$.*

If (a) *or* (b) *is satisfied, then any solution of the equation* (1) *has the form*

$$X = \frac{1}{2}(A^*)^\dagger BA^\dagger A + (A^*)^\dagger B(I - A^\dagger A) + (I - AA^\dagger)Y + AA^\dagger ZA, \qquad (3)$$

where $Z \in \mathcal{L}(\mathcal{K})$ *satisfies* $A^*(Z + Z^*)A = 0$*, and* $Y \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ *is arbitrary.*

Proof. (a)→(b): Obviously, $B^* = B$. Also,

$$\begin{aligned} & (I - A^\dagger A)B(I - A^\dagger A) = (I - A^\dagger A)(A^*X + X^*A)(I - A^\dagger A) \\ = \ & (A^* - (AA^\dagger A)^*)X(I - A^\dagger A) + (I - A^\dagger A)X^*A(I - A^\dagger A) = 0. \end{aligned}$$

(b)→(a): Notice that the condition $(I - A^\dagger A)B(I - A^\dagger A) = 0$ is equivalent to $B = A^\dagger AB + BA^\dagger A - A^\dagger ABA^\dagger A$. Any operator X of the form (3) is a solution of the equation (1).

On the other hand, suppose that X is a solution of the equation (1). Since $\mathcal{R}(A)$ is closed, we have $H = \mathcal{R}(\mathcal{A}^*) \oplus \mathcal{N}(\mathcal{A})$ and $K = \mathcal{R}(\mathcal{A}) \oplus \mathcal{N}(\mathcal{A}^*)$. Now, A has the matrix form $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(\mathcal{A}^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(\mathcal{A}) \\ \mathcal{N}(A^*) \end{bmatrix}$, where A_1 is invertible. Conditions $B = B^*$ and $(I - A^\dagger A)B(I - A^\dagger A) = 0$ imply that B has the form $B = \begin{bmatrix} B_1 & B_2 \\ B_2^* & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(\mathcal{A}) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} R(A) \\ \mathcal{N}(A^*) \end{bmatrix}$, where $B_1^* = B_1$. Let X have the form $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(\mathcal{A}^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(\mathcal{A}) \\ \mathcal{N}(A^*) \end{bmatrix}$. Then $A^*X + X^*A = B$ implies $A_1^*X_{11} + X_{11}^*A_1 = B_1$ and $A_1^*X_{12} = B_2$. Hence, $X_{12} = (A_1^*)^{-1}B_2$. Since A_1 is invertible, from Theorem 2.1 it follows that X_{11} has the form $X_{11} = \frac{1}{2}(A_1^*)^{-1}B_1 + Z_1A_1$, for some operator $Z_1 \in \mathcal{L}(\mathcal{R}(\mathcal{A}))$ satisfying $Z_1^* = -Z_1$. Hence,

$$X = \begin{bmatrix} \frac{1}{2}(A_1^*)^{-1}B_1 + Z_1A_1 & (A_1^*)^{-1}B_2 \\ X_{21} & X_{22} \end{bmatrix},$$

X_{21} and X_{22} can be taken arbitrary. Let $Y = \begin{bmatrix} Y_{11} & Y_{12} \\ X_{21} & X_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(\mathcal{A}^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(\mathcal{A}) \\ \mathcal{N}(A^*) \end{bmatrix}$ and $Z = \begin{bmatrix} Z_1 & Z_{12} \\ -Z_{12}^* & Z_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(\mathcal{A}) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(\mathcal{A}) \\ \mathcal{N}(A^*) \end{bmatrix}$, where Y_{11}, Y_{12} and Z_2 are arbitrary. Notice that $A^*(Z + Z^*)A = 0$.

Then $\frac{1}{2}(A^*)^\dagger BA^\dagger A = \begin{bmatrix} \frac{1}{2}(A_1)^{-1}B_1 & 0 \\ 0 & 0 \end{bmatrix}$, $(A^*)^\dagger B(I - A^\dagger A) = \begin{bmatrix} 0 & (A_1^*)^{-1}B_2 \\ 0 & 0 \end{bmatrix}$, $(I - AA^\dagger)Y = \begin{bmatrix} 0 & 0 \\ X_{21} & X_{22} \end{bmatrix}$ and $AA^\dagger ZA = \begin{bmatrix} Z_1A_1 & 0 \\ 0 & 0 \end{bmatrix}$. Consequently, X has the form (3). $\square$

It is a consequence of the Gelfand-Naimark-Segal theorem and the Harte-Mbekhta theorem [6] that Theorem 2.2 holds in C^*-algebras also.

By exactly similar arguments, we obtain the following analogue of Theorem 2.2, in which equation (1) is replaced by

$$A^*X - X^*A = B. \tag{4}$$

Theorem 2.3 *Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ have a closed range and $B \in \mathcal{L}(\mathcal{H})$. Then the following statements are equivalent:*

(a) *There exists a solution $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ of the equation* (4)*.*
(b) *$B = -B^*$ and $(I - A^\dagger A)B(I - A^\dagger A) = 0$.*

If (a) *or* (b) *is satisfied, then any solution of the equation* (4) *has the form*

$$X = \frac{1}{2}(A^*)^\dagger BA^\dagger A + (A^*)^\dagger B(I - A^\dagger A) + (I - AA^\dagger)Y + AA^\dagger ZA, \tag{5}$$

where $Z \in \mathcal{L}(\mathcal{K})$ satisfies $A^*(Z - Z^*)A = 0$, and $Y \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is arbitrary.

Acknowledgement. I am grateful to both of referees for helpful comments concerning the paper.

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