Characterizations of normal, hyponormal and EP operators

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Abstract

In this paper normal and hyponormal operators with closed ranges, as well as EP operators, are characterized in arbitrary Hilbert spaces. All characterizations involve generalized inverses. Thus, recent results of S. Cheng and Y. Tian (Linear Algebra Appl. 375 (2003), 181–195) are extended to infinite dimensional settings.

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1 Introduction

There are many conditions for a linear bounded operator on a Hilbert space to be normal (see [12, 13, 14]). Recently, S. Chen and Y. Tian (see [8]) obtained several results characterizing normal and EP complex matrices. In this paper we characterize normal and hyponormal operators with closed ranges, as well as EP operators on arbitrary Hilbert spaces. Using properties of operator matrices, we obtain an extension of results from [8] in infinite

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dimensional settings. These characterizations are obtained using the Moore-Penrose or the group inverse of a linear bounded operator with a closed range.

In this paper we use H and K to denote Hilbert spaces and $\mathcal{L}(H,K)$ to denote the set of all linear bounded operators from H to K. The Moore-Penrose inverse of $A \in \mathcal{L}(H,K)$ is denoted by $A^\dagger$ (see [3], page 40). We use $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, to denote the range and the null-space of $A \in \mathcal{L}(H,K)$. For given $A \in \mathcal{L}(H,K)$ the operator $A^\dagger \in \mathcal{L}(K,H)$ exists if and only if $\mathcal{R}(A)$ is closed. If $A^\dagger$ exists, then A is called relatively regular, or Moore-Penrose invertible.

An operator $A \in \mathcal{L}(H)$ is normal, if $A^*A = AA^*$. If $A = A^*$, then A is Hermitian, or selfadjoint. The inner product $(\cdot,\cdot)$ in H defines the natural order of Hermitian operators. Namely, if A and B are Hermitian, then $A \leq B$ if and only if $(Ax,x) \leq (Bx,x)$ for all $x \in H$. If $A = A^*$ and $A \geq 0$, then A is non-negative. If $A \geq 0$ and A is invertible, then A is positive and we write $A > 0$. If $A^*A \geq AA^*$, then A is hyponormal.

The notion of EP operators in well-known (see [2, 3, 6, 7, 8, 9, 16, 17, 18, 19, 20]). An operator $A \in \mathcal{L}(H)$ is EP if $\mathcal{R}(A)$ is closed and $[A, A^\dagger] = 0$. Here $[A,B] = AB - BA$. The class of all normal operators with closed range is a subclass of EP operators, while the class of all hyponomal operators with closed range is not a subclass of EP operators. Also, the class of all EP operators is not contained in the set of all hyponormal operators. An elementary observations shows that a closed range operator A is EP if and only if $\mathcal{R}(A) = \mathcal{R}(A^*)$.

If A is an EP operator, then $A^\dagger$ is also the Drazin inverse of A, or, more precisely, the group inverse of A (see [3] pages 156 and 163). In this case the Drazin index of A (denoted by $\mathrm{ind}A$) is not greater than 1. Moreover, A has the following nice matrix decomposition with respect to the orthogonal sum of subspaces:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}, \qquad (1.1)$$

where A_1 is invertible. In this case $A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$.

More results on generalized inverses can be found in [3, 4, 6, 15].

If $A \in \mathcal{L}(H)$, then the ascent and descent of A, respectively, are denoted by $\mathrm{asc}A$ and $\mathrm{dsc}A$.

We prove several new characterizations of normal, hyponormal and EP operators.

The paper is organized as follows. Section 2 contains some auxiliary results. In Section 3 characterizations of normal operators with closed ranges are given. Section 4 is devoted to hyponormal operators. In Section 4, EP operators on Hilbert spaces are investigated.

Thus, results from [8] are extended to infinite dimensional settings. Notice that in [8] the finite dimensional technique is involved, mostly properties of a rank of complex matrices. In this paper a systematic use is made of operator matrix representations, and of generalized inverses of closed range operators.

2 Auxiliary results

All results are proved using properties of operator matrices and various matrix representations of relatively regular operators. First, we state one useful result.

Lemma 2.1 *Let $A \in \mathcal{L}(H)$ have a closed range. Then the operator A has the following three matrix representations with respect to the orthogonal sums of subspaces:*

(2.1.1) $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}$, *where A_1 is invertible.*

(2.1.2) $A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}$, *where $B = A_1A_1^* + A_2A_2^*$ maps $\mathcal{R}(A)$ into itself and $B > 0$.*

(2.1.3) $A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}$, *where $B = A_1^*A_1 + A_2^*A_2$ maps $\mathcal{R}(A^*)$ into itself and $B > 0$.*

Here A_i denotes different operators in any of these three cases.

The proof of Lemma 2.1 is straightforward.

We need the following result (see [21] for a finite dimensional case, and [11] for an infinite dimensional case). Recall that A has a generalized Drazin inverse A^d if and only if 0 is not an accumulation spectral point of A.

Lemma 2.2 *Let X and Y be Banach spaces, $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y,X)$. Then A and B have generalized Drazin inverses if and only if $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ has the generalized Drazin inverse, which is given by*

$$M^d = \begin{bmatrix} A^d & S \\ 0 & B^d \end{bmatrix}$$

for some $S \in \mathcal{L}(Y,X)$. Moreover, $\mathrm{ind}M < \infty$ *if and only if* $\mathrm{ind}A < \infty$ *and* $\mathrm{ind}B < \infty$.

We need the following important result (see [5, 22]).

Lemma 2.3 *Let $A \in \mathcal{L}(H)$.*

(a) *If* $\mathrm{asc}A < \infty$ *and* $\mathrm{dsc}A < \infty$*, then* $\mathrm{asc}A = \mathrm{dsc}A$.

(b) *If at least one of the quantities* $\dim\mathcal{N}(T)$, $\dim\mathcal{N}(T^*)$ *is finite, then* $\mathrm{asc}(T) < \infty$ *implies* $\dim\mathcal{N}(T) \leq \dim\mathcal{N}(T^*)$*, and* $\mathrm{dsc}(T) < \infty$ *implies* $\dim\mathcal{N}(T^*) \leq \dim\mathcal{N}(T)$.

The following result is proved in [1] for a finite dimensional case. The infinite dimensional case is proved in [10].

Lemma 2.4 *Let $M \in \mathcal{L}(H)$ be selfadjoint and let M have the decomposition (with respect to the orthogonal sum of subspaces of H):*

$$M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix},$$

such that A has a closed range. Then $M \geq 0$ if and only if the following hold:

(1) $A \geq 0$;
(2) $AA^\dagger B = B$;
(3) $C - B^*A^\dagger B \geq 0$.

Analogous conditions can be formulated in the case when C has a closed range.

We also need the following elementary result.

Lemma 2.5 *Let $A \in \mathcal{L}(H)$. If A and $AA^* + A^*A$ have closed ranges, then $(AA^* + A^*A)(AA^* + A^*A)^\dagger AA^* = AA^*$, i.e. $\mathcal{R}(AA^*) \subset \mathcal{R}(AA^* + A^*A)$.*

Proof. Let $M = \begin{bmatrix} AA^* + A^*A & AA^* \\ AA^* & AA^* \end{bmatrix}$. As A is a closed range operator, so is AA^*. Since $AA^* \geq 0$, $AA^*(AA^*)^\dagger AA^* = AA^*$ and $AA^* + A^*A - AA^*(AA^*)^\dagger AA^* = A^*A \geq 0$, by Lemma 2.4 it follows that $M \geq 0$. Applying part (2) of Lemma 2.4 to M, we obtain that $(AA^* + A^*A)(AA^* + A^*A)^\dagger AA^* = AA^*$ is satisfied. $\Box$

The following example (actually, a counter-example) is interesting and it is related to results obtained in Sections 3 and 5. This example illustrates the difference between the finite dimensional case (considered in [8]) and the infinite dimensional case (which is considered in this paper).

Example 2.1 *Consider the real Hilbert space ℓ_2 and let $A \in \mathcal{L}(\ell_2)$ be the left shift, i.e. $A(x_1, x_2, \dots) = (x_2, x_3, \dots)$. Then $A^*(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ and $A^\dagger = A^*$. In this case $AA^\dagger = I$ and $A^\dagger A(x_1, x_2, \dots) = (0, x_2, x_3, \dots)$. Let $B = A^*$. Then $B^* = B^\dagger = A$. In this case $(BB^*)^\dagger = BB^*$. Let $X = (A^*)^2$. Operators A and B are neither normal, nor EP, but they satisfy the following equalities (which in the finite dimensional case would ensure the normality or the EP property) :*

(1) $BB^\dagger B^* BBB^\dagger = BB^*$;
(2) $B^*B(BB^*)^\dagger B^*B = BB^*$;
(3) $AX = A^*$, $(A^\dagger)^* X = A^\dagger$;
(4) $AA^\dagger = A^2(A^\dagger)^2$;
(5) $AAA^\dagger + (AAA^\dagger)^* = A + A^*$.

It is important to mention that $\mathrm{ind}A = \infty$, $\mathrm{asc}A = \infty$ *and* $\mathrm{dsc}B = \infty$ *hold.*

3 Normal operators

In this section we characterize normal operators with closed ranges on arbitrary Hilbert spaces. Recall that a normal operator A has closed range if and only if it has the group inverse $A^\#$.

Theorem 3.1 *Let $A \in \mathcal{L}(H)$ have a closed range. Then the following statements are equivalent:*

(1) *A is normal;*
(2) $A(AA^*A)^\dagger = (AA^*A)^\dagger A$;

(3) $A(A^* + A^\dagger) = (A^* + A^\dagger)A$;
(4) $A^\dagger(A + A^*) = (A + A^*)A^\dagger$;
(5) $\mathrm{ind}A \le 1$ *and* $A^\# A^* = A^* A^\#$;
(6) $\mathrm{dsc}A < \infty$ *and* $A^*A(AA^*)^\dagger A^*A = AA^*$;
(7) $\mathrm{asc}A < \infty$ *and* $AA^*(A^*A)^\dagger AA^* = A^*A$;
(8) *There exists some* $X \in \mathcal{L}(H)$ *such that* $AA^*X = A^*A$ *and* $A^*AX = AA^*$;
(9) $\mathrm{asc}A < \infty$ *and there exists some* $X \in \mathcal{L}(H)$ *such that* $AX = A^*$ *and* $(A^\dagger)^*X = A^\dagger$.

Proof. Property (1) implies conditions (2)-(9); this is either elementary or follow from (1.1).

(2)$\Rightarrow$(1): Let A have the decomposition (2.1.2). Then $AA^* = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$, where $B = A_1A_1^* + A_2A_2^* > 0$. In this case let $C = AA^*A = \begin{bmatrix} BA_1 & BA_2 \\ 0 & 0 \end{bmatrix}$. Then $\mathcal{R}(C) = \mathcal{R}(A)$ is closed and $CC^* = \begin{bmatrix} B^3 & 0 \\ 0 & 0 \end{bmatrix}$. We get $(AA^*A)^\dagger = C^*(CC^*)^\dagger = \begin{bmatrix} A_1^*B^{-2} & 0 \\ A_2^*B^{-2} & 0 \end{bmatrix}$. From $A(AA^*A)^\dagger = (AA^*A)^\dagger A$ we get the following identities: $B^{-1} = A_1^*B^{-2}A_1$ and $A_2^*B^{-2}A_2 = 0$. Hence, $A_2 = 0$ and $B = A_1A_1^* > 0$. It follows that A_1 is right invertible.

Since $A_1 \in \mathcal{L}(\mathcal{R}(A))$ and $\mathcal{R}(A) = \mathcal{R}(A_1^*) \oplus \mathcal{N}(A_1)$, consider the matrix decomposition $A_1 = \begin{bmatrix} U & 0 \\ V & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A_1^*) \\ \mathcal{N}(A_1) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A_1^*) \\ \mathcal{N}(A_1) \end{bmatrix}$. Obviously, A_1 satisfies the identity $A_1(A_1A_1^*A_1)^\dagger = (A_1A_1^*A_1)^\dagger A_1$. Now, $A_1^*A_1 = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$, where $E = U^*U + V^*V : \mathcal{R}(A_1^*) \to \mathcal{R}(A_1^*)$ is positive and invertible. If $D = A_1A_1^*A_1$, then D has closed range and $D^\dagger = (D^*D)^\dagger D^* = \begin{bmatrix} E^{-2}U^* & E^{-2}V^* \\ 0 & 0 \end{bmatrix}$. From $A_1D^\dagger = D^\dagger A_1$ we get the equalities: $UE^{-2}U^* = E^{-1}$ and $VE^{-2}V^* = 0$. Hence, $V = 0$, $A_1 = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}$ and $E = U^*U$ is invertible. It follows that U is left invertible. Since A_1 is right invertible, we conclude that U is invertible and $\mathcal{N}(A_1) = \{0\}$. Finally, A_1 is invertible. Now, from the equality $A_1(A_1A_1^*A_1)^{-1} = (A_1A_1^*A_1)^{-1}A_1$ it follows that A_1 is normal. Consequently, A is normal.

(3)⇒(1): Let A have the form (2.1.2). Then $AA^* = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$, where $B = A_1A_1^* + A_2A_2^*$ is positive, and $A^\dagger = A^*(AA^*)^\dagger = \begin{bmatrix} A_1^*B^{-1} & 0 \\ A_2^*B^{-1} & 0 \end{bmatrix}$. From $A(A^* + A^\dagger) = (A^* + A^\dagger)A$ we obtain $B + I = A_1^*(I + B^{-1})A_1$ and $A_2^*(I + B^{-1})A_2 = 0$. Since $I + B$ and $I + B^{-1}$ are positive, it follows that A_1 is left invertible and $A_2 = 0$. Hence, $B = A_1A_1^*$ is invertible, A_1 is right invertible. Consequently, A_1 is invertible. We have $A_1A_1^* + I = A_1^*A_1 + A_1^*(A_1A_1^*)^{-1}A_1$ and A_1 is normal. Consequently, A is normal.

(4)⇒(1): *Step one.* First, suppose that A is left invertible. If $A^\dagger(A + A^*) = (A + A^*)A^\dagger$ holds, we shall prove that A is actually invertible. Since A is left invertible, we get $A^\dagger A = I$. Suppose that A is not right invertible. Then the decomposition $H = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$ is non-trivial. Then A has the non-trivial form (2.1.2). We have $AA^* = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$, where $B = A_1A_1^* + A_2A_2^* > 0$, and $A^\dagger = \begin{bmatrix} A_1^*B^{-1} & 0 \\ A_2^*B^{-1} & 0 \end{bmatrix}$. Now, we have $AA^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $A^\dagger A = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. Finally, from $A^\dagger(A+A^*) = (A+A^*)A^\dagger$ we get the equality $\begin{bmatrix} A_1^*B^{-1}A_1^* & 0 \\ A_2^*B^{-1}A_1^* & I \end{bmatrix} = \begin{bmatrix} A_1^*A_1^*B^{-1} & 0 \\ A_2^*A_1^*B^{-1} & 0 \end{bmatrix}$, which is impossible. Consequently, $\mathcal{N}(A^*) = \{0\}$ and A is invertible. Hence, A is normal.

Step two. In general, let A have the form (2.1.3). Then $A^*A = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$, where $B = A_1^*A_1 + A_2^*A_2 > 0$. Since $A^\dagger = \begin{bmatrix} B^{-1}A_1^* & B^{-1}A_2^* \\ 0 & 0 \end{bmatrix}$, from $A^\dagger(A + A^*) = (A + A^*)A^\dagger$ we get $B^{-1}A_1^*(A_1 + A_1^*) + B^{-1}A_2^*A_2 = (A_1 + A_1^*)B^{-1}A_1^*$ and $A_2B^{-1}A_2^* = 0$. The fact $B > 0$ implies $A_2 = 0$. Now, $B = A_1^*A_1$ is invertible and A is left invertible. Hence, we obtain $(A_1^*A_1)^{-1}A_1^*(A_1 + A_1^*) = (A_1 + A_1^*)(A_1^*A_1)^{-1}A_1^*$, i.e. $A_1^\dagger(A_1 + A_1^*) = (A_1 + A_1^*)A_1^\dagger$. From *Step one* it follows that A_1 is invertible and normal. Hence, A is normal.

(5)⇒(1): Since $(A^\#)^\# = A$ and $C^\#$ double commutes with C whenever it exists, this implication is trivial.

(6)⇒(1): Let A have the form (2.1.2). Then $B = A_1A_1^* + A_2A_2^* > 0$, $AA^* = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$ and $A^*A = \begin{bmatrix} A_1^*A_1 & A_1^*A_2 \\ A_2^*A_1 & A_2^*A_2 \end{bmatrix}$. From $A^*A(AA^*)^\dagger A^*A =$

AA^* we get the following: $A_1^*A_1B^{-1}A_1^*A_1 = B$ and $A_2^*A_1B^{-1}A_1^*A_2 = 0$. From the first equality we obtain that A_1 is left invertible. Since $\mathrm{dsc}(A) < \infty$, it follows that A_1 is invertible (Lemma 2.3). From the second equality it follows that $(B^{-1/2}A_1^*A_2)^*(B^{-1/2}A_1^*A_2) = 0$ and $B^{-1/2}A_1^*A_2 = 0$. From the invertibility of B and A_1 we get $A_2 = 0$. Now $B^{-1} = (A_1A_1^*)^{-1} = (A_1^*)^{-1}A_1^{-1}$. The equality $A_1^*A_1B^{-1}A_1^*A_1 = B$ is equivalent to $C^2 = I$, where $C = A_1^{-1}A_1^*A_1(A_1^*)^{-1} = (A_1^{-1}A_1^*)(A_1^{-1}A_1^*)^*$. The operator C is positive. Since the square root of a positive operator I is unique, it follows that $C = I$. Consequently, A_1 is normal. We obtain that A is normal also.

Example 2.6 (2) shows that the condition $\mathrm{dsc}(A) < \infty$ cannot be omitted.

(7)$\Rightarrow$(1): This part is dual to the previous one.

(8)$\Rightarrow$(1): *Step one.* Suppose that A is invertible. From $AA^*X = A^*A$ and $A^*AX = AA^*$ we conclude that $AA^*(A^*A)^{-1}AA^* = A^*A$, or, equivalently, $S^2 = I$, where $S = (A^*A)^{-1/2}AA^*(A^*A)^{-1/2} = S^* > 0$. Since the square root of a positive operator I is unique, we conclude $S = I$. Consequently, A is normal.

Step two. In general, suppose that A has a closed range. From $AA^*X = A^*A$ and $A^*AX = AA^*$ we conclude that $\mathcal{R}(A) = \mathcal{R}(A^*)$ and A is an EP operator. Hence, A has the form (1.1), where A_1 is invertible. Let $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$ with respect to the same decomposition of the space. From $AA^*X = A^*A$ and $A^*AX = AA^*$ we conclude that $A_1A_1^*X_1 = A_1^*A_1$ and $A_1^*A_1X_1 = A_1A_1^*$ hold. Using *Step one*, we get that A_1 is normal. Hence, A is normal.

(9)$\Rightarrow$(1): The condition $AX = A^*$ is equivalent to $\mathcal{R}(A^*) \subset \mathcal{R}(A)$ (consequently, $\mathcal{N}(A^*) \subset \mathcal{N}(A)$). Hence, A has the form $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}$ and $A^* = \begin{bmatrix} A_1^* & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}$, where A_1 is onto. The condition $\mathrm{asc}A < \infty$ is equivalent to $\mathrm{asc}A_1 < \infty$. Hence, A_1 is invertible. Let $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}$. From $AX = A^*$ and $(A^\dagger)^*X = A^\dagger$ we get $A_1X_1 = A_1^*$ and $(A_1^{-1})^*X_1 = A_1^{-1}$. We see that $X_1 = A_1^{-1}A_1^* = A_1^*A_1^{-1}$ holds, so A_1 is normal. Consequently, and A is normal.

Example 2.6 (3) shows that the condition $\mathrm{asc}A < \infty$ cannot be omitted. $\square$

For finite rank operators the following result can be proved.

Theorem 3.2 *Let $A \in \mathcal{L}(H)$ be a finite rank operator. Then the following statements are equivalent:*

(1) *A is normal;*
(2) *$AA^\dagger A^* AAA^\dagger = AA^*$;*
(3) *$A^\dagger AA^* AA^\dagger A = AA^*$;*

Proof. (2)⇒(1): Let A have the form (2.1.2). Again, $AA^* = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$, where $B = A_1A_1^* + A_2A_2^* > 0$. Also, $A^\dagger = \begin{bmatrix} A_1^*B^{-1} & 0 \\ A_2B^{-1} & 0 \end{bmatrix}$. From $AA^\dagger A^* AAA^\dagger = AA^*$ we get the equality $A_1^*A_1 = A_1A_1^* + A_2A_2^* > 0$. Hence, A_1 is hyponormal on a finite dimensional space $\mathcal{R}(A)$. It follows that A_1 is normal and $A_2 = 0$. Consequently, A is normal.

The Example 2.6 (1) shows that the condition $\dim\mathcal{R}(A) < \infty$ can not be avoided easily.

Eventually, the additional condition $\mathrm{ind}(A) < \infty$, which is frequently used later (with: A has a closed range, but A is not a finite rank operator), would not imply that A is normal (at least, this is not obvious). Precisely, if $\mathrm{ind}A < \infty$, then $\mathrm{ind}A_1 < \infty$ (Lemma 2.2). From $A_1^*A_1 = A_1A_1^* + A_2A_2^* > 0$ it follows that A_1 is left invertible and hyponormal. From $\mathrm{ind}A_1 < \infty$ we get that A_1 is invertible. However, there exist hyponormal and invertible operators on infinite dimensional Hilbert spaces, which are not normal.

(3)⇒(1): This implication can be proved in the same way as the previous one. □

4 Hyponormal operators

In this section we characterize hyponormal operators with closed ranges.

Theorem 4.1 *Let A and $AA^* + A^*A$ have closed ranges. Then the following statements are equivalent:*

(1) *A is hyponormal;*
(2) *$2AA^*(AA^* + A^*A)^\dagger AA^* \leq AA^*$.*

Proof. (1)$\Rightarrow$(2): Let A be hyponormal, i.e. $A^*A \geq AA^*$. Consider the matrix

$$M = \begin{bmatrix} AA^* + A^*A & AA^* \\ AA^* & \frac{1}{2}AA^* \end{bmatrix}.$$

Since $\frac{1}{2}AA^* \geq 0$,

$$\frac{1}{2}AA^* \left(\frac{1}{2}AA^*\right)^\dagger AA^* = AA^*$$

and

$$AA^* + A^*A - AA^* \left(\frac{1}{2}AA^*\right)^\dagger AA^* = A^*A - AA^* \geq 0,$$

by Lemma 2.4 we get that $M \geq 0$. Applying part (3) of Lemma 2.4 to M we get

$$\frac{1}{2}AA^* - AA^*(AA^* + A^*A)^\dagger AA^* \geq 0$$

and (2) is satisfied.

(2)$\Rightarrow$(1): Suppose that (2) holds. Using Lemma 2.5, we have the following:

$$\begin{aligned} (AA^* + A^*A) \geq 0,\ (AA^* + A^*A)(AA^* + A^*A)^\dagger AA^* &= AA^*, \\ \frac{1}{2}AA^* - AA^*(AA^* + A^*A)^\dagger AA^* &\geq 0. \end{aligned}$$

According to Lemma 2.4 we conclude that that the operator

$$M = \begin{bmatrix} AA^* + A^*A & AA^* \\ AA^* & \frac{1}{2}AA^* \end{bmatrix}$$

is non-negative. Applying again Lemma 2.4 to M, using the opposite blocks, we conclude that $A^*A \geq AA^*$, i.e. A is hyponornal. □

Analogously, the following result can be proved.

Theorem 4.2 *Let A and $AA^* + A^*A$ have closed ranges. Then the following statements are equivalent:*

(1) A^* *is hyponormal;*
(2) $2A^*A(AA^* + A^*A)^\dagger A^*A \leq A^*A$.

Notice that in [8] it is proved that if H is finite dimensional, then A is normal if and only if $2AA^*(AA^*+A^*A)^\dagger AA^* = AA^*$ holds. An easy proof follows. If A is normal, then obviously $2A^*A(AA^*+A^*A)^\dagger A^*A = A^*A$ holds. On the other hand, if $2A^*A(AA^*+A^*A)^\dagger A^*A = A^*A$ holds, then the step (2)$\Rightarrow$(1) from the proof of Theorem 4.1 can be used to see that A is hyponormal. Since H is finite dimensional, it follows that A is normal.

5 EP operators

In this section EP operators on Hilbert spaces are characterized.

Theorem 5.1 *Let $A \in \mathcal{L}(H)$ have a closed range. Then the following statements are equivalent:*

(1) *A is EP;*
(2) $AA^\dagger = A^2(A^\dagger)^2$ *and* $\mathrm{asc}A < \infty$*;*
(3) $A^\dagger A = (A^\dagger)^2 A^2$ *and* $\mathrm{dsc}A < \infty$*.*
(4) $\mathrm{ind}A \le 1$ *and* $AA^\dagger A^*A = A^*AAA^\dagger$*;*
(5) $\mathrm{ind}A \le 1$ *and* $A^\dagger AAA^* = AA^*A^\dagger A$*.*
(6) $\mathrm{ind}A \le 1$ *and* $AA^\dagger(AA^*-A^*A) = (AA^*-A^*A)AA^\dagger$*;*
(7) $\mathrm{ind}A \le 1$ *and* $A^\dagger A(AA^*-A^*A) = (AA^*-A^*A)AA^\dagger$*.*
(8) $A^*A^\#A + AA^\#A^* = 2A^*$*;*
(9) $A^\dagger A^\# A + AA^\# A^\dagger = 2A^\dagger$*.*
(10) $AAA^\dagger + A^\dagger AA = 2A$*.*
(11) $AAA^\dagger + (AAA^\dagger)^* = A + A^*$ *and* $\mathrm{asc}A < \infty$*;*
(12) $A^\dagger AA + (A^\dagger AA)^* = A + A^*$ *and* $\mathrm{dsc}A < \infty$*.*

Proof. Property (1) implies conditions (2)-(12); this is either elementary or follows from (1.1).

(2)$\Rightarrow$(1): Let A have the matrix form (2.1.2). Then $(AA^*)^\dagger = \begin{bmatrix} B^{-1} & 0 \\ 0 & 0 \end{bmatrix}$, $A^\dagger = A^*(AA^*)^\dagger = \begin{bmatrix} A_1^*B^{-1} & 0 \\ A_2^*B^{-1} & 0 \end{bmatrix}$ and $AA^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. From $AA^\dagger = A^2(A^\dagger)^2$ the following equality follows:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1A_1^*B^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $A_1A_1^*B^{-1} = I$ and $A_1A_1^* = B = A_1A_1^* + A_2A_2^*$, implying that $A_2 = 0$ and A_1 is right invertible. Now, $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$. The condition $\mathrm{asc}A < \infty$ is equivalent to $\mathrm{asc}A_1 < \infty$. From Lemma 2.3 it follows that A_1 is invertible. Finally, $A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ and A is EP.

Example 2.6 (4) shows that the condition $\mathrm{asc}A < \infty$ cannot be omitted.

(3)⇒(1): Follows from the previous implication, knowing the fact $(A^\dagger)^\dagger = A$.

In statements (4)-(7) it is only interesting to consider the case $\mathrm{ind}A = 1$. Otherwise, if $\mathrm{ind}A = 0$, then A is invertible, so it is EP. Hence, in (4)-(7) we assume that $\mathrm{ind}A = 1$ holds.

(4)⇒(1): Let A have the decomposition (2.1.2). We have

$$A^*A = \begin{bmatrix} A_1^*A_1 & A_1^*A_2 \\ A_2^*A_1 & A_2^*A_2 \end{bmatrix}.$$

From $AA^\dagger A^*A = A^*AAA^\dagger$ we obtain the equality

$$\begin{bmatrix} A_1^*A_1 & A_1^*A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^*A_1 & 0 \\ A_2^*A_1 & 0 \end{bmatrix},$$

implying $A_1^*A_2 = 0$.

We have $\mathcal{R}(A) = \{A_1u + A_2v : u \in \mathcal{R}(A), v \in \mathcal{N}(A^*)\} = \mathcal{R}(A_1) + \mathcal{R}(A_2)$. Obviously, if $x \in \mathcal{R}(A)$ and $y \in \mathcal{N}(A^*)$, then $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}(A)$ if and only if $A_1x + A_2y = 0$. From $A^2 = \begin{bmatrix} A_1^2 & A_1A_2 \\ 0 & 0 \end{bmatrix}$ it follows that $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}(A^2)$ if and only if $A_1(A_1x + A_2y) = 0$. Since $\mathrm{ind}A = 1$, we have $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}(A)$ if and only if $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}(A^2)$.

Now, let $u \in \mathcal{R}(A)$ and $A_1u = 0$. Then there exist some $x \in \mathcal{R}(A)$ and $y \in \mathcal{N}(A^*)$, such that $u = A_1x + A_2y$. Also, $A_1(A_1x + A_2y) = 0$, implying that $u = A_1x + A_2y = 0$. Hence, A_1 is 1–1 (on $\mathcal{R}(A)$).

From Lemma 2.2, the condition $\mathrm{ind}A = 1$ implies $\mathrm{ind}A_1 < \infty$. Since A_1 is 1–1, we get that A_1 is invertible. From $A_1^*A_2 = 0$ we get $A_2 = 0$ and A is EP.

(5)$\Rightarrow$(1): This part can be proved in a similar way as the previous one, using the (2.1.3) matrix form of A.

(6)$\Rightarrow$(1): Let A have the decomposition (2.1.2). Then $(AA^*)^\dagger = \begin{bmatrix} B^{-1} & 0 \\ 0 & 0 \end{bmatrix}$, $A^\dagger = A^*(AA^*)^\dagger = \begin{bmatrix} A_1B^{-1} & 0 \\ A_2^*B^{-1} & 0 \end{bmatrix}$ and $A^*A = \begin{bmatrix} A_1^*A_1 & A_1^*A_2 \\ A_2^*A_1 & A_2^*A_2 \end{bmatrix}$. From $AA^\dagger(AA^* - A^*A) = (AA^* - A^*A)AA^\dagger$ we obtain $A_1^*A_2 = 0$. Since $\mathrm{ind}A = 1$, we get $\mathrm{ind}A_1 < \infty$ and A_1 is 1–1 (for the same reason as in the proof of (4)$\Rightarrow$(1)). Hence A_1 is invertible, $A_2 = 0$ and A is EP.

(7)$\Rightarrow$(1): This part can be proved in the same way as the previous one, taking the decomposition (2.1.3) for A.

(8)$\Rightarrow$(1): Multiplying the equality $A^*A^\# A + AA^\# A^* = 2A^*$ by A from the left side, we obtain $AA^*(I - A^\# A) = 0$. Since $I - A^\# A$ is a projection onto $\mathcal{N}(A)$, it follows that $\mathcal{N}(A) \subset \mathcal{N}(AA^*) = \mathcal{N}(A^*)$ holds.

Taking conjugates of $A^*A^\# A + AA^\# A^* = 2A^*$, we obtain $A^*(A^*)^\# A + A(A^*)^\# A^* = 2A$. We replace A^* by B and obtain $BB^\# B^* + B^*B^\# B = 2B^*$. In the same way as before we get $\mathcal{N}(B) \subset \mathcal{N}(B^*)$, or, equivalently $\mathcal{N}(A^*) \subset \mathcal{N}(A)$.

Consequently, we get $\mathcal{N}(A) = \mathcal{N}(A^*)$ and $\mathcal{R}(A) = \mathcal{R}(A^*)$. Hence, A is EP.

(9)$\Rightarrow$(1): Multiplying $A^\dagger A^\# A + AA^\# A^\dagger = 2A^\dagger$ by A from the left side, we get $AA^\# = AA^\dagger$. Hence, $AA^\#$ is the orthogonal projection. Multiplying the equality $A^\dagger A^\# A + AA^\# A^\dagger = 2A^\dagger$ by A from the right side, we get $A^\# A = A^\dagger A$ and $A^\# A$ is orthogonal. Consequently, $A^\dagger = A^\#$ and A is EP.

(10)$\Rightarrow$(1): Multiplying $AAA^\dagger + A^\dagger AA = 2A$ by A from the right side, we get $(A^\dagger A)A^2 = A^2$. Since $A^\dagger A$ is a projection onto $\mathcal{R}(A^*)$, it follows that $\mathcal{R}(A^2) \subset \mathcal{R}(A^*)$. Let A have the form (2.1.3). Then $A^2 = \begin{bmatrix} A_1^2 & 0 \\ A_2A_1 & 0 \end{bmatrix}$. Since $\mathcal{R}(A^2) \subset \mathcal{R}(A^*)$, we conclude that $A_2A_1 = 0$ and $A^2 = \begin{bmatrix} A_1^2 & 0 \\ 0 & 0 \end{bmatrix}$. From $(A^*A)^\dagger = \begin{bmatrix} B^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ with $B = A_1^*A_1 + A_2^*A_2 > 0$, we get $A^\dagger = \begin{bmatrix} B^{-1}A_1^* & B^{-1}A_2^* \\ 0 & 0 \end{bmatrix}$. Also, $A^\dagger A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. From $AAA^\dagger + A^\dagger AA = 2A$ we get $A_2 = 0$ and $A_1^2B^{-1}A_1^* = A_1$. Now, $B = A_1^*A_1$ is invertible and $B^{-1} = B^\dagger = A_1^\dagger(A_1^*)^\dagger$, since the reverse order rule for the Moore-Penrose inverse holds in this special case. Particularly, A_1 is left invertible and A_1^* is right invertible. We have $A_1^2 A_1^\dagger (A_1^*)^\dagger A_1^* = A_1$ and consequently $A_1(A_1 A_1^\dagger) = A_1$. Since $A_1 A_1^\dagger$ is a projection onto $\mathcal{R}(A_1)$, we get $\mathrm{dsc} A_1 \leq 1$. By Lemma 2.3 we obtain that A_1 is invertible. Hence, A is EP.

(11)$\Rightarrow$(1): Let A have the form (2.1.2). Then $B = A_1 A_1^* + A_2 A_2^* > 0$, $A^\dagger = \begin{bmatrix} A_1^* B^{-1} & 0 \\ A_2^* B^{-1} & 0 \end{bmatrix}$ and $AA^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Now, from $AAA^\dagger + (AAA^\dagger)^* = A + A^*$, it follows that $A_2 = 0$. Hence, $B = A_1 A_1^*$ is invertible and A_1 is right invertible. Now, the fact $\mathrm{asc} A < \infty$ is equivalent to $\mathrm{asc} A_1 < \infty$. By Lemma 2.3 it follows that A_1 is invertible. We conclude that A is EP.

Example 2.6 (5) shows that the condition $\mathrm{asc} A < \infty$ cannot be omitted.

(12)$\Rightarrow$(1): This part can be proved in the same way as the previous one.
$\square$

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