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Idempotents related to the weighted Moore–Penrose inverse

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Abstract

We investigate necessary and sufficient conditions for $aa^{\dagger}_{e,f} = bb^{\dagger}_{e,f}$ to hold in rings with involution. Here, $a^{\dagger}_{e,f}$ denotes the weighted Moore-Penrose inverse of a, related to invertible and Hermitian elements $e, f \in \mathcal{R}$. Thus, some recent results from [7] are extended to the weighted Moore-Penrose inverse.

1 Introduction

Let $\mathcal{R}$ be an associative ring with the unit 1. An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

An element $a \in \mathcal{R}$ is selfadjoint (or Hermitian) if $a^* = a$. An element $a \in \mathcal{R}$ is regular if there exists some inner inverse (or 1-inverse) $a^- \in \mathcal{R}$ satisfying $aa^-a = a$. The set of all inner inverses (or 1-inverses) is denoted by $a\{1\}$. Hence, a is regular if $a\{1\} \neq \emptyset$. A reflexive inverse a^+ of a is a 1-inverse of a such that $a^+aa^+ = a^+$.

Definition 1.1. Let $\mathcal{R}$ be a ring with involution, and let e, f be invertible Hermitian elements in $\mathcal{R}$. The element $a \in \mathcal{R}$ has the weighted Moore-Penrose inverse (weighted MP-inverse) with weights e, f if there exists $b \in \mathcal{R}$ such that

$$aba = a, \quad bab = b, \quad (eab)^* = eab, \quad (fba)^* = fba.$$

The unique weighted MP-inverse with weights e, f, will be denoted by $a^{\dagger}_{e,f}$ if it exists [4]. The set of all weighted MP-invertible elements of $\mathcal{R}$ with weights e, f, will be denoted by $\mathcal{R}^{\dagger}_{e,f}$. If $e = f = 1$, then the weighted MP-inverse reduces to the ordinary MP-inverse of a, denoted by $a^{\dagger}$.

If $a \in \mathcal{R}^{\dagger}_{e,f}$, then $aa^{\dagger}_{e,f}$ and $a^{\dagger}_{e,f}a$ are idempotents related to a and $a^{\dagger}_{e,f}$.

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Notice that if $\mathcal{R}$ is a C^*-algebra, if e, f are selfadjoint, invertible and *positive* elements in a C^*-algebra $\mathcal{R}$, and if $a \in \mathcal{R}$ is regular, then the following formula holds:

$$a^\dagger_{e,f} = f^{-1/2}(e^{1/2}af^{-1/2})^\dagger e^{1/2}.$$

Hence, the existence of an inner inverse of a implies the existence of the MP-inverse and the weighted MP-inverse of a.

However, if $\mathcal{R}$ is a general ring with involution, then we do not have the existence of a square root of a positive element. Hence, in this case we always have to assume that the weighted MP-inverse of a exists.

Define the mapping $(*, e, f) : x \mapsto x^{*,e,f} = e^{-1}x^*, f$, for all $x \in \mathcal{R}$. Notice that $(*, e, f) : \mathcal{R} \to \mathcal{R}$ is not an involution, because in general $(xy)^{*,e,f} \neq y^{*,e,f}x^{*,e,f}$. Now, we formulate the following result which can be proved directly by the definition of the weighted MP-inverse.

Theorem 1.1. *Let $\mathcal{R}$ be a ring with involution and let e, f be invertible Hermitian elements in $\mathcal{R}$. For any $a \in \mathcal{R}^\dagger_{e,f}$, the following is satisfied:*

(a) $(a^\dagger_{e,f})^\dagger_{f,e} = a$;

(b) $(a^{*,f,e})^\dagger_{f,e} = (a^\dagger_{e,f})^{*,e,f}$;

(c) $a^{*,f,e} = a^\dagger_{e,f}aa^{*,f,e} = a^{*,f,e}aa^\dagger_{e,f}$;

(d) $a^{*,f,e}(a^\dagger_{e,f})^{*,e,f} = a^\dagger_{e,f}a$;

(e) $(a^\dagger_{e,f})^{*,e,f}a^{*,f,e} = aa^\dagger_{e,f}$;

(f) $(a^{*,f,e}a)^\dagger_{f,f} = a^\dagger_{e,f}(a^\dagger_{e,f})^{*,e,f}$;

(g) $(aa^{*,f,e})^\dagger_{e,e} = (a^\dagger_{e,f})^{*,e,f}a^\dagger_{e,f}$;

(h) $a^\dagger_{e,f} = (a^{*,f,e}a)^\dagger_{f,f}a^{*,f,e} = a^{*,f,e}(aa^{*,f,e})^\dagger_{e,e}$;

(i) $(a^{*,e,f})^\dagger_{f,e} = a(a^{*,f,e}a)^\dagger_{f,f} = (aa^{*,f,e})^\dagger_{e,e}a$.

For $a \in \mathcal{R}$ consider two annihilators

$$a^\circ = \{x \in \mathcal{R} : ax = 0\}, \qquad {}^\circ a = \{x \in \mathcal{R} : xa = 0\}.$$

Notice that,

$$(a^*)^\circ = a^\circ \Leftrightarrow {}^\circ(a^*) = {}^\circ a, \qquad a\mathcal{R} = a^*\mathcal{R} \Leftrightarrow \mathcal{R}a = \mathcal{R}a^*.$$

Lemma 1.1. *Let $a \in \mathcal{A}^-$, and let e, f be invertible positive elements in $\mathcal{A}$. Then*

$$a^\dagger_{e,f} = (a^{*,f,e}a + 1 - a^\dagger_{e,f}a)^{-1}a^{*,f,e} = a^{*,f,e}(aa^{*,f,e} + 1 - aa^\dagger_{e,f})^{-1}, \tag{1}$$

$$a^{*,f,e}\mathcal{A}^{-1} = a^\dagger_{e,f}\mathcal{A}^{-1} \text{ and } \mathcal{A}^{-1}a^{*,f,e} = \mathcal{A}^{-1}a^\dagger_{e,f}, \tag{2}$$

$$(a^{*,f,e})^\circ = (a^\dagger_{e,f})^\circ \text{ and } {}^\circ(a^{*,f,e}) = {}^\circ(a^\dagger_{e,f}). \tag{3}$$

Proof. By Theorem 1.1, we can verify

$$a^{*,f,e} = (a^{*,f,e}a + 1 - a^{\dagger}_{e,f}a)a^{\dagger}_{e,f} = a^{\dagger}_{e,f}(aa^{*,f,e} + 1 - aa^{\dagger}_{e,f}),$$

$$(a^{*,f,e}a + 1 - a^{\dagger}_{e,f}a)^{-1} = a^{\dagger}_{e,f}(a^{\dagger}_{e,f})^{*,e,f} + 1 - a^{\dagger}_{e,f}a$$

and

$$(aa^{*,f,e} + 1 - aa^{\dagger}_{e,f})^{-1} = (a^{\dagger}_{e,f})^{*,e,f}a^{\dagger}_{e,f} + 1 - aa^{\dagger}_{e,f}.$$

Thus, the part (1) holds and it implies the equalities (2) and (3). □

Now, we state an useful result from [7].

Lemma 1.2. [7, Lemma 2.1] *Let $a, b \in \mathcal{R}$ be regular elements.*

(1) *There exist $a^- \in a\{1\}$, $b^- \in b\{1\}$ for which $(1 - bb^-)aa^- = 0$ if and only if $(1 - bb^-)aa^- = 0$ for all $a^- \in a\{1\}$, $b^- \in b\{1\}$.*

(2) *There exist $a^- \in a\{1\}$, $b^- \in b\{1\}$ for which $(1 - bb^-)(1 - a^-a) = 0$ if and only if $(1 - bb^-)(1 - a^-a) = 0$ for all $a^- \in a\{1\}$, $b^- \in b\{1\}$.*

In [7], necessary and sufficient conditions for $aa^\dagger = bb^\dagger$ in ring with involution are investigated. In this paper we generalized this results to the weighted Moore-Penrose in rings with involution.

2 Results

A semigroup is a regular, if every elements of that semigroup has an inner generalized inverse. The notion extends to rings also.

In a regular semigroup, the natural partial order is defined by ([2], [5], [6])

$$a \leq_- b \text{ if } aa^- = ba^- \text{ and } a^-a = a^-b \text{ for some inner inverse } a^- \text{ of } a.$$

See also [3] for intuitionistic fuzzy matrices. Notice that $\leq_-$ is a partial order in regular rings.

A semigroup with involution $x \mapsto x^*$ is proper, if the following implication holds:

$$a^*a = a^*b = b^*a = b^*b \implies a = b.$$

Notice that if the semigroup has the zero element 0, then a semigroup is a proper with respect to the involution $x \mapsto x^*$, if and only if $a^*a = 0 \implies a = 0$. The last implication is called $*$-cancellability. For example, every element of a C^*-algebra is $*$-cancellable, so every C^*-algebra is proper (with respect to multiplication).

Drazin [1] presented a partial order on a proper $*$-semigroup in the following way

$$a \leq_* b \text{ if } aa^* = ba^* \text{ and } a^*a = a^*b.$$

If $a \in \mathcal{R}$ is MP invertible, then "$\leq_*$" implies "$\leq_-$". Indeed, $aa^* = ba^* \Rightarrow aa^\dagger = aa^*(a^\dagger)^*a^\dagger = ba^*(a^\dagger)^*a^\dagger = ba^\dagger$ and similarly $a^*a = a^*b \Rightarrow a^\dagger a = a^\dagger b$.

In this paper we introduce the "$\leq_{*,e,f}$" as follows:

$$a \leq_{*,e,f} b \text{ if } aa^{*,e,f} = ba^{*,e,f} \text{ and } a^{*,e,f}a = a^{*,e,f}b.$$

Here e, f are Hermitian invertible elements in a ring $\mathcal{R}$ with involution $x \mapsto x^*$. We like to see that $\leq_{*,e,f}$ is a partial ordering in $\mathcal{R}$.

If $a \in \mathcal{R}^{\dagger}_{e,f}$, then "$\leq_{*,e,f}$" implies "$\leq_{-}$". Indeed, from $aa^{*,e,f} = ba^{*,e,f}$ we get $aa^{\dagger}_{e,f} = aa^{*,e,f}(a^{\dagger}_{e,f})^{*,e,f}a^{\dagger}_{e,f} = ba^{*,e,f}(a^{\dagger}_{e,f})^{*,e,f}a^{\dagger}_{e,f} = ba^{\dagger}_{e,f}$. Similarly, $a^{*,e,f}a = a^{*,e,f}b$ gives $a^{\dagger}_{e,f}a = a^{\dagger}_{e,f}b$.

In the rest of the paper we assume that $e, f \in \mathcal{R}$ are Hermitian end invertible.

The ring $\mathcal{R}$ is $(*, e, f)$-proper if the following implication holds:

$$a^{*,e,f}a = a^{*,e,f}b = b^{*,e,f}a = b^{*,e,f}b \implies a = b.$$

If $\mathcal{R}$ is a C^*-algebra and e, f are positive Hermitian elements in $\mathcal{R}$, then $\mathcal{R}$ is $(*, e, f)$-proper. Indeed, $a^{*,e,f}a = a^{*,e,f}b = b^{*,e,f}a = b^{*,e,f}b$ gives $(a-b)^{*,e,f}(a-b) = 0$ which gives that $[f^{1/2}(a-b)]^*f^{1/2}(a-b) = 0$. Since every element in C^*-algebra is $*$-cancellable, then $f^{1/2}(a-b) = 0$, that is $a = b$.

Theorem 2.1. *Let $\mathcal{R}$ be: $(*, e, f)$-proper, $(*, e, e)$-proper and $(*, f, f)$-proper. Then $\leq_{*,e,f}$ is a partial ordering in $\mathcal{R}$.*

Proof. Since $a \leq_{*,e,f} a$, then "$\leq_{*,e,f}$" is reflexive.

From $a \leq_{*,e,f} b$ and $b \leq_{*,e,f} a$, we get $a^{*,e,f}a = a^{*,e,f}b$ and $b^{*,e,f}a = b^{*,e,f}b$. Observe that

$$a^{*,e,f}a = (a^{*,e,f}a)^{*,e,e} = (a^{*,e,f}b)^{*,e,e} = b^{*,e,f}a \tag{4}$$

So, we deduce $a^{*,e,f}a = a^{*,e,f}b = b^{*,e,f}a = b^{*,e,f}b$ which gives $a = b$.

If $a \leq_{*,e,f} b$ and $b \leq_{*,e,f} c$, we obtain (4) and, applying (4) for b and c instead of a and b, we have $b^{*,e,f}b = c^{*,e,f}b$. Further,

$$c^{*,e,f}(aa^{*,e,f})c = (c^{*,e,f}b)a^{*,e,f}c = b^{*,e,f}(ba^{*,e,f})c = (b^{*,e,f}a)a^{*,e,f}c = a^{*,e,f}aa^{*,e,f}c,$$

$$(a^{*,e,f}a)a^{*,e,f}a = b^{*,e,f}(aa^{*,e,f})a = (b^{*,e,f}b)a^{*,e,f}a = c^{*,e,f}(ba^{*,e,f})a = c^{*,e,f}aa^{*,e,f}a$$

and

$$a^{*,e,f}aa^{*,e,f}a = (a^{*,e,f}aa^{*,e,f}a)^{*,e,e} = (c^{*,e,f}aa^{*,e,f}a)^{*,e,e} = a^{*,e,f}aa^{*,e,f}c.$$

Since $(a^{*,e,f}a)^{*,e,e} = a^{*,e,f}a$ and $(a^{*,e,f}c)^{*,e,e} = c^{*,e,f}a$, by the previous tree equalities, we conclude

$$(a^{*,e,f}a)^{*,e,e}a^{*,e,f}a = (a^{*,e,f}a)^{*,e,e}a^{*,e,f}c = (a^{*,e,f}c)^{*,e,e}a^{*,e,f}a = (a^{*,e,f}c)^{*,e,e}a^{*,e,f}c$$

which implies $a^{*,e,f}a = a^{*,e,f}c$, because ring $\mathcal{R}$ is $*, e, e$-proper. Similarly, by $*, f, f$-proper of $\mathcal{R}$, we can verify that $aa^{*,e,f} = (ca^{*,e,f})^{*,f,f}$ which yields $aa^{*,e,f} = (aa^{*,e,f})^{*,f,f} = ((ca^{*,e,f})^{*,f,f})^{*,f,f} = ca^{*,e,f}$. Thus, $a^{*,e,f}a = a^{*,e,f}c$ and $aa^{*,e,f} = ca^{*,e,f}$ give that $a \leq_{*,e,f} c$. □

In the following theorem, we present some equivalent conditions for $aa^{\dagger}_{e,f} = bb^{\dagger}_{e,f}aa^{\dagger}_{e,f}$ to hold.

Theorem 2.2. *Let $\mathcal{R}$ be a ring with involution, and let e, f be invertible Hermitian elements in $\mathcal{R}$. If $a, b \in \mathcal{R}^{\dagger}_{e,f}$, then the following conditions are equivalent:*

(1) $aa^{\dagger}_{e,f} = bb^{\dagger}_{e,f}aa^{\dagger}_{e,f}$;

(2) $aa^{\dagger}_{e,f} = aa^{\dagger}_{e,f}bb^{\dagger}_{e,f}$;

(3) $a = bb^{\dagger}_{e,f}a$;

(4) $a^{\dagger}_{e,f} = a^{\dagger}_{e,f}bb^{\dagger}_{e,f}$;

(5) $aa^{*,f,e} = bb^{\dagger}_{e,f}aa^{*,f,e}$;

(6) $aa^{*,f,e} = aa^{*,f,e}bb^{\dagger}_{e,f}$;

(7) $a^{*,f,e} = a^{*,f,e}bb^{\dagger}_{e,f}$;

(8) $aa^{-} = bb^{-}aa^{-}$ *for all choices* $a^{-} \in a\{1\}$, $b^{-} \in b\{1\}$;

(9) $aa^{-} = bb^{-}aa^{-}$ *for some* $a^{-} \in a\{1\}$, $b^{-} \in b\{1\}$;

(10) $a = bb^{-}a$ *for all* $b^{-} \in b\{1\}$;

(11) $a = bb^{-}a$ *for some* $b^{-} \in b\{1\}$;

(12) $aa^{*,f,e} = bb^{-}aa^{*,f,e}$ *for all* $b^{-} \in b\{1\}$;

(13) $aa^{*,f,e} = bb^{-}aa^{*,f,e}$ *for some* $b^{-} \in b\{1\}$;

(14) $aa^{\dagger}_{e,f} \leq bb^{\dagger}_{e,f}$;

(15) $aa^{\dagger}_{e,f} \leq_{*,e,e} bb^{\dagger}_{e,f}$;

(16) $a \leq bb^{-}a$ *for all* $b^{-} \in b\{1\}$;

(17) $a \leq bb^{-}a$ *for some* $b^{-} \in b\{1\}$;

(18) $a\mathcal{R} \subseteq bb^{\dagger}_{e,f}a\mathcal{R}$;

(19) $a\mathcal{R} \subseteq b\mathcal{R}$;

(20) $\mathcal{R}a^{\dagger}_{e,f} \subseteq \mathcal{R}a^{\dagger}_{e,f}bb^{\dagger}_{e,f}$;

(21) $\mathcal{R}a^{\dagger}_{e,f} \subseteq \mathcal{R}b^{\dagger}_{e,f}$;

Proof. (1) $\Leftrightarrow$ (2): Applying the involution, the equality $aa^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}$ is equivalent to $(e^{-1}eaa^\dagger_{e,f})^* = (e^{-1}ebb^\dagger_{e,f}e^{-1}eaa^\dagger_{e,f})^*$ which is $eaa^\dagger_{e,f}e^{-1} = eaa^\dagger_{e,f}e^{-1}ebb^\dagger_{e,f}e^{-1}$, i.e. $aa^\dagger_{e,f} = aa^\dagger_{e,f}bb^\dagger_{e,f}$.

(1) $\Leftrightarrow$ (3): Multiplying (1) by a from the right side we get (3), and multiplying (3) by $a^\dagger_{e,f}$ from the right side we obtain (1).

(2) $\Leftrightarrow$ (4): This part can be verified in the same way as (1) $\Leftrightarrow$ (3).

(3) $\Leftrightarrow$ (5): If we multiply (3) by $a^{*,f,e}$ from the right side we obtain (5), and if we multiply (5) by $(a^\dagger_{e,f})^{*,e,f}$ from the right side, by Theorem 1.1(d), we have (3).

(2) $\Leftrightarrow$ (6): By Theorem 1.1, multiplying (2) by $aa^{*,f,e}$ from the left side, we obtain (6). Conversely, multiplying (6) by $(a^\dagger_{e,f})^{*,e,f}a^\dagger_{e,f}$ from the left side, we get (2).

(6) $\Leftrightarrow$ (7): Multiplying (6) by $a^\dagger_{e,f}$ from the left side, we obtain (7) and multiplying (7) by a from the left side, we get (6).

(1) $\Leftrightarrow$ (8): The assumption $aa^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}$ is equivalent to $(1 - bb^\dagger_{e,f})aa^\dagger_{e,f} = 0$. Applying Lemma 1.2, we obtain this equivalence.

(8) $\Leftrightarrow$ (9): By Lemma 1.2.

(8) $\Leftrightarrow$ (10), (9) $\Leftrightarrow$ (11): Obviously.

(10) $\Leftrightarrow$ (12): Multiplying (10) by $a^{*,f,e}$ from the right side, we obtain (12). On the other hand, multiplying (12) from the right side by $(a^\dagger_{e,f})^{*,e,f}$, we get (10).

(11) $\Leftrightarrow$ (13): See the previous part.

(1) $\Leftrightarrow$ (14): We can easy verify that $(aa^\dagger_{e,f})^\dagger_{e,e} = aa^\dagger_{e,f}$. Now, for $(aa^\dagger_{e,f})^+ = (aa^\dagger_{e,f})^\dagger_{e,e}$, we have $aa^\dagger_{e,f} \leq bb^\dagger_{e,f}$ if and only if $aa^\dagger_{e,f}(aa^\dagger_{e,f})^\dagger_{e,e} = bb^\dagger_{e,f}(aa^\dagger_{e,f})^\dagger_{e,e}$ and $(aa^\dagger_{e,f})^\dagger_{e,e}aa^\dagger_{e,f} = (aa^\dagger_{e,f})^\dagger_{e,e}bb^\dagger_{e,f}$, which is equivalent to $aa^\dagger_{e,f} = bb^\dagger_{e,f}aa^\dagger_{e,f}$ and $aa^\dagger_{e,f} = aa^\dagger_{e,f}bb^\dagger_{e,f}$.

(1) $\Leftrightarrow$ (15): Since $(aa^\dagger_{e,f})^{*,e,e} = e^{-1}(e^{-1}eaa^\dagger_{e,f})^*, e = aa^\dagger_{e,f}$, we show this equivalence in the same way as (1) $\Leftrightarrow$ (14).

(10) $\Rightarrow$ (16): For $a^+ = a^\dagger_{e,f}$, we already proved this part.

(16) $\Rightarrow$ (17): Obviously.

(17) $\Rightarrow$ (11): Suppose that $a \leq bb^- a$ for some $b^- \in b\{1\}$. Then, for some a^+, we have $aa^+ = bb^- aa^+$, so $a = bb^- a$.

(3) $\Rightarrow$ (18) $\Rightarrow$ (19): Obviously.

(19) $\Rightarrow$ (3): The hypothesis $a\mathcal{R} \subseteq b\mathcal{R}$ gives $a = bx$, for some $x \in \mathcal{R}$. Therefore, $a = bb^\dagger_{e,f}(bx) = bb^\dagger_{e,f}a$.

(4) $\Rightarrow$ (20) $\Rightarrow$ (21) $\Rightarrow$ (4): Similarly as (3) $\Rightarrow$ (18) $\Rightarrow$ (19) $\Rightarrow$ (3). $\square$

Theorem 2.3. *Let $\mathcal{R}$ be a ring with involution, and let e, f be invertible Hermitian elements in $\mathcal{R}$. If $a, b \in \mathcal{R}^\dagger_{e,f}$, then the following conditions are equivalent:*

(1) $aa^\dagger_{e,f} = bb^\dagger_{e,f}$;

(2) $aa^\dagger_{e,f} = aa^\dagger_{e,f}bb^\dagger_{e,f}$ *and* $u = aa^\dagger_{e,f} + 1 - bb^\dagger_{e,f} \in \mathcal{R}^{-1}$;

(3) $aa^\dagger_{e,f} = aa^\dagger_{e,f}bb^\dagger_{e,f}$ *and* $v = aa^{*,f,e} + 1 - bb^\dagger_{e,f} \in \mathcal{R}^{-1}$;

(4) $aa^{\dagger}_{e,f} = aa^{\dagger}_{e,f}bb^{\dagger}_{e,f}$ *and* $\forall b^{-} \in b\{1\}$ $w = aa^{*,f,e} + 1 - bb^{-} \in \mathcal{R}^{-1}$*;*

(5) $aa^{\dagger}_{e,f} = aa^{\dagger}_{e,f}bb^{\dagger}_{e,f}$ *and* $\exists b^{-} \in b\{1\}$ $w = aa^{*,f,e} + 1 - bb^{-} \in \mathcal{R}^{-1}$*;*

(6) $aa^{\dagger}_{e,f}bb^{\dagger}_{e,f} = bb^{\dagger}_{e,f}aa^{\dagger}_{e,f}$, $u = aa^{\dagger}_{e,f} + 1 - bb^{\dagger}_{e,f} \in \mathcal{R}^{-1}$ *and* $l = bb^{\dagger}_{e,f} + 1 - aa^{\dagger}_{e,f} \in \mathcal{R}^{-1}$*;*

(7) $aa^{\dagger}_{e,f}bb^{\dagger}_{e,f} = bb^{\dagger}_{e,f}aa^{\dagger}_{e,f}$, $v = aa^{*,f,e} + 1 - bb^{\dagger}_{e,f} \in \mathcal{R}^{-1}$ *and* $k = bb^{*,f,e} + 1 - aa^{\dagger}_{e,f} \in \mathcal{R}^{-1}$*;*

Proof. (1) $\Rightarrow$ (2): It is easy to check.

(2) $\Leftrightarrow$ (3): Using Theorem 2.2, $(aa^{\dagger}_{e,f} + 1 - bb^{\dagger}_{e,f})(aa^{*,f,e} + 1 - aa^{\dagger}_{e,f}) = aa^{*,f,e} + 1 - bb^{\dagger}_{e,f}$. By Lemma 1.1, $aa^{*,f,e} + 1 - aa^{\dagger}_{e,f} \in \mathcal{R}^{-1}$ and then $u \in \mathcal{R}^{-1} \Leftrightarrow v \in \mathcal{R}^{-1}$.

(3) $\Rightarrow$ (1): Observe that, by Theorem 2.2, $vaa^{\dagger}_{e,f} = aa^{*,f,e} = vbb^{\dagger}_{e,f}$. Since $v \in \mathcal{R}^{-1}$, we have $aa^{\dagger}_{e,f} = bb^{\dagger}_{e,f}$.

(3) $\Rightarrow$ (4): By Theorem 2.2, we have $aa^{*,f,e} = bb^{\dagger}_{e,f}aa^{*,f,e} = bb^{\dagger}_{e,f}aa^{*,f,e}bb^{\dagger}_{e,f}$. Now, by [8, Proposition 3], $v = aa^{*,f,e} + 1 - bb^{\dagger}_{e,f} = bb^{\dagger}_{e,f}aa^{*,f,e}bb^{\dagger}_{e,f} + 1 - bb^{\dagger}_{e,f} \in \mathcal{R}^{-1}$ if and only if $bb^{\dagger}_{e,f}aa^{*,f,e}bb^{-} + 1 - bb^{-} \in \mathcal{R}^{-1}$, $\forall b^{-} \in b\{1\}$, i.e. $1 - (-bb^{\dagger}_{e,f}aa^{*,f,e} + 1)bb^{-} \in \mathcal{R}^{-1}$ for all $b^{-} \in b\{1\}$, which is equivalent to $1 - bb^{-}(-bb^{\dagger}_{e,f}aa^{*,f,e} + 1) = w \in \mathcal{R}^{-1}$, $\forall b^{-} \in b\{1\}$.

(4) $\Rightarrow$ (3) $\wedge$ (5): Obviously.

(5) $\Rightarrow$ (4): From $w = aa^{*,f,e} + 1 - bb^{-} = 1 - bb^{-}(-aa^{*,f,e} + 1) \in \mathcal{R}^{-1}$, we deduce that $1 - (-aa^{*,f,e} + 1)bb^{-} = bb^{-}aa^{*,f,e}bb^{-} + 1 - bb^{-} \in \mathcal{R}^{-1}$. Then, by [8, Proposition 3], $bb^{-}aa^{*,f,e}bb^{=} + 1 - bb^{=} = 1 - (-aa^{*,f,e} + 1)bb^{=} \in \mathcal{R}^{-1}$, for all $b^{=} \in \{1\}$, which gives $1 - bb^{=}(-aa^{*,f,e} + 1) = bb^{=}aa^{*,f,e} + 1 - bb^{=} = aa^{*,f,e} + 1 - bb^{=} \in \mathcal{R}^{-1}$.

(1) $\Rightarrow$ (6): Obviously.

(6) $\Rightarrow$ (1): Since, by $aa^{\dagger}_{e,f}bb^{\dagger}_{e,f} = bb^{\dagger}_{e,f}aa^{\dagger}_{e,f}$, $bb^{\dagger}_{e,f}u = bb^{\dagger}_{e,f}aa^{\dagger}_{e,f} = bb^{\dagger}_{e,f}aa^{\dagger}_{e,f}u$ and $u \in \mathcal{R}^{-1}$, then $bb^{\dagger}_{e,f} = bb^{\dagger}_{e,f}aa^{\dagger}_{e,f}$. Similarly, $laa^{\dagger}_{e,f} = bb^{\dagger}_{e,f}aa^{\dagger}_{e,f} = lbb^{\dagger}_{e,f}aa^{\dagger}_{e,f}$ and $l \in \mathcal{R}^{-1}$ give $aa^{\dagger}_{e,f} = bb^{\dagger}_{e,f}aa^{\dagger}_{e,f}$. Thus, $aa^{\dagger}_{e,f} = bb^{\dagger}_{e,f}$.

(1) $\Rightarrow$ (7): By Lemma 1.1.

(7) $\Rightarrow$ (3): The equality $aa^{\dagger}_{e,f}bb^{\dagger}_{e,f} = bb^{\dagger}_{e,f}aa^{\dagger}_{e,f}$ implies $aa^{\dagger}_{e,f}k = aa^{\dagger}_{e,f}bb^{*,f,e} = bb^{\dagger}_{e,f}aa^{\dagger}_{e,f}k$. Because $k \in \mathcal{R}^{-1}$, then $aa^{\dagger}_{e,f} = bb^{\dagger}_{e,f}aa^{\dagger}_{e,f}$ and the condition (3) holds. $\square$

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