Reverse order law for the group inverse in rings

Dijana Mosić and Dragan S. Djordjević[*]

Abstract

We investigate some necessary and sufficient conditions for the reverse order law for the group inverse in rings.

Key words and phrases: Group inverse; Reverse order law.
2010 *Mathematics subject classification*: 16B99, 15A09, 46L05.

1 Introduction

Let $\mathcal{R}$ be an associative ring with the unit 1, and let $a \in \mathcal{R}$. Then a is *group invertible* if there is $a^\# \in \mathcal{R}$ such that

$$(1)\ aa^\# a = a, \quad (2)\ a^\# aa^\# = a^\#, \quad (5)\ aa^\# = a^\# a;$$

$a^\#$ is a group inverse of a and it is uniquely determined by these equations. The group inverse $a^\#$ double commutes with a, that is, $ax = xa$ implies $a^\# x = xa^\#$ [1]. Denote by $\mathcal{R}^\#$ the set of all group invertible elements of $\mathcal{R}$.

Notice that we omitted the equations (3) and (4) by purpose, since these equations are related to the Moore-Penrose inverse in rings with involutions. Since we do not consider the Moore-Penrose inverse in this paper, we skip the notation.

If $\delta \subset \{1, 2, 5\}$ and b satisfies the equations (i) for all $i \in \delta$, then b is an δ–inverse of a. The set of all δ–inverse of a is denote by $a\{\delta\}$. Notice that $a\{1, 2, 5\} = \{a^\#\}$. If a is invertible, then $a^\#$ coincides with the ordinary inverse of a. The set of all invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{-1}$.

For $a \in \mathcal{R}$ consider two annihilators

$$a^\circ = \{x \in \mathcal{R} : ax = 0\}, \qquad {}^\circ a = \{x \in \mathcal{R} : xa = 0\}.$$

[*]The authors are supported by the Ministry of Science, Republic of Serbia, grant no. 174007.

If $a, b \in \mathcal{R}$ are invertible, then ab is also invertible, and the inverse of the product ab satisfied the reverse order law $(ab)^{-1} = b^{-1}a^{-1}$. Since this formula cannot trivially be extended to various generalized inverse of the product ab, the reverse order law for various generalized inverses yields a class of interesting problems that are fundamental in the theory of generalized inverses. Many authors studied these problems [1, 2, 3, 4, 5, 6].

C.Y. Deng [3] presented some necessary and sufficient conditions concerning the reverse order law for the group inverse of linear bounded operators on Hilbert spaces. He used the matrix form of operators induced by some natural decomposition of Hilbert spaces.

In this paper we investigate equivalent conditions which are related to the reverse order law for the group inverse in rings. Thus we extend the recent results [3] to more general settings, giving some new conditions and providing simpler and more transparent proofs to already existing conditions.

2 Reverse order law for the group inverse

In the following theorem, under the assumption $ba = a^2$, we study the reverse order law for the group inverse of product of two elements in ring.

Theorem 2.1. *Let $a, b, ab \in \mathcal{R}^{\#}$ and let $ba = a^2$. Then the following statements are equivalent:*

(i) $ab(ab)^{\#} = abb^{\#}a^{\#}$,

(ii) $(ab)^{\#} = b^{\#}a^{\#} = a^{\#}b^{\#} = (a^{\#})^2$.

Proof. (i) $\Rightarrow$ (ii): The assumption $ba = a^2$ gives $baa^{\#} = a$ and $b^{\#}a = b^{\#}baa^{\#}$. Now, we obtain

$$\begin{aligned} abb^{\#}a^{\#} &= ab(b^{\#}a)(a^{\#})^2 = abb^{\#}baa^{\#}(a^{\#})^2 = ab(a^{\#})^2 \\ &= a(ba)(a^{\#})^3 = aa^2(a^{\#})^3 = aa^{\#}. \end{aligned} \tag{1}$$

From the equality $ba = a^2$, we also have $b^{\#}ba^2 = b^{\#}bba = ba = a^2$ and

$$aa^{\#} = a^2(a^{\#})^2 = b^{\#}ba^2(a^{\#})^2 = b^{\#}(ba)a^{\#} = b^{\#}a^2a^{\#} = b^{\#}a. \tag{2}$$

Using $(ab)^{\#}ab = ab(ab)^{\#} = abb^{\#}a^{\#}$, (1) and (2), we get

$$\begin{aligned} (ab)^{\#} &= (ab)^{\#}(ab(ab)^{\#}) = ((ab)^{\#}ab)b^{\#}a^{\#} = (abb^{\#}a^{\#})b^{\#}a^{\#} \\ &= aa^{\#}b^{\#}a^{\#} = aa^{\#}(b^{\#}a)(a^{\#})^2 = aa^{\#}aa^{\#}(a^{\#})^2 = (a^{\#})^2. \end{aligned}$$

Observe that, by (2), $(a^\#)^2 = (aa^\#)(a^\#)^2 = b^\# a(a^\#)^2 = b^\# a^\#$. Hence, $(ab)^\# = (a^\#)^2 = b^\# a^\#$ and $abb^\# a^\# = b^\# a^\# ab$. By (1) and (2), we have

$$aa^\# = abb^\# a^\# = b^\# a^\# ab = (b^\# a)a^\# b = aa^\# a^\# b = a^\# b,$$

which yields $ab = a^2(a^\# b) = a^2 aa^\# = a^2 = ba$. Since b commutes with a and the group inverse $a^\#$ double commutes with a, then b commutes with $a^\#$. Similarly, because $a^\#$ commutes with b, we deduce that $a^\#$ commutes with $b^\#$, that is $a^\# b^\# = b^\# a^\# = (ab)^\#$.

(ii) $\Rightarrow$ (i): Obviously. □

Necessary and sufficient conditions for $(ab)^\# = (a^\# ab)^\# a^\#$ to hold are investigated in the following result.

Theorem 2.2. *If $b \in \mathcal{R}$, and if $a, ab, a^\# ab \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(ab)^\# = (a^\# ab)^\# a^\#$,

(ii) $(a^\# ab)^\# a^\# \in (ab)\{1,5\}$,

(iii) $abaa^\# = ab$ *and* $(a^\# ab)^\# aa^\# ba = ab(a^\# ab)^\#$,

(iv) $(a^\# ab)\{1,5\} \cdot a\{1,5\} \subseteq (ab)\{1,5\}$.

Proof. (i) $\Rightarrow$ (ii): This is obvious.

(ii) $\Rightarrow$ (iii): Since $(a^\# ab)^\# a^\# \in (ab)\{1,5\}$, then $ab = ab(a^\# ab)^\# a^\# ab$ and $ab(a^\# ab)^\# a^\# = (a^\# ab)^\# a^\# ab$. Therefore,

$$abaa^\# = abab(a^\# ab)^\# a^\# aa^\# = abab(a^\# ab)^\# a^\# = ab$$

and

$$\begin{aligned}(a^\# ab)^\# aa^\# ba &= ((a^\# ab)^\# a^\# ab)a = ab(a^\# ab)^\# a^\# a \\ &= a(a^\# ab(a^\# ab)^\#)a^\# a = a(a^\# ab)^\# a^\# (aba^\# a) \\ &= a((a^\# ab)^\# a^\# ab) = aa^\# ab(a^\# ab)^\# = ab(a^\# ab)^\#.\end{aligned}$$

(iii) $\Rightarrow$ (i): The equalities $abaa^\# = ab$ and $(a^\# ab)^\# aa^\# ba = ab(a^\# ab)^\#$ imply

$$(a^\# ab)^\# a^\# (ab) = ((a^\# ab)^\# a^\# aba)a^\# = ab(a^\# ab)^\# a^\#.$$

So, $(a^\# ab)^\# a^\# \in (ab)\{5\}$. Notice that, from

$$ab(a^\# ab)^\# a^\# ab = a(a^\# ab(a^\# ab)^\# a^\# ab) = aa^\# ab = ab,$$

and

$$((a^{\#}ab)^{\#}a^{\#}ab(a^{\#}ab)^{\#})a^{\#} = (a^{\#}ab)^{\#}a^{\#}$$

$(a^{\#}ab)^{\#}a^{\#} \in (ab)\{1,2\}$ and the condition (i) holds.

(iii) $\Rightarrow$ (iv): Suppose that $abaa^{\#} = ab$ and $(a^{\#}ab)^{\#}aa^{\#}ba = ab(a^{\#}ab)^{\#}$. Let $(a^{\#}ab)^{(1,5)} \in (a^{\#}ab)\{1,5\}$ and $a^{(1,5)} \in a\{1,5\}$. Observe that

$$aa^{(1,5)} = a^{\#}a(aa^{(1,5)}) = a^{\#}(aa^{(1,5)}a) = a^{\#}a, \tag{3}$$

which yields

$$ab(a^{\#}ab)^{(1,5)}(a^{(1,5)}a)b = a(a^{\#}ab(a^{\#}ab)^{(1,5)}a^{\#}ab) = aa^{\#}ab = ab.$$

Hence, $(a^{\#}ab)^{(1,5)}a^{(1,5)} \in (ab)\{1\}$. Applying the equality (3) for $a^{\#}ab$ instead of a, we have $a^{\#}ab(a^{\#}ab)^{(1,5)} = a^{\#}ab(a^{\#}ab)^{\#}$. From this and (iii), we obtain that $(a^{\#}ab)^{(1,5)}a^{(1,5)} \in (ab)\{5\}$:

$$\begin{aligned} ab(a^{\#}ab)^{(1,5)}a^{(1,5)} &= a(a^{\#}ab(a^{\#}ab)^{(1,5)})a^{(1,5)} = aa^{\#}ab(a^{\#}ab)^{\#}a^{(1,5)} \\ &= (ab(a^{\#}ab)^{\#})a^{(1,5)} = (a^{\#}ab)^{\#}aa^{\#}b(aa^{(1,5)}) \\ &= (a^{\#}ab)^{\#}a^{\#}(abaa^{\#}) = (a^{\#}ab)^{\#}a^{\#}ab \\ &= (a^{\#}ab)^{(1,5)}(a^{\#}a)b = (a^{\#}ab)^{(1,5)}a^{(1,5)}ab. \end{aligned}$$

Thus, for any $(a^{\#}ab)^{(1,5)} \in (a^{\#}ab)\{1,5\}$ and $a^{(1,5)} \in a\{1,5\}$, $(a^{\#}ab)^{(1,5)}a^{(1,5)} \in (ab)\{1,5\}$ and the statement (iv) is satisfied.

(iv) $\Rightarrow$ (ii): It follows by $(a^{\#}ab)^{\#} \in (a^{\#}ab)\{1,5\}$ and $a^{\#} \in a\{1,5\}$. $\square$

The following theorem concerning $(ab)^{\#} = b^{\#}(abb^{\#})^{\#}$ can be proved in the same way as Theorem 2.2.

Theorem 2.3. *If $a \in \mathcal{R}$, and if $b, ab, abb^{\#} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)^{\#} = b^{\#}(abb^{\#})^{\#}$,

(ii) $b^{\#}(abb^{\#})^{\#} \in (ab)\{1,5\}$,

(iii) $bb^{\#}ab = ab$ *and* $babb^{\#}(abb^{\#})^{\#} = (abb^{\#})^{\#}ab$,

(iv) $b\{1,5\} \cdot (abb^{\#})\{1,5\} \subseteq (ab)\{1,5\}$.

In the following theorems, we consider the conditions which ensure that $(a^{\#}ab)^{\#} = b^{\#}a^{\#}a$ and $(abb^{\#})^{\#} = bb^{\#}a^{\#}$ hold.

Theorem 2.4. *If $a, b, a^{\#}ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(a^\# ab)^\# = b^\# a^\# a$,

(ii) $a^\# ab = ba^\# a$.

Proof. (i) $\Rightarrow$ (ii): The hypothesis $(a^\# ab)^\# = b^\# a^\# a$ implies

$$a^\# abb^\# a^\# a = b^\# a^\# aa^\# ab = b^\# a^\# ab \tag{4}$$

and

$$b^\# a^\# a = b^\# a^\# a(a^\# ab)b^\# a^\# a = b^\# (a^\# abb^\# a^\# a) = b^\# b^\# a^\# ab. \tag{5}$$

By the equalities (4) and (5), we get

$$ba^\# a = b^2(b^\# a^\# a) = b^2 b^\# b^\# a^\# ab = bb^\# a^\# ab. \tag{6}$$

and

$$(a^\# abb^\# a^\# a)b = b^\# a^\# abb = b(b^\# b^\# a^\# ab)b = bb^\# a^\# ab. \tag{7}$$

Since

$$a^\# ab = a^\# ab(a^\# ab)^\# a^\# ab = a^\# abb^\# a^\# aa^\# ab = a^\# abb^\# a^\# ab,$$

by (7) and (6), we obtain

$$a^\# ab = bb^\# a^\# ab = ba^\# a.$$

Thus, the statements (ii) is satisfied.

(ii) $\Rightarrow$ (i): Let $a^\# ab = ba^\# a$. Since the group inverse $b^\#$ double commutes with b, we obtain $a^\# ab^\# = b^\# a^\# a$ and $a^\# abb^\# = bb^\# a^\# a$. Now, we can easy check that $b^\# a^\# a \in (a^\# ab)\{1, 2, 5\}$. $\square$

In the same way as in the proof of Theorem 2.4, we can verify the following result.

Theorem 2.5. *If $a, b, abb^\# \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(abb^\#)^\# = bb^\# a^\#$,

(ii) $abb^\# = bb^\# a$.

Assuming that elements a, b and $a^\# ab$ are group invertible, we give the following equivalent condition for $(a^\# ab)^\# a^\# = b^\# a^\#$.

Theorem 2.6. *If $a, b, a^{\#}ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(a^{\#}ab)^{\#}a^{\#} = b^{\#}a^{\#}$,

(ii) $ba^{\#}a = a^{\#}aba^{\#}a$,

(iii) $ba^{\#}a\mathcal{R} \subset a\mathcal{R}$ (*or* ${}^{\circ}a \subset {}^{\circ}(ba^{\#}a)$).

Proof. (i) $\Rightarrow$ (ii): Suppose that $(a^{\#}ab)^{\#}a^{\#} = b^{\#}a^{\#}$. Then

$$a^{\#}ab = a^{\#}ab(a^{\#}ab)^{\#}a^{\#}ab = ((a^{\#}ab)^{\#}a^{\#})aba^{\#}ab = b^{\#}a^{\#}aba^{\#}ab$$

gives

$$\begin{aligned}(a^{\#}ab)a^{\#}a &= b^{\#}a^{\#}aba^{\#}aba^{\#}a = b^{\#}b(b^{\#}a^{\#}aba^{\#}ab)a^{\#}a \\ &= b^{\#}ba^{\#}aba^{\#}a = b(b^{\#}a^{\#})aba^{\#}a = b((a^{\#}ab)^{\#}a^{\#}ab)a^{\#}a \\ &= ba^{\#}ab((a^{\#}ab)^{\#}a^{\#})a = ba^{\#}abb^{\#}a^{\#}a.\end{aligned}$$

Using this equality and

$$b^{\#}a^{\#} = (a^{\#}ab)^{\#}a^{\#} = (a^{\#}ab)^{\#}a^{\#}ab(a^{\#}ab)^{\#}a^{\#} = b^{\#}a^{\#}abb^{\#}a^{\#},$$

we observe that (ii) holds:

$$a^{\#}aba^{\#}a = ba^{\#}abb^{\#}a^{\#}a = b^2(b^{\#}a^{\#}abb^{\#}a^{\#})a = b^2b^{\#}a^{\#}a = ba^{\#}a.$$

(ii) $\Rightarrow$ (i): If $ba^{\#}a = a^{\#}aba^{\#}a$, we have

$$\begin{aligned}b^{\#}a^{\#} &= b^{\#}b^{\#}(baa^{\#})a^{\#} = b^{\#}b^{\#}(a^{\#}ab)a^{\#}aa^{\#} = b^{\#}b^{\#}a^{\#}ab((a^{\#}ab)^{\#}a^{\#}ab)a^{\#} \\ &= b^{\#}b^{\#}(a^{\#}aba^{\#}a)b(a^{\#}ab)^{\#}a^{\#} = b^{\#}b^{\#}ba^{\#}ab(a^{\#}ab)^{\#}a^{\#} \\ &= b^{\#}a^{\#}ab(a^{\#}ab)^{\#}a^{\#}. \qquad (8)\end{aligned}$$

From the condition (ii) and the equality

$$(a^{\#}ab)^{\#} = a^{\#}ab[(a^{\#}ab)^{\#}]^2 = a^{\#}a(a^{\#}ab[(a^{\#}ab)^{\#}]^2) = a^{\#}a(a^{\#}ab)^{\#},$$

we obtain

$$\begin{aligned}(a^{\#}ab)^{\#}a^{\#} &= (a^{\#}aba^{\#}a)b[(a^{\#}ab)^{\#}]^3a^{\#} = ba^{\#}ab[(a^{\#}ab)^{\#}]^3a^{\#} \\ &= bb^{\#}(ba^{\#}a)b[(a^{\#}ab)^{\#}]^3a^{\#} = bb^{\#}a^{\#}aba^{\#}ab[(a^{\#}ab)^{\#}]^3a^{\#} \\ &= bb^{\#}a^{\#}a(a^{\#}aba^{\#}ab[(a^{\#}ab)^{\#}]^3)a^{\#} = b^{\#}(ba^{\#}a)(a^{\#}ab)^{\#}a^{\#} \\ &= b^{\#}a^{\#}ab(a^{\#}a(a^{\#}ab)^{\#})a^{\#} = b^{\#}a^{\#}ab(a^{\#}ab)^{\#}a^{\#}.\end{aligned}$$

Hence, by (8), $(a^{\#}ab)^{\#}a^{\#} = b^{\#}a^{\#}$.

(ii) $\Leftrightarrow$ (iii): By the equality $ba^{\#}a = aa^{\#}ba^{\#}a$, obviously $ba^{\#}a\mathcal{R} \subset a\mathcal{R}$. Conversely, from $ba^{\#}a\mathcal{R} \subset a\mathcal{R}$, we observe that $ba^{\#}a = ax$ for some $x \in \mathcal{R}$. Now, $ba^{\#}a = ax = aa^{\#}(ax) = aa^{\#}ba^{\#}a$. $\square$

The following result can be proved in the similar manner as Theorem 2.6.

Theorem 2.7. *If $a, b, abb^{\#} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $b^{\#}(abb^{\#})^{\#} = b^{\#}a^{\#}$,

(ii) $bb^{\#}a = bb^{\#}abb^{\#}$,

(iii) $\mathcal{R}bb^{\#}a \subset \mathcal{R}b$ *(or* $b^{\circ} \subset (bb^{\#}a)^{\circ}$*).*

Combining the conditions of Theorem 2.2 and Theorem 2.6 (or Theorem 2.3 and Theorem 2.7), we can get the sufficient conditions for the reverse order law $(ab)^{\#} = b^{\#}a^{\#}$ to be satisfied.

Theorem 2.8. *If $a, b, ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $b^{\#} = (ab)^{\#}a$,

(ii) $b = aa^{\#}b = baa^{\#}$ *and* $abb^{\#} = bb^{\#}a$,

(iii) $b\mathcal{R} \subset a\mathcal{R}$, $aa^{\#}b = baa^{\#}$ *and* $abb^{\#} = bb^{\#}a$,

(iv) $a^{\circ} \subset b^{\circ}$, $aa^{\#}b = baa^{\#}$ *and* $abb^{\#} = bb^{\#}a$.

Proof. (i) $\Rightarrow$ (ii): First, the assumption $b^{\#} = (ab)^{\#}a$ implies

$$b = b^{\#}b^2 = ((ab)^{\#}ab)b = ab(ab)^{\#}b = aa^{\#}(ab(ab)^{\#}b) = aa^{\#}b$$

and

$$abb^{\#} = (ab(ab)^{\#})a = ((ab)^{\#}a)ba = b^{\#}ba.$$

Since $abb^{\#} = b^{\#}ba$ and the group inverse $a^{\#}$ double commutes with a, then $a^{\#}bb^{\#} = b^{\#}ba^{\#}$ and $a^{\#}abb^{\#} = b^{\#}ba^{\#}a$. Further,

$$baa^{\#} = b(b^{\#}baa^{\#}) = b(a^{\#}ab)b^{\#} = bbb^{\#} = b.$$

Thus, the condition (ii) holds.

(ii) $\Rightarrow$ (i): Assume that $b = aa^{\#}b = baa^{\#}$ and $abb^{\#} = bb^{\#}a$. It follows that $(ab)^{\#}a \in \ b\{1, 2\}$ from

$$b(ab)^{\#}ab = a^{\#}ab(ab)^{\#}ab = a^{\#}ab = b$$

and

$$(ab)^{\#}ab(ab)^{\#}a = (ab)^{\#}a.$$

Then, by

$$\begin{aligned} b(ab)^{\#}a &= b[(ab)^{\#}]^2aba = b([(ab)^{\#}]^2ab)bb^{\#}a = b(ab)^{\#}(bb^{\#}a) \\ &= b(ab)^{\#}abb^{\#} = a^{\#}(ab(ab)^{\#}ab)b^{\#} = (a^{\#}ab)b^{\#} = bb^{\#} \end{aligned}$$

and

$$ab(ab)^{\#} = (abb^{\#})b(ab)^{\#} = bb^{\#}ab(ab)^{\#} = b^{\#}(b(ab)^{\#}ab) = b^{\#}b,$$

we conclude that $b(ab)^{\#}a = ab(ab)^{\#}$. So, $(ab)^{\#}a \in\ b\{5\}$ and the condition (i) is satisfied.

(ii) $\Leftrightarrow$ (iii): We only need to prove that $b = aa^{\#}b$ is equivalent to $b\mathcal{R} \subset a\mathcal{R}$. By $b = aa^{\#}b$, we have $b\mathcal{R} \subset a\mathcal{R}$. Conversely, if $b\mathcal{R} \subset a\mathcal{R}$, then, for some $x \in R$, $b = ax$ which gives $b = aa^{\#}(ax) = aa^{\#}b$.

(ii) $\Leftrightarrow$ (iv): This part follows because $b = baa^{\#}$ iff $b(1 - aa^{\#}) = 0$ iff $(1 - aa^{\#})\mathcal{R} \subset b^{\circ}$ iff $a^{\circ} \subset b^{\circ}$. $\square$

In the same way as in Theorem 2.8, we prove the following theorem.

Theorem 2.9. *If $a, b, ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $a^{\#} = b(ab)^{\#}$,

(ii) $aa^{\#}b = baa^{\#}$ *and* $a = abb^{\#} = bb^{\#}a$,

(iii) $a\mathcal{R} \subset b\mathcal{R}$, $aa^{\#}b = baa^{\#}$ *and* $abb^{\#} = bb^{\#}a$,

(iv) ${}^{\circ}b \subset {}^{\circ}a$, $aa^{\#}b = baa^{\#}$ *and* $abb^{\#} = bb^{\#}a$.

We can verify that $b^{\#} = (ab)^{\#}a$ or $a^{\#} = b(ab)^{\#}$ implies $(ab)^{\#} = b^{\#}a^{\#}$, by Theorem 2.8 and Theorem 2.9.

Theorem 2.10. *If $a, b, ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $a^{\#}ab = b(ab)^{\#}ab$,

(ii) $baba = aa^{\#}baba$,

(iii) $baba\mathcal{R} = a\mathcal{R}$ (*or* ${}^{\circ}a \subset {}^{\circ}(baba)$).

Proof. (i) ⇒ (ii): Using the hypothesis $a^{\#}ab = b(ab)^{\#}ab$, we observe that

$$(aa^{\#}b)aba = b((ab)^{\#}abab)a = baba.$$

Hence, the statements (ii) is satisfied.

(ii) ⇒ (i): The equality $baba = aa^{\#}baba$ gives

$$a^{\#}ab = a^{\#}abab(ab)^{\#} = (aa^{\#}baba)b[(ab)^{\#}]^2 = babab[(ab)^{\#}]^2 = b(ab)^{\#}ab.$$

(ii) ⇒ (iii): Obviously.

(iii) ⇒ (ii): The condition $baba\mathcal{R} = a\mathcal{R}$ give $baba = ax$, for $x \in \mathcal{R}$, which implies $baba = ax = aa^{\#}(ax) = aa^{\#}baba$.

□

The next result follows analogously to Theorem 2.10.

Theorem 2.11. *If $a, b, ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $abb^{\#} = ab(ab)^{\#}a$,

(ii) $baba = bababb^{\#}$,

(iii) $\mathcal{R}baba = \mathcal{R}b$ (*or* $b^{\circ} \subset (baba)^{\circ}$).

Some equivalent conditions to the reverse order law $(ab)^{\#} = b^{\#}a^{\#}$ in settings of a ring are presented in the following theorem.

Theorem 2.12. *If $a, b, ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)^{\#} = b^{\#}a^{\#}$,

(ii) $(ab)^{\#}a = b^{\#}a^{\#}a$ *and* $a^{\#}ab = a^{\#}aba^{\#}a$,

(iii) $b(ab)^{\#} = bb^{\#}a^{\#}$ *and* $abb^{\#} = bb^{\#}abb^{\#}$.

Proof. (i) ⇒ (ii): From the equality $(ab)^{\#} = b^{\#}a^{\#}$, we get $(ab)^{\#}a = b^{\#}a^{\#}a$ and

$$\begin{aligned} a^{\#}ab &= a^{\#}ab((ab)^{\#}ab) = a^{\#}abab(ab)^{\#} = a^{\#}ababb^{\#}a^{\#} \\ &= a^{\#}(ababb^{\#}a^{\#})aa^{\#} = a^{\#}abaa^{\#}. \end{aligned}$$

So, the condition (ii) holds.

(ii) $\Rightarrow$ (i): If $(ab)^{\#}a = b^{\#}a^{\#}a$ and $a^{\#}ab = a^{\#}aba^{\#}a$, we verify that $b^{\#}a^{\#} \in (ab)\{5\}$:

$$\begin{aligned}(b^{\#}a^{\#}a)b &= (ab)^{\#}ab = (ab)^{\#}a(a^{\#}ab) = (ab)^{\#}aa^{\#}aba^{\#}a \\ &= ((ab)^{\#}ab)a^{\#}a = ab((ab)^{\#}a)a^{\#} = abb^{\#}a^{\#}aa^{\#} \\ &= abb^{\#}a^{\#}.\end{aligned}$$

Then, by the equalities

$$ab = ab((ab)^{\#}a)b = abb^{\#}a^{\#}ab$$

and

$$\begin{aligned}b^{\#}a^{\#} &= (b^{\#}a^{\#}a)a^{\#} = (ab)^{\#}aa^{\#} = ((ab)^{\#}a)b((ab)^{\#}a)a^{\#} \\ &= b^{\#}a^{\#}abb^{\#}a^{\#}aa^{\#} = b^{\#}a^{\#}abb^{\#}a^{\#},\end{aligned}$$

we deduce that $b^{\#}a^{\#} \in (ab)\{1,2\}$, i.e. $(ab)^{\#} = b^{\#}a^{\#}$.

(i) $\Leftrightarrow$ (iii): This equivalence can be proved similarly as (i) $\Leftrightarrow$ (ii). $\square$

Now, we study necessary and sufficient condition which involve $a^{\#}ab \in \mathcal{R}^{\#}$ to ensure that $(ab)^{\#} = b^{\#}a^{\#}$ holds.

Theorem 2.13. *If $a, b, ab, a^{\#}ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)^{\#} = b^{\#}a^{\#}$,

(ii) $(ab)^{\#}a = b^{\#}a^{\#}a = (a^{\#}ab)^{\#}$,

(iii) $(ab)^{\#}a = b^{\#}a^{\#}a$ *and* $a^{\#}ab = ba^{\#}a$,

(iv) $(ab)^{\#} = (a^{\#}ab)^{\#}a^{\#}$ *and* $(a^{\#}ab)^{\#} = b^{\#}a^{\#}a$.

Proof. (i) $\Rightarrow$ (ii): The assumption $(ab)^{\#} = b^{\#}a^{\#}$ gives that $(ab)^{\#}a = b^{\#}a^{\#}a$ and $a^{\#}ab = a^{\#}aba^{\#}a$, by Theorem 2.12. Observe that

$$(ab)^{\#} = ab[(ab)^{\#}]^2 = aa^{\#}(ab[(ab)^{\#}]^2) = aa^{\#}(ab)^{\#},$$

and then

$$\begin{aligned}b^{\#}a^{\#}a &= (ab)^{\#}a = aa^{\#}(ab)^{\#}a = aa^{\#}b^{\#}a^{\#}a = (a^{\#}ab)b^{\#}b^{\#}a^{\#}a \\ &= (a^{\#}ab)^{\#}(a^{\#}aba^{\#}a)(bb^{\#}b^{\#})a^{\#}a = (a^{\#}ab)^{\#}a^{\#}abb^{\#}a^{\#}a. \quad (9)\end{aligned}$$

Since

$$\begin{aligned}(a^{\#}ab)^{\#} &= [(a^{\#}ab)^{\#}]^2(a^{\#}ab) = [(a^{\#}ab)^{\#}]^2a^{\#}aba^{\#}a \\ &= [(a^{\#}ab)^{\#}]^2(a^{\#}ab)b^{\#}ba^{\#}a = ([(a^{\#}ab)^{\#}]^2a^{\#}ab)a^{\#}ab^{\#}ba^{\#}a \\ &= (a^{\#}ab)^{\#}a^{\#}ab^{\#}ba^{\#}a,\end{aligned}$$

by (9), we have $(a^{\#}ab)^{\#} = b^{\#}a^{\#}a$.

(ii) $\Rightarrow$ (i): Let $(ab)^{\#}a = b^{\#}a^{\#}a = (a^{\#}ab)^{\#}$. Notice that $b^{\#}a^{\#} \in (ab)\{1,2\}$, by

$$ab(b^{\#}a^{\#}a)b = ab(ab)^{\#}ab = ab,$$

and

$$\begin{aligned}b^{\#}a^{\#}abb^{\#}a^{\#} &= (b^{\#}a^{\#}a)a^{\#}ab(b^{\#}a^{\#}a)a^{\#} = (a^{\#}ab)^{\#}a^{\#}ab(a^{\#}ab)^{\#}a^{\#} \\ &= (a^{\#}ab)^{\#}a^{\#} = b^{\#}a^{\#}aa^{\#} = b^{\#}a^{\#}.\end{aligned}$$

The condition $b^{\#}a^{\#}a = (a^{\#}ab)^{\#}$ implies

$$a^{\#}abb^{\#}a^{\#}a = b^{\#}a^{\#}aa^{\#}ab = b^{\#}a^{\#}ab.$$

which yields

$$\begin{aligned}abb^{\#}a^{\#} &= ab(b^{\#}a^{\#}a)a^{\#} = ab(ab)^{\#}aa^{\#} = ((ab)^{\#}a)baa^{\#} = (b^{\#}a^{\#}ab)aa^{\#} \\ &= a^{\#}abb^{\#}a^{\#}aaa^{\#} = a^{\#}abb^{\#}a^{\#}a = b^{\#}a^{\#}ab.\end{aligned}$$

Thus, $b^{\#}a^{\#} \in (ab)\{5\}$ and (i) is satisfied.

(ii) $\Leftrightarrow$ (iii): This part follows by Theorem 2.4.

(i) $\Rightarrow$ (iv): By (i) $\Rightarrow$ (ii), we see that $(ab)^{\#} = b^{\#}a^{\#}$ implies $(a^{\#}ab)^{\#} = b^{\#}a^{\#}a$. Therefore,

$$(a^{\#}ab)^{\#}a^{\#} = b^{\#}a^{\#}aa^{\#} = b^{\#}a^{\#} = (ab)^{\#}.$$

(iv) $\Rightarrow$ (i): The equality $(ab)^{\#} = (a^{\#}ab)^{\#}a^{\#}$ and $(a^{\#}ab)^{\#} = b^{\#}a^{\#}a$ give

$$(ab)^{\#} = (a^{\#}ab)^{\#}a^{\#} = b^{\#}a^{\#}aa^{\#} = b^{\#}a^{\#}.$$

Hence, the reverse order law $(ab)^{\#} = b^{\#}a^{\#}$ holds. □

Similarly as in Theorem 2.13, we obtain the following theorem related to the reverse order law for the group inverse which contains the hypothesis $abb^{\#} \in \mathcal{R}^{\#}$.

Theorem 2.14. *If $a, b, ab, abb^{\#} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)^\# = b^\# a^\#$,

(ii) $b(ab)^\# = bb^\# a^\# = (abb^\#)^\#$,

(iii) $b(ab)^\# = bb^\# a^\#$ *and* $abb^\# = bb^\# a$,

(iv) $(ab)^\# = b^\#(abb^\#)^\#$ *and* $(abb^\#)^\# = bb^\# a^\#$.

Both equalities in the condition (iv) of Theorem 2.13 (Theorem 2.14) can be replaced by some equivalent condition of Theorem 2.2 and Theorem 2.4 (Theorem 2.3 and Theorem 2.5).

Theorem 2.15. (i) *If $a, ab \in \mathcal{R}^\#$, then*

$$1 + a^\#(b-a) \in \mathcal{R}^{-1} \Leftrightarrow ab(ab)^\# a = a.$$

(ii) *If $a, ba \in \mathcal{R}^\#$, then*

$$1 + a^\#(b-a) \in \mathcal{R}^{-1} \Leftrightarrow a(ba)^\# ba = a$$

Proof. Since $a \in \mathcal{R}^\#$, then a, b and ab have the matrix representations

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix} \quad \textit{and} \quad ab = \begin{bmatrix} a_1 b_1 & a_1 b_3 \\ 0 & 0 \end{bmatrix}.$$

The group inverse of a is given by

$$a^\# = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

a_1 is invertible. In the similar way as in Theorem [7, Theorem 1] for the matrix case, we have $ab \in \mathcal{R}^\#$ if and only if $a_1 b_1 \in \mathcal{R}^\#$ and $a_1 b_1 (a_1 b_1)^\# a_1 b_3 = a_1 b_3$. In this case,

$$(ab)^\# = \begin{bmatrix} (a_1 b_1)^\# & [(a_1 b_1)^\#]^2 a_1 b_3 \\ 0 & 0 \end{bmatrix}.$$

Further, $ab(ab)^\# a = a$ iff $a_1 b_1 (a_1 b_1)^\# a_1 = a_1$ iff $a_1 b_1 (a_1 b_1)^\# = 1$ iff $a_1 b_1$ is invertible iff b_1 is invertible. From

$$1 + a^\#(b-a) = \begin{bmatrix} a_1^{-1} b_1 & a_1^{-1} b_3 \\ 0 & 1 \end{bmatrix},$$

we conclude that $1 + a^\#(b-a) \in \mathcal{R}^{-1}$ iff b_1 is invertible . $\square$

Several equivalent conditions for $aa^\# = bb^\#$ to hold are given in the following theorem in a ring.

Theorem 2.16. *If $a, b \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $aa^\# = bb^\#$,

(ii) $a\mathcal{R} = b\mathcal{R}$ *and* $a^\circ = b^\circ$,

(iii) $a + 1 - bb^\# \in \mathcal{R}^{-1}$ *and* $aa^\# = aa^\# bb^\# = bb^\# aa^\#$,

(iv) $a + 1 - bb^\#, 1 - aa^\# + bb^\# \in \mathcal{R}^{-1}$ *and* $abb^\# = bb^\# a$,

(v) $a + 1 - bb^\#, 1 - aa^\# + bb^\# \in \mathcal{R}^{-1}$ *and* $aa^\# b = baa^\#$,

(vi) $a + 1 - bb^\#, 1 - aa^\# + bb^\# \in \mathcal{R}^{-1}$ *and* $aa^\# bb^\# = bb^\# aa^\#$,

(vii) $1 - aa^\# + b \in \mathcal{R}^{-1}$ *and* $b = aa^\# b = baa^\#$,

(viii) $1 + a^\#(b - a) \in \mathcal{R}^{-1}$ *and* $b = aa^\# b = baa^\#$.

Proof. (i) $\Rightarrow$ (ii)-(viii): Obviously, if we observe that $(a + 1 - aa^\#)(a^\# + 1 - aa^\#) = 1$ and $(1 + a^\#(b - a))(1 + (b^\# - a^\#)a) = (1 + (b^\# - a^\#)a)(1 + a^\#(b - a)) = 1$ imply $a + 1 - aa^\# \in \mathcal{R}^{-1}$ and $1 + a^\#(b - a) \in \mathcal{R}^{-1}$.

(ii) $\Rightarrow$ (i): Let $a\mathcal{R} = b\mathcal{R}$ and $a^\circ = b^\circ$. Then $b = ax$ for $x \in \mathcal{R}$ and, by $b^\circ = (1 - bb^\#)\mathcal{R}$, $a^\circ = (1 - bb^\#)\mathcal{R}$. Consequently, $b = aa^\#(ax) = aa^\# b$ and $a(1 - bb^\#) = 0$. Therefore, $bb^\# = aa^\# bb^\# = a^\#(abb^\#) = a^\# a$.

(iii) $\Rightarrow$ (i): Suppose that $a+1-bb^\# \in \mathcal{R}^{-1}$ and $aa^\# = aa^\# bb^\# = bb^\# aa^\#$. Then, from

$$(a + 1 - bb^\#)bb^\# = abb^\# + bb^\# - bb^\# = abb^\#$$

and

$$(a + 1 - bb^\#)bb^\# aa^\# = a(bb^\# aa^\#) = aaa^\# bb^\# = abb^\#,$$

we conclude that $bb^\# = bb^\# aa^\#$ which gives $bb^\# = aa^\#$.

(iv) $\Rightarrow$ (iii): Since $abb^\# = bb^\# a$, and the group inverse $a^\#$ double commutes with a, then $a^\# bb^\# = bb^\# a^\#$ and $aa^\# bb^\# = bb^\# aa^\#$. The equalities

$$(1 - aa^\# + bb^\#)aa^\# = aa^\# - aa^\# + bb^\# aa^\# = bb^\# aa^\#,$$

$$(1 - aa^\# + bb^\#)aa^\# bb^\# = bb^\#(aa^\# bb^\#) = bb^\# aa^\#,$$

and the assumption $1 - aa^\# + bb^\# \in \mathcal{R}^{-1}$, imply $aa^\# = aa^\# bb^\#$. Thus, the condition (iii) is satisfied.

(v) $\Rightarrow$ (iii): It follows similarly as (iv) $\Rightarrow$ (iii).

(vi) $\Rightarrow$ (i): In the same way as (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(vii) $\Rightarrow$ (i): Assume that $1 - aa^{\#} + b \in \mathcal{R}^{-1}$ and $b = aa^{\#}b = baa^{\#}$. By double commutativity of the group inverse, we conclude that $aa^{\#}b^{\#} = b^{\#}aa^{\#}$ and $aa^{\#}bb^{\#} = bb^{\#}aa^{\#}$. Further, from

$$(1 - aa^{\#} + b)aa^{\#} = aa^{\#} - aa^{\#} + baa^{\#} = baa^{\#}$$

and

$$(1 - aa^{\#} + b)aa^{\#}bb^{\#} = b(aa^{\#}bb^{\#}) = bbb^{\#}aa^{\#} = baa^{\#},$$

we have $aa^{\#} = aa^{\#}bb^{\#}$. Therefore, $aa^{\#} = bb^{\#}$.

(viii) $\Rightarrow$ (i): Using the hypothesis $b = aa^{\#}b = baa^{\#}$, we obtain

$$(1 + a^{\#}(b - a))a^{\#}a = a^{\#}a + a^{\#}(ba^{\#}a) - a^{\#}a = a^{\#}b$$

and

$$(1 + a^{\#}(b - a))a^{\#}abb^{\#} = a^{\#}bbb^{\#} = a^{\#}b.$$

Now, by the condition $1 + a^{\#}(b - a) \in \mathcal{R}^{-1}$, we have $a^{\#}a = a^{\#}abb^{\#}$. So, $a^{\#}a = (a^{\#}ab)b^{\#} = bb^{\#}$.

$\square$

Finally, we give some examples to illustrate our results.

Example 2.1. Let $\mathcal{R}$ be an associative ring with the unit 1. If $a = \begin{bmatrix} p & 0 \\ 0 & c \end{bmatrix} \in \mathcal{R}$ and $b = \begin{bmatrix} d & 0 \\ 0 & 1-p \end{bmatrix} \in \mathcal{R}$ relative to the idempotent $p \in \mathcal{R}$, and if d and c are group invertible, then $a^{\#} = \begin{bmatrix} p & 0 \\ 0 & c^{\#} \end{bmatrix}$ and $b = \begin{bmatrix} d^{\#} & 0 \\ 0 & 1-p \end{bmatrix}$. It follows that statements of Theorem 2.12 (or Theorem 2.13 or Theorem 2.14) are satisfied, implying $ab = \begin{bmatrix} d & 0 \\ 0 & c \end{bmatrix}$ and $(ab)^{\#} = \begin{bmatrix} d^{\#} & 0 \\ 0 & c^{\#} \end{bmatrix} = b^{\#}a^{\#}$.

Example 2.2. In a ring $\mathcal{R}$, if $a = \begin{bmatrix} p & c \\ 0 & 0 \end{bmatrix} \in \mathcal{R}$ and $b = \begin{bmatrix} d & 0 \\ 0 & 1-p \end{bmatrix} \in \mathcal{R}$ relative to the idempotent $p \in \mathcal{R}$, and if $dc = c$ and d is invertible, then $a^{\#} = a$ and $b^{\#} = \begin{bmatrix} d^{-1} & 0 \\ 0 & 1-p \end{bmatrix}$. Since statements of Theorem 2.12 (or

Theorem 2.13 or Theorem 2.14) are satisfied, we obtain $ab = \begin{bmatrix} d & c \\ 0 & 0 \end{bmatrix}$ and $(ab)^\# = \begin{bmatrix} d^{-1} & c \\ 0 & 0 \end{bmatrix} = b^\# a^\#$.

Acknowledgement. We are grateful to the anonymous referees for carefully reading of the paper.

References

[1] A. Ben-Israel, T.N.E. Greville, *Generalized Inverses: Theory and Applications*, 2nd ed., Springer, New York, 2003.

[2] C. Cao, X. Zhang, X. Tang, *Reverse order law of group inverses of products of two matrices*, Appl. Math. Comput. 158 (2004), 489-495.

[3] C.Y. Deng, *Reverse order law for the group inverses*, J. Math. Anal. Appl. 382 (2) (2011), 663-671.

[4] D. S. Djordjević, *Unified approach to the reverse order rule for generalized inverses*, Acta Sci. Math. (Szeged), 67 (2001), 761-776.

[5] T.N.E. Greville, *Note on the generalized inverse of a matrix product*, SIAM Rev. 8 (1966), 518–521.

[6] S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tohoku Math. J. 34 (1982), 43–52.

[7] R.E. Hartwig, J. Shoaf *Group inverses and Drazin inverses of bidiagonal and triangular Toeplitz matrices*, J. Austral. Math. Soc. 24(A) (1977), 10–34.

Address:

Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia

E-mail

D. Mosić: `dijana@pmf.ni.ac.rs`
D. S. Djordjević: `dragan@pmf.ni.ac.rs`