ON LEFT-RIGHT CONSISTENCY IN RINGS

By

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Abstract *A bounded linear operator T is said to be "left-right consistent" if the products ST and TS always have the same spectrum: this notion lies behind a spectral property of positive operators. Extended to Banach algebras, the same notion helps to delineate the closure of the invertible group.*

In a C^*-algebra the spectrum of the product of a positive and a self adjoint element is always real. This simple observation is the tip of a curious iceberg, built on a sort of "left-right consistency" relative to invertibility.

Suppose A is a semigroup, assumed by default to have an identity 1, with invertible group $A^{-1} = A^{-1}_{left} \cap A^{-1}_{right}$; more generally much of what we have to say extends to an abstract category. Elements $a \in A$ induce left and right multiplications on A,

$$L_a : x \mapsto ax \ ; \ R_a : x \mapsto xa \ .$$

It is the relationship between these operators which gives rise to the left-right consistency behind the positive operator phenomenon:

Definition 1. *If $K \subseteq A$ is arbitrary write*

1.1 $$\varpi(K) = \{a \in A : L_a^{-1}(K) = R_a^{-1}(K)\}$$

for the set of K-(left-right) consistent elements of A.

Evidently $\varpi(K)$ is always a sub-semigroup:

1.2 $$\varpi(K) \cdot \varpi(K) \subseteq \varpi(K) \ .$$

In this note we determine $\varpi(K)$ for the invertible group $K = A^{-1}$ and for the semigroups A^{-1}_{left} and A^{-1}_{right} of left and of right invertibles:

Theorem 2. *For an arbitrary semigroup A there is equality*

2.1 $$\varpi(A^{-1}) = A^{-1} \cup \big(A \setminus (A^{-1}_{left} \cup A^{-1}_{right})\big) = \varpi(\{1\}) \ .$$

If $K \subseteq A^{-1}$ is a normal subgroup then also

2.2 $$\varpi(K) = \varpi(A^{-1}) \ .$$

Proof. We claim, if $K \subseteq A^{-1}$ is a normal subgroup,

2.3 $$a \in A^{-1} \Longrightarrow L_a^{-1}(K) = a^{-1}K = Ka^{-1} = R_a^{-1}(K) \ ;$$

2.4 $$a \in A \setminus (A^{-1}_{left} \cup A^{-1}_{right}) \Longrightarrow L_a^{-1}(A^{-1}) = \emptyset = R_a^{-1}(A^{-1}) \ ;$$

2.5 $$a \in A^{-1}_{left} \setminus A^{-1}_{right} \Longrightarrow L_a^{-1}(1) = A^{-1}_{left} \neq \emptyset = R_a^{-1}(A^{-1}) \ ;$$

2.6 $$a \in A^{-1}_{right} \setminus A^{-1}_{left} \Longrightarrow L_a^{-1}(A^{-1}) = \emptyset \neq A^{-1}_{right} = R_a^{-1}(1) \ .$$

To see all this observe

2.7 $$\begin{gathered} L_a^{-1}(1) \subseteq A^{-1}_{left} \Longrightarrow R_a^{-1}(A^{-1}_{left}) \neq \emptyset \Longrightarrow a \in A^{-1}_{left} \\ \Longrightarrow L_a^{-1}(A^{-1}_{left}) = A^{-1}_{left} \subseteq R_a^{-1}(A^{-1}_{left}) \end{gathered}$$

and

2.8 $$\begin{gathered} R_a^{-1}(1) \subseteq A^{-1}_{right} \Longrightarrow L_a^{-1}(A^{-1}_{right}) \neq \emptyset \Longrightarrow a \in A^{-1}_{right} \\ \Longrightarrow R_a^{-1}(A^{-1}_{right}) = A^{-1}_{right} \subseteq L_a^{-1}(A^{-1}_{right}) \end{gathered} \bullet$$

For example it is trivial that whenever the senigroup A is commutative

2.9 $$\varpi(A^{-1}) = A \ ;$$

this also holds [12] in finite dimensional rings A, where $A^{-1}_{left} \cup A^{-1}_{right} = A^{-1}$.

This is very simple ([6] Corollary 1.2) in a C^*-algebra:

Corollary 3. *If A is a C^*-algebra then necessary and sufficient for $a \in \varpi(A^{-1})$ is that*

3.1 $$\textit{either } \{a^*a, aa^*\} \subseteq A^{-1} \textit{ or } \{a^*a, aa^*\} \cap A^{-1} = \emptyset \ .$$

In particular normal elements are left-right consistent.

Proof. $a \in A^{-1}_{left} \Longleftrightarrow a^*a \in A^{-1}$ $\bullet$

Djordjevic ([5] Theorem 2.1) has obtained Theorem 2 for the ring $A = BL(X, X)$ of bounded operators on a Banach space X, and also ([5] Theorem 2.4) for the Calkin algebra $A = BL(X, X)/KL(X, X)$. Gong and Han ([6] Theorem 1.1) had the same result for Hilbert space; they seem to have been motivated by the observation of Hladnik and Omladic ([11] Proposition 1) - which ironically does not use

Corollary 3. Of course in a linear algebra A, where the Jacobson lemma guarantees that if $0 \neq \lambda \in \mathbf{C}$ then

3.2 $$\textit{either } \{\lambda - ba, \lambda - ab\} \subseteq A^{-1} \textit{ or } \{\lambda - ba, \lambda - ab\} \cap A^{-1} = \emptyset \ ,$$

the condition $a \in \varpi(A^{-1})$ can be reproduced in terms of the spectrum σ:

3.3 $$x \in A \Longrightarrow \sigma(ax) = \sigma(xa) \ .$$

Corollary 3 therefore says that certain relatives of $a \in A$, derived from its "polar decomposition", share its spectrum: whenever we can write $a = u|a|$ with $|a| = (a^*a)^{\frac{1}{2}}$ and $u = uu^*u$ then

3.4 $$\sigma(u|a|) = \sigma(|a|u) = \sigma(|a|^{1/2}u|a|^{1/2}) \ .$$

The normal subgroup condition (2.2) applies in particular if the semigroup A has a topology for which multiplication is separately continuous, and inversion is continuous, when $K = A_0^{-1}$ is the connected component of $1 \in A^{-1}$.

There is an interaction between left-right consistency and generalized inverses. We recall [7],[8] the "regular" elements of a semigroup, or more generally a category:

3.5 $$\ulcorner A \urcorner = \{a \in A : a \in aAa\} \ ,$$

elements $a = aba$ with *generalized inverses* $b \in A$, together with the "decomposably regular" elements

3.6 $$\overline{A} = \{a \in A : a \in aA^{-1}a\} \ ,$$

elements with invertible generalized inverses $b \in A^{-1}$.

Theorem 4. *If A is a semigroup with identity, then there is inclusion*

4.1 $$\overline{A} \subseteq \varpi(A^{-1}) \ .$$

Proof. In a general semigroup or category we have

4.2 $$\overline{A} \cap (A^{-1}_{left} \cup A^{-1}_{right}) = A^{-1} \ .$$

By (4.2) and (2.1)

4.3 $$\overline{A} \cap \varpi(A^{-1}) = A^{-1} \cup \big(\overline{A} \setminus (A^{-1}_{left} \cup A^{-1}_{right})\big) = A^{-1} \cup (\overline{A} \setminus A^{-1}) = \overline{A} \ \bullet$$

If for example $A = BL(X, X)$ is the bounded operators on a Banach space then [10]

4.4 $$\{a \in A : a^{-1}(0) \cong X/\mathrm{cl}\ a(X)\} \subseteq \varpi(A^{-1}) \ :$$

operators "of index zero" are left-right consistent.

In a ring A we can look for a stabilized version of this, in particular relative to a special kind of two sided ideal:

Definition 5. *We shall call the two sided ideal $J \subseteq A$ completely regular if there is inclusion*

5.1 $$J \subseteq \ulcorner A \urcorner \ ,$$

completely decomposably regular if there is inclusion

5.2 $$J \subseteq \llcorner A \lrcorner \ ,$$

regular if there is inclusion

5.3 $$1 + J \subseteq \ulcorner A \urcorner \ ,$$

and decomposably regular if there is inclusion

5.4 $$1 + J \subseteq \llcorner A \lrcorner \ .$$

If A is a Banach algebra we shall say that J is weakly Riesz if there is inclusion

5.5 $$1 + J \subseteq \mathrm{cl}\ A^{-1} \ .$$

The archetype is the finite rank operators in the ring of all bounded operators on a normed space. With the help of a lemma of Atkinson ([8] Theorems 7.3.2, 7.3.3) it follows that

5.6 $$J \textit{ completely regular } \Longrightarrow \ulcorner A \urcorner + J \subseteq \ulcorner A \urcorner \ ;$$

the analogue for completely decomposably regular ideals is not so clear [9]. In the other direction it follows from (5.5) that also

5.7 $$J \subseteq \mathrm{cl}\ A^{-1} \ ,$$

since (5.5) puts $\lambda + J \subseteq \mathrm{cl}\ A^{-1}$ for arbitrary scalars λ. The point about weakly Riesz ideals is that in a Banach algebra A there is ([7] Theorem 1.1; [8] Theorem 7.3.4) equality

5.8 $$\ulcorner A \urcorner \cap \mathrm{cl}\ A^{-1} = \llcorner A \lrcorner \ .$$

Since ([8] Theorems 3.10.5, 3.10.6) the boundary of the invertibles lies among the topological zero divisors we also have

5.9 $$(A^{-1}_{left} \cup A^{-1}_{right}) \cap \mathrm{cl}\ A^{-1} = A^{-1} \ .$$

From (5.9) and (2.1) it is clear that, in a Banach algebra A,

5.10 $$\mathrm{cl}\ A^{-1} \subseteq \varpi(A^{-1}) \ .$$

We can improve on this:

Theorem 6. *If $J \subseteq A$ is a completely regular weakly Riesz two-sided ideal in a Banach algebra A then there is inclusion*

6.1 $$\mathrm{cl}\ A^{-1} \subseteq \overline{A} \cup \big(A \setminus \ulcorner A \urcorner\big) \subseteq \bigcap_{d \in J} (\varpi(A^{-1}) + d)\ .$$

Proof. The first inclusion follows at once from (5.8). Towards the second observe ([9] Theorem 7) that if $J \subseteq A$ is a two-sided ideal then

6.2 $$J \textit{ regular and weakly Riesz } \Longrightarrow J \textit{ decomposably regular } ;$$

6.3 $$\begin{matrix} J \textit{ completely regular and weakly Riesz} \\ \Longrightarrow\ J \textit{ completely decomposably regular} \end{matrix}\ ;$$

6.4 $$J \textit{ weakly Riesz and completely decomposably regular } \Longrightarrow \overline{A} + J \subseteq \overline{A}\ .$$

This mostly also follows from (5.8): immediately in the case of (6.2), while for (6.3) recall (5.7):

$$1 + J \subseteq \mathrm{cl}\ A^{-1} \Longrightarrow J \subseteq \mathrm{cl}\ A^{-1} + J \subseteq \mathrm{cl}(A^{-1} + J) \subseteq \mathrm{cl}\ \mathrm{cl}\ A^{-1} = \mathrm{cl}\ A^{-1}\ .$$

Finally for (6.4) (cf [4] Theorem 4) note that from (5.5) it also follows $A^{-1} + J \subseteq \mathrm{cl}\ A^{-1}$, so that

$$\overline{A} + J \subseteq \ulcorner A \urcorner \cap \mathrm{cl}(A^{-1} + J) \subseteq \ulcorner A \urcorner \cap \mathrm{cl}\ \mathrm{cl}\ A^{-1} = \overline{A}\ .$$

Now for (6.1), if $a \in \overline{A}$ then $a + J \subseteq \overline{A} \subseteq \varpi(A^{-1})$ using (4.1), while if $a \in A \setminus \ulcorner A \urcorner$ then $a + J \subseteq A \setminus \ulcorner A \urcorner \subseteq A \setminus (A^{-1}_{left} \cup A^{-1}_{right}) \subseteq \varpi(A^{-1})$ •

In the particular case of the finite rank operators $J \subseteq A$ among the bounded operators on a separable Hilbert space both these inclusions become equality; the first was noticed by Bouldin ([1] Theorem 3; [3] (3.5)), while the second is given by Gong and Han ([6] Theorem 3.2). Another characterization, due to Wu, ([13] Theorem 1.1; [6] Theorem 3.1) says that the same set consists of all finite products of normal operators.

We can also consider left-right consistency separately relative to left and to right invertibility. We need to recall the "mixed invertible" elements of A,

6.5 $$A^{-1}_{mixed} = \{a \in A : 1 \in AaA\}\ :$$

Theorem 7. *For arbitrary A there is equality*

7.1 $$\varpi(A^{-1}_{left}) = A^{-1} \cup \big(A \setminus A^{-1}_{mixed}\big) = \varpi(A^{-1}_{right})\ .$$

Proof. It is clear that if $a \in A^{-1}$ is invertible then

7.2 $$L_a^{-1}(A^{-1}_{left}) = A^{-1}_{left} = R_a^{-1}(A^{-1}_{left})$$

while if $a \in A \setminus A^{-1}_{mixed}$ then

7.3 $$L_a^{-1}(A^{-1}_{left}) = \emptyset = R_a^{-1}(A^{-1}_{left}) \ .$$

If $a \in A^{-1}_{left} \setminus A^{-1}$ then

7.4 $$a'a = 1 \Longrightarrow a' \in R_a^{-1}(A^{-1}_{left}) \setminus L_a^{-1}(A^{-1}_{left}) \ ;$$

finally if $a \in A^{-1}_{mixed} \setminus A^{-1}_{left}$ then $R_a^{-1}(A^{-1}_{left})$ is empty while $L_a^{-1}(A^{-1}_{left})$ is not. This gives the first equality in (7.1), and therefore also the second •

Notice that if $J \subseteq A$ is a proper two-sided ideal then A^{-1}_{mixed} is disjoint from the "J inessential elements"

7.5 $$\mathrm{Hull}(J) = \{a \in A : a + J \in \mathrm{Radical}(A/J) \ \ :$$

certainly if $1 \in AaA$ and also $a \in \mathrm{Radical}(A)$, so that also $1 - AaA \subseteq A^{-1}$, then $1 = 0$ giving $A = \{0\}$: now apply this to the quotient A/J. Thus in particular when $A = B(X)$ then [5] A^{-1}_{mixed} excludes all *strictly singular* and *strictly co-singular* operators.

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