Mixed-type reverse order laws for generalized inverses in rings with involution

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Abstract

We investigate mixed-type reverse order laws for the Moore–Penrose inverse in rings with involution. We extend some well-known results to more general settings, and also prove some new results.

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1 Introduction

Many authors have studied the equivalent conditions for the reverse order law $(ab)^\dagger = b^\dagger a^\dagger$ to hold in setting of matrices, operators, $C^*$-algebras or rings [2, 9, 3, 5, 8, 10, 12, 16, 17]. This formula cannot trivially be extended to the other generalized inverses of the product $ab$. Since the reverse order law $(ab)^\dagger = b^\dagger a^\dagger$ does not always holds, it is not easy to simplify various expressions that involve the Moore-Penrose inverse of a product. In addition to $(ab)^\dagger = b^\dagger a^\dagger$, $(ab)^\dagger$ may be expressed as $(ab)^\dagger = b^\dagger (a^\dagger a b^\dagger) a^\dagger$, $(ab)^\dagger = b^\dagger (a^\dagger a b^\dagger) a^\dagger$, $(ab)^\dagger = b^\dagger a^\dagger - b^\dagger [(1 - b^\dagger b^\dagger)](1 - a^\dagger a^\dagger)] a^\dagger$, etc. These equalities are called mixed-type reverse order laws for the Moore-Penrose inverse of a product and some of them are in fact equivalent (see [4, 12, 14]). In this paper we study necessary and sufficient conditions for mixed-type reverse order laws of the form: $(ab)^\dagger = (a^\dagger a b^\dagger) a^\dagger$, $(ab)^\dagger = b^\dagger (a^\dagger a b^\dagger) a^\dagger$, $(ab)^\dagger = (a^\dagger a b^\dagger) a^\dagger$, $(ab)^\dagger = b^\dagger (a^\dagger a b^\dagger) a^\dagger$ in rings with involution.

Let $R$ be an associative ring with the unit $1$. An involution $a \mapsto a^*$ in a ring $R$ is an anti-isomorphism of degree $2$, that is,

$$(a^*)^* = a, \quad (a + b)^* = a^* + b^*, \quad (ab)^* = b^* a^*.$$
An element \( a \in \mathcal{R} \) is selfadjoint if \( a^* = a \).

The Moore–Penrose inverse (or MP-inverse) of \( a \in \mathcal{R} \) is the element \( b \in \mathcal{R} \), such that the following equations hold [13]:

\[
(1) \quad aba = a, \quad (2) \quad bab = b, \quad (3) \quad (ab)^* = ab, \quad (4) \quad (ba)^* = ba.
\]

There is at most one \( b \) such that above conditions hold (see [13]), and such \( b \) is denoted by \( a^\dagger \). The set of all Moore–Penrose invertible elements of \( \mathcal{R} \) will be denoted by \( \mathcal{R}^\dagger \). If \( a \) is invertible, then \( a^\dagger \) coincides with the ordinary inverse of \( a \).

If \( \delta \subset \{1,2,3,4\} \) and \( b \) satisfies the equations (i) for all \( i \in \delta \), then \( b \) is an \( \delta \)-inverse of \( a \). The set of all \( \delta \)-inverse of \( a \) is denote by \( a^{\{\delta\}} \). Notice that \( a^{\{1,2,3,4\}} = \{a^\dagger\} \). If \( a^{\{1\}} \neq \emptyset \), then \( a \) is regular.

Now, we state the following useful result.

**Theorem 1.1.** [6, 11] For any \( a \in \mathcal{R}^\dagger \), the following is satisfied:

(a) \( (a^\dagger)^\dagger = a \);
(b) \( (a^*)^\dagger = (a^\dagger)^* \);
(c) \( (a^*a)^\dagger = a^\dagger(a^\dagger)^* \);
(d) \( (aa^*)^\dagger = (a^\dagger)^*a^\dagger \);
(e) \( a^* = a^\dagger aa^* = a^*aa^\dagger \);
(f) \( a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger \);
(g) \( (a^*)^\dagger = a(a^*a)^\dagger = (aa^*)^\dagger a \).

The following result is well-known for complex matrices [1] and linear bounded Hilbert space operators [18], and it is equally true in rings with involution.

**Lemma 1.1.** If \( a, b \in \mathcal{R} \) such that \( a \) is regular, then

(a) \( b \in a^{\{1,3\}} \iff a^*ab = a^* \);
(b) \( b \in a^{\{1,4\}} \iff baa^* = a^* \).

**Proof.** (a) Let \( b \in a^{\{1,3\}} \), then we get \( a^*ab = a^*(ab)^* = (aba)^* = a^* \).

Conversely, the equality \( a^*ab = a^* \) implies

\[
(ab)^* = b^*a^* = b^*a^*ab = (ab)^*ab \quad \text{is selfadjoint}
\]
and
\[ aba = (ab)^*a = (a^*ab)^* = (a^*)^* = a. \]

Hence, \( b \in a\{1,3\}. \)

Similarly, we can verify the second statement.

The reverse-order law \((ab)^\dagger = b^\dagger(a^\dagger ab^\dagger)^\dagger a^\dagger\) was first studied by Galperin and Waksman [7]. A Hilbert space version of their result was given by Isumino [9]. Many results concerning the reverse order law \((ab)^\dagger = b^\dagger(a^\dagger ab^\dagger)^\dagger a^\dagger\) for complex matrices appeared in Tian’s papers [14] and [15], where the author used mostly properties of the rank of a complex matrices. In [12], a set of equivalent conditions for this reverse order rule for the Moore-Penrose inverse in the setting of \(C^*-\)algebra is studied.

Xiong and Qin [18] investigated the following mixed-type reverse order laws for the Moore-Penrose inverse of a product of Hilbert space operators:
\[
(ab)^\dagger = (a^\dagger ab^\dagger)^\dagger a^\dagger, \quad (ab)^\dagger = b^\dagger(ab^\dagger)^\dagger, \quad (ab)^\dagger = b^\dagger(a^\dagger ab^\dagger)^\dagger a^\dagger. \]
They used the technique of block operator matrices. We extend results from [18] to more general settings.

This paper is organized as follows. In Section 2, we extend the results from [18] to settings of rings with involution without the hypothesis corresponding to \(R(A^*AB) \subseteq R(B)\). In Section 3, we consider the following mixed-type reverse order laws for the Moore-Penrose inverse in rings with involution: \((ab)^\dagger = (a^*ab)^\dagger a^*, \quad (ab)^\dagger = b^*(abb^*)^\dagger\) and \((ab)^\dagger = b^*(a^*ab^*)^\dagger a^*. \)

In this paper we apply a purely algebraic technique.

## 2 Reverse order laws

\((a^\dagger ab)^\dagger a^\dagger = (ab)^\dagger, \quad b^\dagger(ab^\dagger)^\dagger = (ab)^\dagger\)

In this section, we consider necessary and sufficient conditions for reverse order laws \((a^\dagger ab)^\dagger a^\dagger = (ab)^\dagger, \quad b^\dagger(ab^\dagger)^\dagger = (ab)^\dagger\) and \(b^\dagger(a^\dagger ab^\dagger)^\dagger a^\dagger = (ab)^\dagger\) to be satisfied in rings with involution. The results in [18] for linear bounded Hilbert space operators are generalized, since we do not use any hypothesis corresponding to the condition \(R(A^*AB) \subseteq R(B)\) from [18].

**Theorem 2.1.** If \(a, b, a^\dagger ab \in \mathcal{R}^\dagger\), then the following statements are equivalent:

1. \( a^*ab\mathcal{R} \subseteq a^\dagger ab\mathcal{R}; \)
2. \( (a^\dagger ab)^\dagger a^\dagger \in (ab)\{1,3\}; \)
3. \( (a^\dagger ab)^\dagger a^\dagger = (ab)^\dagger; \)
Proof. (2) ⇒ (1): Since \((a^\dagger ab)^\dagger a^\dagger \in (ab)^\dagger\{1,3\}\), then \(ab = ab(a^\dagger ab)^\dagger a^\dagger ab\) and 
\[
ab(a^\dagger ab)^\dagger a^\dagger = (ab(a^\dagger ab)^\dagger a^\dagger)^* = (aa^\dagger ab(a^\dagger ab)^\dagger a^\dagger)^*
\]
\[
= (a^\dagger)^*a^\dagger ab(a^\dagger ab)^\dagger a^* \text{,}
\]
which gives
\[
a^*ab = a^*(ab(a^\dagger ab)^\dagger a^\dagger)ab = a^*(a^\dagger)^*a^\dagger ab(a^\dagger ab)^\dagger a^*ab
\]
\[
= a^\dagger aa^\dagger ab(a^\dagger ab)^\dagger a^\dagger ab = a^\dagger ab(a^\dagger ab)^\dagger a^*ab.
\]
Therefore, \(a^*ab R = a^\dagger ab(a^\dagger ab)^\dagger a^\dagger ab R \subseteq a^\dagger ab R\).

(1) ⇒ (4): The assumption \(a^*ab R \subseteq a^\dagger ab R\) implies that \(a^*ab = a^\dagger ab x\),
for some \(x \in R\). Now, for any \((a^\dagger ab)^{(1,3)} \in (a^\dagger ab)^\dagger\{1,3\}\) and \(a^{(1,3)} \in a\{1,3\}\),
\[
a^*ab = a^\dagger ab x = a^\dagger ab(a^\dagger ab)^{(1,3)}(a^\dagger ab x) = a^\dagger ab(a^\dagger ab)^{(1,3)}a^*ab. \tag{1}
\]
Applying the involution to (1), we obtain
\[
b^*a^*a = b^*a^*aa^\dagger ab(a^\dagger ab)^{(1,3)} = b^*a^*ab(a^\dagger ab)^{(1,3)}. \tag{2}
\]
Multiplying the equality (2) by \(a^{(1,3)}\) from the right side, we get
\[
b^*a^* = b^*a^*ab(a^\dagger ab)^{(1,3)}a^{(1,3)}, \tag{3}
\]
by \(a^*aa^{(1,3)} = a^*(aa^{(1,3)})^* = (aa^{(1,3)}a)^* = a^*\). From the equality (3) and Lemma 1.1, we deduce that
\((a^\dagger ab)^{(1,3)} a^{(1,3)} \in (ab)^\dagger\{1,3\}\), for any \((a^\dagger ab)^{(1,3)} \in (a^\dagger ab)^\dagger\{1,3\}\) and \(a^{(1,3)} \in a\{1,3\}\). So,
\((a^\dagger ab)^\dagger\{1,3\} \cdot a\{1,3\} \subseteq (ab)^\dagger\{1,3\}\).

(4) ⇒ (2): Obviously, because \((a^\dagger ab)^\dagger \in (a^\dagger ab)^\dagger\{1,3\}\) and \(a^\dagger \in a\{1,3\}\).

(2) ⇔ (3): It is easy to check this equivalence.

Using Lemma 1.1(b), we can prove the following theorem in the same way as Theorem 2.1.

**Theorem 2.2.** If \(a, b, abb^\dagger \in R^\dagger\), then the following statements are equivalent:

(1) \(bb^*a^* R \subseteq bb^\dagger a^* R\);

(2) \(b^\dagger(ab^\dagger)^\dagger \in (ab)^\dagger\{1,4\}\);

(3) \(b^\dagger(ab^\dagger)^\dagger = (ab)^\dagger\);
(4) \( b\{1, 4\} \cdot (ab\dagger)\{1, 4\} \subseteq (ab)\{1, 4\} \).

In the following result, we consider some equivalent conditions for mixed-type reverse order law \((ab)^\dagger = b\dagger(a^\dagger ab\dagger)^\dagger a\dagger\) to hold.

**Theorem 2.3.** If \(a, b, a^\dagger ab\dagger \in R^\dagger\), then the following statements are equivalent:

1. \(a^*abR \subseteq a^\dagger abR\) and \(bb^*a^*R \subseteq bb^\dagger a^*R\);
2. \(b\dagger(a^\dagger ab\dagger)^\dagger a^\dagger \in (ab)\{1, 3, 4\}\);
3. \(b\dagger(a^\dagger ab\dagger)^\dagger a^\dagger = (ab)^\dagger\);
4. \(b\{1, 3\} \cdot (a^\dagger ab\dagger)^\dagger \{1, 3\} \cdot a\{1, 3\} \subseteq (ab)\{1, 3\}\) and \(b\{1, 4\} \cdot (a^\dagger ab\dagger)^\dagger \{1, 4\} \cdot a\{1, 4\} \subseteq (ab)\{1, 4\}\).

**Proof.** (2) \(\implies\) (1): The condition \(b\dagger(a^\dagger ab\dagger)^\dagger a^\dagger \in (ab)\{3\}\) gives

\[
abb\dagger(a^\dagger ab\dagger)^\dagger a^\dagger = (abb\dagger(a^\dagger ab\dagger)^\dagger a^\dagger)^* = (aa\dagger abb\dagger(a^\dagger ab\dagger)^\dagger a^\dagger)^*
\]

\[
= (a^\dagger)^* a\dagger abb\dagger(a^\dagger ab\dagger)^\dagger a^*.
\]

Using this equality and the hypothesis \(b\dagger(a^\dagger ab\dagger)^\dagger a^\dagger \in (ab)\{1\}\), we have

\[
a^*ab = a^*(abb\dagger(a^\dagger ab\dagger)^\dagger a^\dagger)ab = a^*(a^\dagger)^* a\dagger abb\dagger(a^\dagger ab\dagger)^\dagger a^*ab
\]

\[
= a^\dagger aa\dagger abb\dagger(a^\dagger ab\dagger)^\dagger a^*ab = a\dagger abb\dagger(a^\dagger ab\dagger)^\dagger a^*ab,
\]

which yields \(a^*abR \subseteq a^\dagger abR\).

Similarly, we can prove that \(b\dagger(a^\dagger ab\dagger)^\dagger a^\dagger \in (ab)\{1, 4\}\) implies \(bb^*a^*R \subseteq bb^\dagger a^*R\).

(1) \(\implies\) (4): From \(a^*abR \subseteq a^\dagger abR\), by \(bR = bb^\dagger R\), we get \(a^*abb\dagger R \subseteq a^\dagger ab\dagger R\). Thus, \(a^*abb\dagger = a^\dagger ab\dagger x\), for some \(x \in R\). Then, for any \((a^\dagger ab\dagger)^\{1, 3\}\) \(\subseteq (a^\dagger ab\dagger)^\{1, 3\}\), \(a\{1, 3\} \subseteq (a^\dagger ab\dagger)^\{1, 3\}\) and \(b\{1, 3\} \subseteq b\{1, 3\}\), we obtain

\[
a^*abb\dagger = a^\dagger ab\dagger(a^\dagger ab\dagger)^\{1, 3\}(a^\dagger ab\dagger x) = a^\dagger ab\dagger(a^\dagger ab\dagger)^\{1, 3\} a^*abb\dagger. \quad (4)
\]

If we apply the involution to (4), we see that

\[
bb\dagger a^*a = bb\dagger a^*aa\dagger abb\dagger(a^\dagger ab\dagger)^\{1, 3\} = bb\dagger a^*abb\dagger(a^\dagger ab\dagger)^\{1, 3\}. \quad (5)
\]

Multiplying the equality (5) from the left side by \(b^*\) and from the right side by \(a\{1, 3\}\), it follows

\[
b^*a^* = b^* a^* abb\dagger(a^\dagger ab\dagger)^\{1, 3\} a\{1, 3\}.
\]

5
Notice that this equality and
\[ bb^{(1,3)} = (ab)^* = (ab)^\dagger bb^{(1,3)} = bb^{(1,3)}bb^\dagger = bb^\dagger \] (6)

imply
\[ b^\dagger a^* = b^\dagger a^* ab(ab^\dagger a)^* (a^\dagger ab)^* a^{(1,3)}a^{(1,3)} . \] (7)

By (7) and Lemma 1.1, we observe that \([ab]((a^\dagger ab)^* a^{(1,3)}) \subseteq (ab)^\{1, 3\}\),
for any \((a^\dagger ab)^* a^{(1,3)} \subseteq (ab)^\{1, 3\}\) and \(b^{(1,3)} \in b\{1, 3\}\).

Hence, \((ab)^\{1, 3\}\cdot (a^\dagger ab^\dagger \{1, 3\} \cdot a\{1, 3\} \subseteq (ab)^\{1, 3\}.\)

In the similar way, we can show that \(bb^\dagger a^* R \subseteq bb^\dagger a^* R\) gives \(b^\dagger a^* = b^{(1,4)}(a^\dagger ab)^* a^{(1,4)}ab^\dagger a^*\),
for any \((a^\dagger ab)^* a^{(1,4)} \subseteq (ab)^\{1, 4\}\), \(a^{(1,4)} \subseteq a\{1, 4\}\) and \(b^{(1,4)} \subseteq b\{1, 4\}\), i.e. \((ab)^\{1, 4\}\cdot (a^\dagger ab)^* a^{(1,4)} \subseteq (ab)^\{1, 4\}.

(4) \implies (2) \implies (3): Obviously.

\[ \square \]

3 Reverse order laws \((a^* ab)^* a^* \equiv (ab)^\dagger, b^\dagger \equiv (ab)^\dagger\) and \(b^\dagger (a^* ab)^* a^* = (ab)^\dagger\)

In this section, we give the equivalent conditions related to reverse order laws \((a^* ab)^\dagger a^* = (ab)^\dagger, b^\dagger (a^* ab)^* a^* = (ab)^\dagger\) in settings of rings with involution.

**Theorem 3.1.** If \(a, b, a^* ab \in R^\dagger\), then the following statements are equivalent:

1. \(a^\dagger ab^\dagger R \subseteq a^* ab^\dagger R;\)
2. \((a^\dagger ab)^\dagger a^* \subseteq (ab)^\{1, 3\};\)
3. \((a^\dagger ab)^\dagger a^* = (ab)^\dagger;\)
4. \((a^* ab)^\{1, 3\}\cdot (a^\dagger)^* \{1, 3\} \subseteq (ab)^\{1, 3\}.)

**Proof.** (2) \implies (1): Using the assumption \((a^* ab)^\dagger a^* \subseteq (ab)^\{1, 3\}\), we have
\[
ab(a^* ab)^\dagger a^* = (ab(a^* ab)^\dagger a^*)^* = (aa^\dagger ab(a^* ab)^\dagger a^*)^* = ((a^\dagger)^* a^* ab(a^* ab)^\dagger a^*)^* = aa^* ab(a^* ab)^\dagger a^* ,
\]
and
\[
a^\dagger ab = a^\dagger(ab(a^* ab)^\dagger a^*)ab = a^\dagger aa^* ab(a^* ab)^\dagger a^\dagger ab = a^\dagger ab(a^* ab)^\dagger a^\dagger ab .
\]

6
Thus, the condition (1) is satisfied.

(1) ⇒ (4): First, by the inclusion \( a^\dagger ab \mathcal{R} \subseteq a^{*}ab\mathcal{R} \), we conclude that \( a^\dagger ab = a^{*}aby \), for some \( y \in \mathcal{R} \). Further, for any \( (a^{*}ab)^{(1,3)} \in (a^{*}ab)\{1,3\} \) and \( a' \in (a^{*})^*\{1,3\} \), we get

\[
a^\dagger ab = a^{*}aby = a^{*}ab(a^{*}ab)^{(1,3)}(a^{*}aby) = a^{*}ab(a^{*}ab)^{(1,3)}a^\dagger ab. \tag{8}
\]

When we apply the involution to (8), we observe that

\[
b^{*}a^{\dagger}a = b^{*}a^{\dagger}aa^{*}ab(a^{*}ab)^{(1,3)} = b^{*}a^{*}ab(a^{*}ab)^{(1,3)}. \tag{9}
\]

Since \( a' \in (a^{\dagger})^*\{1,3\} \), by the equality (6) and Theorem 1.1,

\[
a^{\dagger}aa' = a^{*}[(a^{\dagger})^{*}a'] = a^{*}(a^{\dagger})^{*}[(a^{\dagger})^{*}]^{\dagger} = a^{\dagger}aa^{*} = a^{*}. \tag{10}
\]

If we multiply the equality (9) from the right side by \( a' \) and use (10), we obtain

\[
b^{*}a^{*} = b^{*}a^{*}ab(a^{*}ab)^{(1,3)}a',
\]

which implies, by Lemma 1.1, \((a^{*}ab)^{(1,3)}a' \in (ab)\{1,3\}\), for any \((a^{*}ab)^{(1,3)} \in (a^{*}ab)\{1,3\}\) and \( a' \in (a^{\dagger})^*\{1,3\} \), that is, the condition (4) holds.

(4) ⇒ (2): By Theorem 1.1, \( a^{*} = [(a^{\dagger})^*]^{*} = [(a^{\dagger})^*]^{\dagger} \in (a^{\dagger})^*\{1,3\} \) and this implication follows.

(2) ⇔ (3): Obviously.

In the same manner as in the proof of Theorem 3.1, we can verify the following results.

**Theorem 3.2.** If \( a, b, abb^{*} \in \mathcal{R}^{\dagger} \), then the following statements are equivalent:

1. \( bb^{\dagger}a^{*} \mathcal{R} \subseteq bb^{*}a^{*} \mathcal{R} \);
2. \( b^{*}(abb^{*})^{\dagger} \in (ab)\{1,4\} \);
3. \( b^{*}(abb^{*})^{\dagger} = (ab)^{\dagger} \);
4. \( (b^{\dagger})^{*}\{1,4\} \cdot (abb^{*})\{1,4\} \subseteq (ab)\{1,4\} \).

Necessary and sufficient conditions related to the reverse order law \((ab)^{\dagger} = b^{*}(a^{*}abb^{*})^{\dagger}a^{*} \) are studied in the next result.

**Theorem 3.3.** If \( a, b, a^{*}abb^{*} \in \mathcal{R}^{\dagger} \), then the following statements are equivalent:
Proof. (2) ⇒ (1): From $b^*(a^*abb^*)^\dagger a^* \in \{ab\}\{1,3,4\}$,

$$(1) \ a^1 ab \R \subseteq a^*ab \R \text{ and } bb^1 a^* \R \subseteq bb^* a^* \R;$$

$$(2) \ b^*(a^*abb^*)^\dagger a^* \in \{ab\}\{1,3,4\};$$

$$(3) \ b^*(a^*abb^*)^\dagger a^* = (ab)^\dagger;$$

$$(4) \ (b^1)^*\{1,3\} \cdot (a^*abb^*)^*\{1,3\} \subseteq (ab)^*\{1,3\} \text{ and } (b^1)^*\{1,4\} \cdot (a^*abb^*)^*\{1,4\} \subseteq (ab)^*\{1,4\}.$$ 

Proof. (2) ⇒ (1): From $b^*(a^*abb^*)^\dagger a^* \in (ab)\{3\}$,

$$ab^*(a^*abb^*)^\dagger a^* = (ab^*(a^*abb^*)^\dagger a^*)^* = ((a^1)^*a^*abb^*(a^*abb^*)^\dagger a^*)^* = aa^*abb^*(a^*abb^*)^\dagger a^\dagger.$$ 

Now, by $b^*(a^*abb^*)^\dagger a^* \in (ab)\{1\}$,

$$a^1 ab = a^1 (ab^*(a^*abb^*)^\dagger a^*)ab = a^1 aa^*abb^*(a^*abb^*)^\dagger a^\dagger ab = a^*abb^*(a^*abb^*)^\dagger a^\dagger ab$$ 

implying $a^1 ab \R \subseteq a^*ab \R$.

Analogously, we can prove the implication $b^*(a^*abb^*)^\dagger a^* \in (ab)\{1,4\} \Rightarrow bb^1 a^* \R \subseteq bb^* a^* \R$.

(1) ⇒ (4): If $a^1 ab \R \subseteq a^*ab \R$, by $b \R = bb^* \R$, we see $a^1 abb^* \R \subseteq a^*abb^* \R$ and $a^1 abb^* = a^*abb^* y$, for some $y \in \R$. For any $(a^*ab)^{\{1,3\}} \in (a^*ab)\{1,3\}$, $a^1 \in (a^1)^*\{1,3\}$ and $b^1 \in (b^1)^*\{1,3\}$, then

$$a^1 abb^* = a^*abb^*(a^*abb^*)^{\{1,3\}}(a^*abb^* y) = a^*abb^*(a^*abb^*)^{\{1,3\}}a^\dagger abb^*.$$ 

Applying the involution to (11), it follows

$$bb^* a^\dagger a = bb^* a^\dagger aa^*abb^*(a^*abb^*)^{\{1,3\}} = bb^* a^*abb^*(a^*abb^*)^{\{1,3\}}.$$ 

From the condition $b^1 \in (b^1)^*\{1,3\}$ and the equality (10), we obtain

$$bb^1 = b(b^1 bb^1) = bb^*.$$ 

Now, multiplying (12) from the left side by $b^1$ and from the right side by $a^1$, we get, by (10) and the last equality,

$$b^1 a^* = b^1 a^*abb^*(a^*abb^*)^{\{1,3\}} a^1.$$ 

Thus, by Lemma 1.1, $b^1(a^*abb^*)^{\{1,3\}} a^1 \in (ab)\{1,3\}$, for any $(a^*ab)^{\{1,3\}} \in (a^*ab)\{1,3\}$, $a^1 \in (a^1)^*\{1,3\}$ and $b^1 \in (b^1)^*\{1,3\}$, which is equivalent to

$$(b^1)^*\{1,3\} \cdot (a^*abb^*)^*\{1,3\} \subseteq (ab)^*\{1,3\}.$$ 

Similarly, we show that $bb^5 a^* \R \subseteq bb^* a^* \R$ gives $(b^1)^*\{1,4\} \cdot (a^*abb^*)^*\{1,4\} \cdot (a^1)^*\{1,4\} \subseteq (ab)^*\{1,4\}$.

(4) ⇒ (2) ⇔ (3): These parts can be check easy. □
If we state in the proved results the elements $a^*$, $(a^\dagger)^*$, $a^\dagger$, $b^*$, $(b^\dagger)^*$ or $b^\dagger$ instead $a$ or $b$, we obtain various mixed-type reverse order laws for the Moore–Penrose inverses in rings with involution.

By the results presenting in Section 2 and Section 3, we can get the following consequence.

**Corollary 3.1.** If $a, b, ab, a^\dagger ab, abb^\dagger, a^* ab, abb^*, a^* abb^* \in R^\dagger$. Then the following statements are equivalent:

1. $(ab)^\dagger = b^\dagger (a^\dagger a b b^\dagger)^1 a^\dagger$;
2. $(ab)^\dagger = (a^\dagger ab)^1 a^\dagger = b^\dagger (ab b^\dagger)^1$;
3. $(ab)^\dagger = b^* (a^* abb^*)^1 a^*$;
4. $(ab)^\dagger = (a^* ab)^1 a^* = b^* (abb^*)^1$;
5. $a^* ab R \subseteq a^\dagger ab R$ and $bb^* a^* R \subseteq bb^\dagger a^* R$;
6. $b^\dagger (a^\dagger abb^\dagger)^1 a^\dagger \in (ab) \{1, 3, 4\}$;
7. $b \{1, 3\} \cdot (a^\dagger abb^\dagger) \{1, 3\} \cdot a \{1, 3\} \subseteq (ab) \{1, 3\}$ and $b \{1, 4\} \cdot (a^\dagger abb^\dagger) \{1, 4\} \cdot a \{1, 4\} \subseteq (ab) \{1, 4\}$;
8. $(a^\dagger ab)^1 a^\dagger \in (ab) \{1, 3\}$ and $b^\dagger (abb^\dagger)^1 \in (ab) \{1, 4\}$;
9. $(a^\dagger ab) \{1, 3\} \cdot a \{1, 3\} \subseteq (ab) \{1, 3\}$ and $b \{1, 4\} \cdot (abb^\dagger) \{1, 4\} \subseteq (ab) \{1, 4\}$;
10. $a^\dagger ab R \subseteq a^* ab R$ and $bb^\dagger a^* R \subseteq bb^* a^* R$;
11. $b^* (a^* abb^*)^1 a^* \in (ab) \{1, 3, 4\}$;
12. $(b^\dagger)^* \{1, 3\} \cdot (a^* ab b^*) \{1, 3\} \cdot (a^\dagger)^* \{1, 3\} \subseteq (ab) \{1, 3\}$ and $(b^\dagger)^* \{1, 4\} \cdot (a^* ab b^*) \{1, 4\} \cdot (a^\dagger)^* \{1, 4\} \subseteq (ab) \{1, 4\}$;
13. $(a^* ab)^1 a^* \in (ab) \{1, 3\}$ and $b^* (abb^*)^1 \in (ab) \{1, 4\}$;
14. $(a^* ab) \{1, 3\} \cdot (a^\dagger)^* \{1, 3\} \subseteq (ab) \{1, 3\}$ and $(b^\dagger)^* \{1, 4\} \cdot (a^* ab b^*) \{1, 4\} \subseteq (ab) \{1, 4\}$.

**Proof.** The equivalences of conditions (1)-(4) follow as in [12, Theorem 2.6] for elements of $C^*$-algebras. The rest follows from these equivalences and theorems in Section 2 and Section 3.
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