Mixed-type reverse order laws for generalized inverses in rings with involution

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Abstract

We investigate mixed-type reverse order laws for the Moore–Penrose inverse in rings with involution. We extend some well-known results to more general settings, and also prove some new results.

Key words and phrases: Moore-Penrose inverse, reverse order law, ring with involution.

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1 Introduction

Many authors have studied the equivalent conditions for the reverse order law $(ab)^\dagger = b^\dagger a^\dagger$ to hold in setting of matrices, operators, C^*-algebras or rings [2, 9, 3, 5, 8, 10, 12, 16, 17]. This formula cannot trivially be extended to the other generalized inverses of the product ab. Since the reverse order law $(ab)^\dagger = b^\dagger a^\dagger$ does not always holds, it is not easy to simplify various expressions that involve the Moore-Penrose inverse of a product. In addition to $(ab)^\dagger = b^\dagger a^\dagger$, $(ab)^\dagger$ may be expressed as $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$, $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$, $(ab)^\dagger = b^\dagger a^\dagger - b^\dagger[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^\dagger$, etc. These equalities are called mixed-type reverse order laws for the Moore-Penrose inverse of a product and some of them are in fact equivalent (see [4, 12, 14]). In this paper we study necessary and sufficient conditions for mixed-type reverse order laws of the form: $(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger$, $(ab)^\dagger = b^\dagger(abb^\dagger)^\dagger$, $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$, $(ab)^\dagger = (a^*ab)^\dagger a^*$, $(ab)^\dagger = b^*(abb^*)^\dagger$ and $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$ in rings with involution.

Let $\mathcal{R}$ be an associative ring with the unit 1. An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

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An element $a \in \mathcal{R}$ is selfadjoint if $a^* = a$.

The *Moore–Penrose inverse* (or *MP-inverse*) of $a \in \mathcal{R}$ is the element $b \in \mathcal{R}$, such that the following equations hold [13]:

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (3)\ (ab)^* = ab, \quad (4)\ (ba)^* = ba.$$

There is at most one b such that above conditions hold (see [13]), and such b is denoted by $a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$. If a is invertible, then $a^\dagger$ coincides with the ordinary inverse of a.

If $\delta \subset \{1, 2, 3, 4\}$ and b satisfies the equations (i) for all $i \in \delta$, then b is an δ–inverse of a. The set of all δ–inverse of a is denote by $a\{\delta\}$. Notice that $a\{1, 2, 3, 4\} = \{a^\dagger\}$. If $a\{1\} \neq \emptyset$, then a is regular.

Now, we state the following useful result.

Theorem 1.1. [6, 11] *For any $a \in \mathcal{R}^\dagger$, the following is satisfied:*

(a) $(a^\dagger)^\dagger = a$;

(b) $(a^*)^\dagger = (a^\dagger)^*$;

(c) $(a^*a)^\dagger = a^\dagger(a^\dagger)^*$;

(d) $(aa^*)^\dagger = (a^\dagger)^*a^\dagger$;

(e) $a^* = a^\dagger aa^* = a^*aa^\dagger$;

(f) $a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger$;

(g) $(a^*)^\dagger = a(a^*a)^\dagger = (aa^*)^\dagger a$.

The following result is well-known for complex matrices [1] and linear bounded Hilbert space operators [18], and it is equally true in rings with involution.

Lemma 1.1. *If $a, b \in \mathcal{R}$ such that a is regular, then*

(a) $b \in a\{1, 3\} \Leftrightarrow a^*ab = a^*$;

(b) $b \in a\{1, 4\} \Leftrightarrow baa^* = a^*$.

Proof. (a) Let $b \in a\{1, 3\}$, then we get $a^*ab = a^*(ab)^* = (aba)^* = a^*$.

Conversely, the equality $a^*ab = a^*$ implies

$$(ab)^* = b^*a^* = b^*a^*ab = (ab)^*ab \text{ is selfadjoint}$$

and

$$aba = (ab)^*a = (a^*ab)^* = (a^*)^* = a.$$

Hence, $b \in a\{1, 3\}$.

Similarly, we can verify the second statement. $\square$

The reverse-order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ was first studied by Galperin and Waksman [7]. A Hilbert space version of their result was given by Isumino [9]. Many results concerning the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ for complex matrices appeared in Tian's papers [14] and [15], where the author used mostly properties of the rank of a complex matrices. In [12], a set of equivalent conditions for this reverse order rule for the Moore-Penrose inverse in the setting of C^*–algebra is studied.

Xiong and Qin [18] investigated the following mixed-type reverse order laws for the Moore-Penrose inverse of a product of Hilbert space operators: $(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger$, $(ab)^\dagger = b^\dagger(abb^\dagger)^\dagger$, $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$. They used the technique of block operator matrices. We extend results from [18] to more general settings.

This paper is organized as follows. In Section 2, we extend the results from [18] to settings of rings with involution without the hypothesis corresponding to $R(A^*AB) \subseteq R(B)$. In Section 3, we consider the following mixed-type reverse order laws for the Moore-Penrose inverse in rings with involution: $(ab)^\dagger = (a^*ab)^\dagger a^*$, $(ab)^\dagger = b^*(abb^*)^\dagger$ and $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$. In this paper we apply a purely algebraic technique.

2 Reverse order laws $(a^\dagger ab)^\dagger a^\dagger = (ab)^\dagger$, $b^\dagger(abb^\dagger)^\dagger = (ab)^\dagger$ and $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = (ab)^\dagger$

In this section, we consider necessary and sufficient conditions for reverse order laws $(a^\dagger ab)^\dagger a^\dagger = (ab)^\dagger$, $b^\dagger(abb^\dagger)^\dagger = (ab)^\dagger$ and $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = (ab)^\dagger$ to be satisfied in rings with involution. The results in [18] for linear bounded Hilbert space operators are generalized, since we do not use any e hypothesis corresponding to the condition $R(A^*AB) \subseteq R(B)$ from [18].

Theorem 2.1. *If $a, b, a^\dagger ab \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(1) $a^*ab\mathcal{R} \subseteq a^\dagger ab\mathcal{R}$;

(2) $(a^\dagger ab)^\dagger a^\dagger \in (ab)\{1, 3\}$;

(3) $(a^\dagger ab)^\dagger a^\dagger = (ab)^\dagger$;

(4) $(a^\dagger ab)\{1,3\}\cdot a\{1,3\}\subseteq (ab)\{1,3\}$.

Proof. (2) $\Rightarrow$ (1): Since $(a^\dagger ab)^\dagger a^\dagger \in (ab)\{1,3\}$, then $ab = ab(a^\dagger ab)^\dagger a^\dagger ab$ and

$$\begin{aligned} ab(a^\dagger ab)^\dagger a^\dagger &= (ab(a^\dagger ab)^\dagger a^\dagger)^* = (aa^\dagger ab(a^\dagger ab)^\dagger a^\dagger)^* \\ &= (a^\dagger)^* a^\dagger ab(a^\dagger ab)^\dagger a^*, \end{aligned}$$

which gives

$$\begin{aligned} a^*ab &= a^*(ab(a^\dagger ab)^\dagger a^\dagger)ab = a^*(a^\dagger)^* a^\dagger ab(a^\dagger ab)^\dagger a^*ab \\ &= a^\dagger aa^\dagger ab(a^\dagger ab)^\dagger a^*ab = a^\dagger ab(a^\dagger ab)^\dagger a^*ab. \end{aligned}$$

Therefore, $a^*ab\mathcal{R} = a^\dagger ab(a^\dagger ab)^\dagger a^*ab\mathcal{R}\subseteq a^\dagger ab\mathcal{R}$.

(1) $\Rightarrow$ (4): The assumption $a^*ab\mathcal{R}\subseteq a^\dagger ab\mathcal{R}$ implies that $a^*ab = a^\dagger abx$, for some $x\in\mathcal{R}$. Now, for any $(a^\dagger ab)^{(1,3)}\in (a^\dagger ab)\{1,3\}$ and $a^{(1,3)}\in a\{1,3\}$,

$$a^*ab = a^\dagger abx = a^\dagger ab(a^\dagger ab)^{(1,3)}(a^\dagger abx) = a^\dagger ab(a^\dagger ab)^{(1,3)}a^*ab. \tag{1}$$

Applying the involution to (1), we obtain

$$b^*a^*a = b^*a^*aa^\dagger ab(a^\dagger ab)^{(1,3)} = b^*a^*ab(a^\dagger ab)^{(1,3)}. \tag{2}$$

Multiplying the equality (2) by $a^{(1,3)}$ from the right side, we get

$$b^*a^* = b^*a^*ab(a^\dagger ab)^{(1,3)}a^{(1,3)}, \tag{3}$$

by $a^*aa^{(1,3)} = a^*(aa^{(1,3)})^* = (aa^{(1,3)}a)^* = a^*$. From the equality (3) and Lemma 1.1, we deduce that $(a^\dagger ab)^{(1,3)}a^{(1,3)}\in (ab)\{1,3\}$, for any $(a^\dagger ab)^{(1,3)}\in (a^\dagger ab)\{1,3\}$ and $a^{(1,3)}\in a\{1,3\}$. So, $(a^\dagger ab)\{1,3\}\cdot a\{1,3\}\subseteq (ab)\{1,3\}$.

(4) $\Rightarrow$ (2): Obviously, because $(a^\dagger ab)^\dagger\in (a^\dagger ab)\{1,3\}$ and $a^\dagger\in a\{1,3\}$.

(2) $\Leftrightarrow$ (3): It is easy to check this equivalence.

□

Using Lemma 1.1(b), we can prove the following theorem in the same way as Theorem 2.1.

Theorem 2.2. *If $a, b, abb^\dagger\in\mathcal{R}^\dagger$, then the following statements are equivalent:*

(1) $bb^*a^*\mathcal{R}\subseteq bb^\dagger a^*\mathcal{R}$;

(2) $b^\dagger(abb^\dagger)^\dagger\in (ab)\{1,4\}$;

(3) $b^\dagger(abb^\dagger)^\dagger = (ab)^\dagger$;

(4) $b\{1,4\} \cdot (abb^\dagger)\{1,4\} \subseteq (ab)\{1,4\}$.

In the following result, we consider some equivalent conditions for mixed-type reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ to hold.

Theorem 2.3. *If $a, b, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(1) $a^*ab\mathcal{R} \subseteq a^\dagger ab\mathcal{R}$ *and* $bb^*a^*\mathcal{R} \subseteq bb^\dagger a^*\mathcal{R}$;

(2) $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{1,3,4\}$;

(3) $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = (ab)^\dagger$;

(4) $b\{1,3\} \cdot (a^\dagger abb^\dagger)\{1,3\} \cdot a\{1,3\} \subseteq (ab)\{1,3\}$ *and* $b\{1,4\} \cdot (a^\dagger abb^\dagger)\{1,4\} \cdot a\{1,4\} \subseteq (ab)\{1,4\}$.

Proof. (2) $\Rightarrow$ (1): The condition $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{3\}$ gives

$$\begin{aligned} abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger &= (abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)^* = (aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)^* \\ &= (a^\dagger)^* a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^*. \end{aligned}$$

Using this equality and the hypothesis $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{1\}$, we have

$$\begin{aligned} a^*ab &= a^*(abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)ab = a^*(a^\dagger)^* a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^*ab \\ &= a^\dagger aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^*ab = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^*ab, \end{aligned}$$

which yields $a^*ab\mathcal{R} \subseteq a^\dagger ab\mathcal{R}$.

Similarly, we can prove that $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{1,4\}$ implies $bb^*a^*\mathcal{R} \subseteq bb^\dagger a^*\mathcal{R}$.

(1) $\Rightarrow$ (4): From $a^*ab\mathcal{R} \subseteq a^\dagger ab\mathcal{R}$, by $b\mathcal{R} = bb^\dagger\mathcal{R}$, we get $a^*abb^\dagger\mathcal{R} \subseteq a^\dagger abb^\dagger\mathcal{R}$. Thus, $a^*abb^\dagger = a^\dagger abb^\dagger x$, for some $x \in \mathcal{R}$. Then, for any $(a^\dagger abb^\dagger)^{(1,3)} \in (a^\dagger abb^\dagger)\{1,3\}$, $a^{(1,3)} \in a\{1,3\}$ and $b^{(1,3)} \in b\{1,3\}$, we obtain

$$a^*abb^\dagger = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^{(1,3)}(a^\dagger abb^\dagger x) = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^{(1,3)}a^*abb^\dagger. \qquad (4)$$

If we apply the involution to (4), we see that

$$bb^\dagger a^*a = bb^\dagger a^*aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^{(1,3)} = bb^\dagger a^*abb^\dagger(a^\dagger abb^\dagger)^{(1,3)}. \qquad (5)$$

Multiplying the equality (5) from the left side by b^* and from the right side by $a^{(1,3)}$, it follows

$$b^*a^* = b^*a^*abb^\dagger(a^\dagger abb^\dagger)^{(1,3)}a^{(1,3)}.$$

Notice that this equality and

$$bb^{(1,3)} = (bb^{(1,3)})^* = (bb^\dagger bb^{(1,3)})^* = bb^{(1,3)}bb^\dagger = bb^\dagger \tag{6}$$

imply

$$b^*a^* = b^*a^*abb^{(1,3)}(a^\dagger abb^\dagger)^{(1,3)}a^{(1,3)}. \tag{7}$$

By (7) and Lemma 1.1, we observe that $b^{(1,3)}(a^\dagger abb^\dagger)^{(1,3)}a^{(1,3)} \in (ab)\{1,3\}$, for any $(a^\dagger abb^\dagger)^{(1,3)} \in (a^\dagger abb^\dagger)\{1,3\}$, $a^{(1,3)} \in a\{1,3\}$ and $b^{(1,3)} \in b\{1,3\}$. Hence, $b\{1,3\} \cdot (a^\dagger abb^\dagger)\{1,3\} \cdot a\{1,3\} \subseteq (ab)\{1,3\}$.

In the similar way, we can show that $bb^*a^*\mathcal{R} \subseteq bb^\dagger a^*\mathcal{R}$ gives $b^*a^* = b^{(1,4)}(a^\dagger abb^\dagger)^{(1,4)}a^{(1,4)}abb^*a^*$, for any $(a^\dagger abb^\dagger)^{(1,4)} \in (a^\dagger abb^\dagger)\{1,4\}$, $a^{(1,4)} \in a\{1,4\}$ and $b^{(1,4)} \in b\{1,4\}$, i.e. $b\{1,4\} \cdot (a^\dagger abb^\dagger)\{1,4\} \cdot a\{1,4\} \subseteq (ab)\{1,4\}$.

(4) $\Rightarrow$ (2) $\Leftrightarrow$ (3): Obviously.

$\square$

3 Reverse order laws $(a^*ab)^\dagger a^* = (ab)^\dagger$, $b^*(abb^*)^\dagger = (ab)^\dagger$ and $b^*(a^*abb^*)^\dagger a^* = (ab)^\dagger$

In this section, we give the equivalent conditions related to reverse order laws $(a^*ab)^\dagger a^* = (ab)^\dagger$, $b^*(abb^*)^\dagger = (ab)^\dagger$ and $b^*(a^*abb^*)^\dagger a^* = (ab)^\dagger$ in settings of rings with involution.

Theorem 3.1. *If $a, b, a^*ab \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(1) $a^\dagger ab\mathcal{R} \subseteq a^*ab\mathcal{R}$;

(2) $(a^*ab)^\dagger a^* \in (ab)\{1,3\}$;

(3) $(a^*ab)^\dagger a^* = (ab)^\dagger$;

(4) $(a^*ab)\{1,3\} \cdot (a^\dagger)^*\{1,3\} \subseteq (ab)\{1,3\}$.

Proof. (2) $\Rightarrow$ (1): Using the assumption $(a^*ab)^\dagger a^* \in (ab)\{1,3\}$, we have

$$\begin{aligned} ab(a^*ab)^\dagger a^* &= (ab(a^*ab)^\dagger a^*)^* = (aa^\dagger ab(a^*ab)^\dagger a^*)^* \\ &= ((a^\dagger)^*a^*ab(a^*ab)^\dagger a^*)^* = aa^*ab(a^*ab)^\dagger a^\dagger, \end{aligned}$$

and

$$\begin{aligned} a^\dagger ab &= a^\dagger(ab(a^*ab)^\dagger a^*)ab = a^\dagger aa^*ab(a^*ab)^\dagger a^\dagger ab \\ &= a^*ab(a^*ab)^\dagger a^\dagger ab. \end{aligned}$$

Thus, the condition (1) is satisfied.

(1) $\Rightarrow$ (4): First, by the inclusion $a^\dagger ab\mathcal{R} \subseteq a^*ab\mathcal{R}$, we conclude that $a^\dagger ab = a^*aby$, for some $y \in \mathcal{R}$. Further, for any $(a^*ab)^{(1,3)} \in (a^*ab)\{1,3\}$ and $a' \in (a^\dagger)^*\{1,3\}$, we get

$$a^\dagger ab = a^*aby = a^*ab(a^*ab)^{(1,3)}(a^*aby) = a^*ab(a^*ab)^{(1,3)}a^\dagger ab. \tag{8}$$

When we apply the involution to (8), we observe that

$$b^*a^\dagger a = b^*a^\dagger aa^*ab(a^*ab)^{(1,3)} = b^*a^*ab(a^*ab)^{(1,3)}. \tag{9}$$

Since $a' \in (a^\dagger)^*\{1,3\}$, by the equality (6) and Theorem 1.1,

$$a^\dagger aa' = a^*[(a^\dagger)^*a'] = a^*(a^\dagger)^*[(a^\dagger)^*]^\dagger = a^\dagger aa^* = a^*. \tag{10}$$

If we multiply the equality (9) from the right side by a' and use (10), we obtain

$$b^*a^* = b^*a^*ab(a^*ab)^{(1,3)}a',$$

which implies, by Lemma 1.1, $(a^*ab)^{(1,3)}a' \in (ab)\{1,3\}$, for any $(a^*ab)^{(1,3)} \in (a^*ab)\{1,3\}$ and $a' \in (a^\dagger)^*\{1,3\}$, that is, the condition (4) holds.

(4) $\Rightarrow$ (2): By Theorem 1.1, $a^* = [((a^\dagger)^\dagger]^* = [((a^\dagger)^*]^\dagger \in (a^\dagger)^*\{1,3\}$ and this implication follows.

(2) $\Leftrightarrow$ (3): Obviously.

□

In the same manner as in the proof of Theorem 3.1, we can verify the following results.

Theorem 3.2. *If $a, b, abb^* \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(1) $bb^\dagger a^*\mathcal{R} \subseteq bb^*a^*\mathcal{R}$;

(2) $b^*(abb^*)^\dagger \in (ab)\{1,4\}$;

(3) $b^*(abb^*)^\dagger = (ab)^\dagger$;

(4) $(b^\dagger)^*\{1,4\} \cdot (abb^*)\{1,4\} \subseteq (ab)\{1,4\}$.

Necessary and sufficient conditions related to the reverse order law $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$ are studied in the next result.

Theorem 3.3. *If $a, b, a^*abb^* \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(1) $a^\dagger ab\mathcal{R} \subseteq a^*ab\mathcal{R}$ *and* $bb^\dagger a^*\mathcal{R} \subseteq bb^*a^*\mathcal{R}$*;*

(2) $b^*(a^*abb^*)^\dagger a^* \in (ab)\{1,3,4\}$*;*

(3) $b^*(a^*abb^*)^\dagger a^* = (ab)^\dagger$*;*

(4) $(b^\dagger)^*\{1,3\} \cdot (a^*abb^*)\{1,3\} \cdot (a^\dagger)^*\{1,3\} \subseteq (ab)\{1,3\}$ *and* $(b^\dagger)^*\{1,4\} \cdot (a^*abb^*)\{1,4\} \cdot (a^\dagger)^*\{1,4\} \subseteq (ab)\{1,4\}$.

Proof. (2) $\Rightarrow$ (1): From $b^*(a^*abb^*)^\dagger a^* \in (ab)\{3\}$,

$$\begin{aligned} abb^*(a^*abb^*)^\dagger a^* &= (abb^*(a^*abb^*)^\dagger a^*)^* = ((a^\dagger)^*a^*abb^*(a^*abb^*)^\dagger a^*)^* \\ &= aa^*abb^*(a^*abb^*)^\dagger a^\dagger. \end{aligned}$$

Now, by $b^*(a^*abb^*)^\dagger a^* \in (ab)\{1\}$,

$$\begin{aligned} a^\dagger ab &= a^\dagger(abb^*(a^*abb^*)^\dagger a^*)ab = a^\dagger aa^*abb^*(a^*abb^*)^\dagger a^\dagger ab \\ &= a^*abb^*(a^*abb^*)^\dagger a^\dagger ab \end{aligned}$$

implying $a^\dagger ab\mathcal{R} \subseteq a^*ab\mathcal{R}$.

Analogously, we can prove the implication $b^*(a^*abb^*)^\dagger a^* \in (ab)\{1,4\} \Rightarrow bb^\dagger a^*\mathcal{R} \subseteq bb^*a^*\mathcal{R}$.

(1) $\Rightarrow$ (4): If $a^\dagger ab\mathcal{R} \subseteq a^*ab\mathcal{R}$, by $b\mathcal{R} = bb^*\mathcal{R}$, we see $a^\dagger abb^*\mathcal{R} \subseteq a^*abb^*\mathcal{R}$ and $a^\dagger abb^* = a^*abb^*y$, for some $y \in \mathcal{R}$. For any $(a^*ab)^{(1,3)} \in (a^*ab)\{1,3\}$, $a' \in (a^\dagger)^*\{1,3\}$ and $b' \in (b^\dagger)^*\{1,3\}$, then

$$a^\dagger abb^* = a^*abb^*(a^*abb^*)^{(1,3)}(a^*abb^*y) = a^*abb^*(a^*abb^*)^{(1,3)}a^\dagger abb^*. \quad (11)$$

Applying the involution to (11), it follows

$$bb^*a^\dagger a = bb^*a^\dagger aa^*abb^*(a^*abb^*)^{(1,3)} = bb^*a^*abb^*(a^*abb^*)^{(1,3)}. \quad (12)$$

From the condition $b' \in (b^\dagger)^*\{1,3\}$ and the equality (10), we obtain

$$bb' = b(b^\dagger bb') = bb^*.$$

Now, multiplying (12) from the left side by $b^\dagger$ and from the right side by a', we get, by (10) and the last equality,

$$b^*a^* = b^*a^*abb'(a^*abb^*)^{(1,3)}a'.$$

Thus, by Lemma 1.1, $b'(a^*abb^*)^{(1,3)}a' \in (ab)\{1,3\}$, for any $(a^*ab)^{(1,3)} \in (a^*ab)\{1,3\}$, $a' \in (a^\dagger)^*\{1,3\}$ and $b' \in (b^\dagger)^*\{1,3\}$, which is equivalent to $(b^\dagger)^*\{1,3\} \cdot (a^*abb^*)\{1,3\} \cdot (a^\dagger)^*\{1,3\} \subseteq (ab)\{1,3\}$.

Similarly, we show that $bb^\dagger a^*\mathcal{R} \subseteq bb^*a^*\mathcal{R}$ gives $(b^\dagger)^*\{1,4\} \cdot (a^*abb^*)\{1,4\} \cdot (a^\dagger)^*\{1,4\} \subseteq (ab)\{1,4\}$.

(4) $\Rightarrow$ (2) $\Leftrightarrow$ (3): These parts can be check easy. $\square$

If we state in the proved results the elements a^*, $(a^\dagger)^*$, $a^\dagger$, b^*, $(b^\dagger)^*$ or $b^\dagger$ instead a or b, we obtain various mixed-type reverse order laws for the Moore–Penrose inverses in rings with involution.

By the results presenting in Section 2 and Section 3, we can get the following consequence.

Corollary 3.1. *If $a, b, ab, a^\dagger ab, abb^\dagger, a^\dagger abb^\dagger, a^*ab, abb^*, a^*abb^* \in \mathcal{R}^\dagger$. Then the following statements are equivalent:*

(1) $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$;

(2) $(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger = b^\dagger(abb^\dagger)^\dagger$;

(3) $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$;

(4) $(ab)^\dagger = (a^*ab)^\dagger a^* = b^*(abb^*)^\dagger$;

(5) $a^*ab\mathcal{R} \subseteq a^\dagger ab\mathcal{R}$ *and* $bb^*a^*\mathcal{R} \subseteq bb^\dagger a^*\mathcal{R}$;

(6) $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{1, 3, 4\}$;

(7) $b\{1, 3\} \cdot (a^\dagger abb^\dagger)\{1, 3\} \cdot a\{1, 3\} \subseteq (ab)\{1, 3\}$ *and* $b\{1, 4\} \cdot (a^\dagger abb^\dagger)\{1, 4\} \cdot a\{1, 4\} \subseteq (ab)\{1, 4\}$;

(8) $(a^\dagger ab)^\dagger a^\dagger \in (ab)\{1, 3\}$ *and* $b^\dagger(abb^\dagger)^\dagger \in (ab)\{1, 4\}$;

(9) $(a^\dagger ab)\{1, 3\} \cdot a\{1, 3\} \subseteq (ab)\{1, 3\}$ *and* $b\{1, 4\} \cdot (abb^\dagger)\{1, 4\} \subseteq (ab)\{1, 4\}$;

(10) $a^\dagger ab\mathcal{R} \subseteq a^*ab\mathcal{R}$ *and* $bb^\dagger a^*\mathcal{R} \subseteq bb^*a^*\mathcal{R}$;

(11) $b^*(a^*abb^*)^\dagger a^* \in (ab)\{1, 3, 4\}$;

(12) $(b^\dagger)^*\{1, 3\} \cdot (a^*abb^*)\{1, 3\} \cdot (a^\dagger)^*\{1, 3\} \subseteq (ab)\{1, 3\}$ *and* $(b^\dagger)^*\{1, 4\} \cdot (a^*abb^*)\{1, 4\} \cdot (a^\dagger)^*\{1, 4\} \subseteq (ab)\{1, 4\}$;

(13) $(a^*ab)^\dagger a^* \in (ab)\{1, 3\}$ *and* $b^*(abb^*)^\dagger \in (ab)\{1, 4\}$;

(14) $(a^*ab)\{1, 3\} \cdot (a^\dagger)^*\{1, 3\} \subseteq (ab)\{1, 3\}$ *and* $(b^\dagger)^*\{1, 4\} \cdot (abb^*)\{1, 4\} \subseteq (ab)\{1, 4\}$.

Proof. The equivalences of conditions (1)-(4) follow as in [12, Theorem 2.6] for elements of C^*-algebras. The rest follows from these equivalences and theorems in Section 2 and Section 3. □

References

[1] A. Ben-Israel, T.N.E. Greville, *Generalized Inverses: Theory and Applications*, 2nd ed., Springer, New York, 2003.

[2] R.H. Bouldin, *The pseudo-inverse of a product*, SIAM J. Appl. Math. 25 (1973), 489–495.

[3] D.S. Djordjević, *Further results on the reverse order law for generalized inverses*, SIAM J. Matrix Anal. Appl. 29 (4) (2007), 1242-1246.

[4] N.Č. Dinčić, D.S. Djordjević, D. Mosić, *Mixed-type reverse order law and its equivalencies*, Studia Math. (to appear).

[5] D.S. Djordjević, N.Č. Dinčić, *Reverse order law for the Moore–Penrose inverse*, J. Math. Anal. Appl. 361 (1) (2010), 252-261.

[6] D.S. Djordjević, V. Rakočević, *Lectures on generalized inverses*, Faculty of Sciences and Mathematics, University of Niš, 2008.

[7] A.M. Galperin, Z. Waksman, *On pseudo-inverses of operator products*, Linear Algebra Appl. 33 (1980), 123–131.

[8] T.N.E. Greville, *Note on the generalized inverse of a matrix product*, SIAM Rev. 8 (1966), 518–521.

[9] S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tohoku Math. J. 34 (1982), 43–52.

[10] J.J. Koliha, D.S. Djordjević, D. Cvetković, *Moore–Penrose inverse in rings with involution*, Linear Algebra Appl. 426 (2007), 371–381.

[11] D. Mosić, D.S. Djordjević, *Moore-Penrose-invertible normal and Hermitian elements in rings*, Linear Algebra Appl. 431 (5-7) (2009), 732–745.

[12] D. Mosić, D.S. Djordjević, *Reverse order law for the Moore–Penrose inverse in C^*-algebras*, Electron. J. Linear Algebra 22 (2011), 92–111.

[13] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. 51 (1955), 406–413.

[14] Y. Tian, *The reverse-order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ and its equivalent equalities*, J. Math. Kyoto. Univ. 45-4 (2005), 841–850.

[15] Y. Tian, *On mixed-type reverse-order laws for Moore-Penrose inverse of a matrix product*, Int. J. Math. Math. Sci. 58 (2004), 3103–3116.

[16] Y. Tian, *Using rank formulas to characterize equalities for Moore-Penrose inverses of matrix product*, Appl. Math. Comput 147 (2004), 581–600.

[17] Y. Tian, *The equvalence between* $(AB)^\dagger = B^\dagger A^\dagger$ *and other mixed-type reverse-order laws*, Internat. J. Math. Edu. Sci. Technol. 37(3) (2006), 331–339.

[18] Z. Xiong, Y. Qin, *Mixed-type reverse-order laws for the generalized inverses of an operator product*, Arab. J. Sci. Eng. (DOI 10.1007/s13369-011-0046-8).

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