

# Further results on partial isometries and EP elements in rings with involution

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## Abstract

We investigate elements in rings with involution which are EP or partial isometries. Some well-known results are generalized.

*Key words and phrases:* Partial isometry, Moore-Penrose inverse, group inverse, EP element, ring with involution.

2010 *Mathematics subject classification:* 16W10, 15A09.

## 1 Introduction

In this paper we consider both Moore-Penrose invertible and group invertible elements in rings with involution. Our aim is to study partial isometries and EP elements in terms of equations involving their adjoints, Moore-Penrose and group inverse. Some recent results from [1] and [11] follow as corollaries. Notice that the Moore-Penrose inverse and the group inverse are useful in solving overdetermined systems of linear equations.

Let  $\mathcal{R}$  be an associative ring with the unit 1, and let  $a \in \mathcal{R}$ . We say that  $a$  is *group invertible* if there is  $a^\# \in \mathcal{R}$  such that

$$aa^\#a = a, \quad a^\#aa^\# = a^\#, \quad aa^\# = a^\#a.$$

The element  $a^\#$  is called the group inverse and it is uniquely determined by previous equations [2]. Denote by  $\mathcal{R}^\#$  the set of all group invertible elements of  $\mathcal{R}$ . If  $a$  is invertible, then  $a^\#$  coincides with the ordinary inverse  $a^{-1}$  of  $a$ .

An involution  $a \mapsto a^*$  in a ring  $\mathcal{R}$  is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

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\*The authors are supported by the Ministry of Science of Serbia, grant no. 174007.

In the rest of the paper we assume that  $\mathcal{R}$  is a ring with involution. An element  $a \in \mathcal{R}$  satisfying  $aa^* = a^*a$  is called *normal*. An element  $a \in \mathcal{R}$  satisfying  $a = a^*$  is called *Hermitian* (or *symmetric*).

An element  $a^\dagger$  is called the *Moore–Penrose inverse* (or *MP-inverse*) of  $a$ , if [12]:

$$aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad (aa^\dagger)^* = aa^\dagger, \quad (a^\dagger a)^* = a^\dagger a.$$

Recall that, if  $a^\dagger$  exists, then it is uniquely determined [5, 7, 12], and  $a$  is called Moore–Penrose invertible. The set of all Moore–Penrose invertible elements of  $\mathcal{R}$  is denoted by  $\mathcal{R}^\dagger$ . If  $a$  is invertible, then  $a^\dagger$  coincides with the ordinary inverse of  $a$ .

By the analogy with linear bounded operators on a Hilbert space, an element  $a \in \mathcal{R}^\dagger$  satisfying  $a^* = a^\dagger$  is called a *partial isometry*.

The following result is well-known and frequently used in the rest of the paper.

**Theorem 1.1.** [4, 10] *For any  $a \in \mathcal{R}^\dagger$ , the following is satisfied:*

- (a)  $(a^\dagger)^\dagger = a$ ;
- (b)  $(a^*)^\dagger = (a^\dagger)^*$ ;
- (c)  $(a^*a)^\dagger = a^\dagger(a^\dagger)^*$ ;
- (d)  $(aa^*)^\dagger = (a^\dagger)^*a^\dagger$ ;
- (e)  $a^* = a^\dagger aa^* = a^* aa^\dagger$ ;
- (f)  $a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger = (a^*a)^\# a^* = a^*(aa^*)^\#$ ;
- (g)  $(a^*)^\dagger = a(a^*a)^\dagger = (aa^*)^\dagger a$ .

Now we state the definition of EP elements [3], [8], [9], and also a basic characterization of EP elements.

**Definition 1.1.** An element  $a$  of a ring  $\mathcal{R}$  with involution is said to be *EP* if  $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$  and  $a^\# = a^\dagger$ .

**Lemma 1.1.** *An element  $a \in \mathcal{R}$  is EP if and only if  $a \in \mathcal{R}^\dagger$  and  $aa^\dagger = a^\dagger a$ .*

We observe that  $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$  if and only if  $a^* \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$  (see [8]) and  $a$  is EP if and only if  $a^*$  is EP. In [8], the equality  $(a^*)^\# = (a^\#)^*$  is proved. The following theorem is very useful tool to investigate EP elements in ring with involution.

**Theorem 1.2.** [8] *An element  $a \in \mathcal{R}$  is EP if and only if  $a$  is group invertible and  $a^\#a$  is symmetric.*

In [1] it is demonstrated the usefulness of the representation of complex matrices provided in [6] to explore various classes of matrices, such as partial isometries and EP. In [11], characterizations of partial isometries and EP elements in rings with involution are investigated, applying a purely algebraic technique and extending some results in [1] to more general settings. In this paper we present a number of new characterization of partial isometries and EP elements in rings with involution. As a consequence, we obtain some results from [1] and [11].

## 2 Characterizations of partial isometries and EP elements

In this section, we use the setting of rings with involution to give new characterizations of partial isometries and EP elements.

In the following theorem we assume that an element  $a$  of a ring  $\mathcal{R}$  with involution is both Moore–Penrose invertible and group invertible. We investigate some necessary and sufficient conditions for element  $a$  to be a partial isometry. Theorem 2.1 generalizes (Theorem 1 in [1]) and (Theorem 2.1 and Theorem 2.2 in [11]). If  $n = 1$  in the following result, then we get mentioned theorems in [11] as a corollaries.

We use  $\mathbb{N}$  to denote the set of all positive integers.

**Theorem 2.1.** *Suppose that  $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ , and let  $n \in \mathbb{N}$ . Then  $a$  is a partial isometry, if and only if one of the following equivalent conditions holds:*

- (i)  $a^n a^* = a^n a^\dagger$ ;
- (ii)  $a^* a^n = a^\dagger a^n$ ;
- (iii)  $a^* (a^\#)^n = a^\dagger (a^\#)^n$ ;
- (iv)  $(a^\#)^n a^* = (a^\#)^n a^\dagger$ ;
- (v)  $aa^* (a^\#)^n = (a^\#)^n$ ;
- (vi)  $(a^\#)^n a^* a = (a^\#)^n$ .

*Proof.* If  $a$  is a partial isometry, then  $a^* = a^\dagger$ . It is not difficult to check that conditions (i)-(vi) hold.

Conversely, to conclude that  $a$  is a partial isometry, we show that either the condition  $a^* = a^\dagger$  is satisfied, or one of the preceding already established condition of this theorem holds.

(i) Using the hypothesis  $a^n a^* = a^n a^\dagger$ , we obtain

$$a^* = a^\dagger a a^* = a^\dagger (a^\#)^{n-1} (a^n a^*) = a^\dagger (a^\#)^{n-1} a^n a^\dagger = a^\dagger a a^\dagger = a^\dagger.$$

So, the element  $a$  is a partial isometry.

(ii) Applying the involution to  $a^* a^n = a^\dagger a^n$ , using  $(a^\dagger)^* = (a^*)^\dagger$ , we have

$$(a^*)^n (a^*)^* = (a^*)^n (a^*)^\dagger,$$

i.e. the element  $a^*$  satisfies the condition (i). Thus,  $a^*$  is a partial isometry and, applying involution to  $(a^*)^* = (a^*)^\dagger$ , we deduce that  $a$  is a partial isometry.

(iii) From the equality  $a^* (a^\#)^n = a^\dagger (a^\#)^n$ , we get

$$a^* = a^* a a^\dagger = (a^* (a^\#)^n) a^{n+1} a^\dagger = a^\dagger (a^\#)^n a^{n+1} a^\dagger = a^\dagger a a^\dagger = a^\dagger.$$

(iv) Applying the involution to  $(a^\#)^n a^* = (a^\#)^n a^\dagger$ , and using  $(a^\#)^* = (a^*)^\#$  [8], we observe that

$$(a^*)^* [(a^*)^\#]^n = (a^*)^\dagger [(a^*)^\#]^n.$$

Hence, the condition (iii) is satisfied for  $a^*$  instead of  $a$ , and  $a^*$  is a partial isometry. Consequently,  $a$  is a partial isometry.

(v) Multiplying the assumption  $a a^* (a^\#)^n = (a^\#)^n$  by  $a^\dagger$  from the left side, we obtain

$$a^* (a^\#)^n = a^\dagger (a^\#)^n.$$

So, the condition (iii) holds, and  $a$  is a partial isometry.

(vi) Applying the involution to  $(a^\#)^n a^* a = (a^\#)^n$ , we get

$$a^* (a^*)^* [(a^*)^\#]^n = [(a^*)^\#]^n.$$

Thus, the condition (v) is satisfied for  $a^*$ , and  $a^*$  is a partial isometry.  $\square$

In the following result we present new equivalent conditions which ensure that an element  $a$  of a ring with involution is both a partial isometry and EP. These conditions involve elements  $a$ ,  $a^*$ ,  $a^\dagger$ ,  $a^\#$ , and also powers of these elements. If  $n = 1$ , then the following theorem gives as a consequence (Theorem 2.3 in [11]).

**Theorem 2.2.** *Suppose that  $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ , and let  $n \in \mathbb{N}$ . Then  $a$  is a partial isometry and EP, if and only if one of the following equivalent conditions holds:*

- (i)  $a$  is partial isometry and  $a^*a^n = a^n a^*$ ;
- (ii)  $a^n a^* = a^\dagger a^n$ ;
- (iii)  $a^* a^n = a^n a^\dagger$ ;
- (iv)  $a^n a^* = a^n a^\#$ ;
- (v)  $a^* a^n = a^n a^\#$ ;
- (vi)  $a^*(a^\dagger)^n = a^\dagger(a^\#)^n$ ;
- (vii)  $(a^\dagger)^n a^* = (a^\#)^n a^\dagger$ ;
- (viii)  $(a^\dagger)^n a^* = a^\dagger(a^\#)^n$ ;
- (ix)  $a^*(a^\dagger)^n = (a^\#)^n a^\dagger$ ;
- (x)  $a^*(a^\#)^n = (a^\#)^n a^\dagger$ ;
- (xi)  $a^*(a^\dagger)^n = (a^\#)^{n+1}$ ;
- (xii)  $a^*(a^\#)^n = (a^\dagger)^{n+1}$ ;
- (xiii)  $a^*(a^\#)^n = (a^\#)^{n+1}$ ;
- (xiv)  $aa^*(a^\dagger)^n = (a^\#)^n$ ;
- (xv)  $aa^*(a^\#)^n = (a^\dagger)^n$ ;
- (xvi)  $a^*a^{n+1} = a^n$ ;
- (xvii)  $a^{n+1}a^* = a^n$ ;
- (xviii)  $a(a^\dagger)^n a^* = (a^\#)^n$ ;
- (xix)  $a^*(a^\dagger)^n a = (a^\#)^n$ .

*Proof.* If  $a$  is a partial isometry and EP, then  $a^* = a^\dagger = a^\#$ . It is not difficult to verify that conditions (i)-(xix) hold.

Conversely, we know that  $a \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$  if and only if  $a^* \in \mathcal{R}^\# \cap \mathcal{R}^\dagger$ , and  $a$  is EP if and only if  $a^*$  is EP. We prove that  $a$  is a partial isometry and EP,

or we show that the element  $a$  or  $a^*$  satisfies one of the preceding already established conditions of this theorem.

(i) Since  $a$  is a partial isometry and  $a^*a^n = a^n a^*$ , then

$$aa^\# = a^n(a^\#)^n = a^n a^\dagger a(a^\#)^n = (a^n a^*)a(a^\#)^n = a^* a^n a(a^\#)^n = a^\dagger a.$$

Since  $a^\dagger a$  is symmetric, we get that  $aa^\#$  is symmetric also, and  $a$  is EP, by Theorem 1.2.

(ii) By the condition  $a^n a^* = a^\dagger a^n$ , we get

$$\begin{aligned} a^* &= a^\dagger a a^* = a^\dagger (a^\#)^{n-1} (a^n a^*) a a^\dagger = a^\dagger (a^\#)^{n-1} a^\dagger a^n a a^\dagger \\ &= a^\dagger (a^\#)^n a a^\dagger a a^n a^\dagger = a^\dagger (a^\#)^n a^{n+1} a^\dagger = a^\dagger a a^\dagger = a^\dagger, \end{aligned}$$

i.e.  $a$  is a partial isometry, and  $a^n a^* = a^* a^n$ , which implies that  $a$  satisfies the condition (i).

(iii) Applying the involution to  $a^* a^n = a^n a^\dagger$ , we obtain

$$(a^*)^n (a^*)^* = (a^\dagger)^* (a^*)^n = (a^*)^\dagger (a^*)^n,$$

by Theorem 1.1. So, the condition (ii) holds for  $a^*$ .

(iv) Using the equality  $a^n a^* = a^n a^\#$ , we get

$$(1) \quad a^n a^* = (a^n a^*) a a^\dagger = a^n a^\# a a^\dagger = a^n a^\dagger,$$

which gives, by Theorem 2.1 (i), that  $a$  is a partial isometry. The equalities (iv) and (1) imply  $a^n a^\dagger = a^n a^\#$  and multiplying this expression by  $(a^\#)^{n-1}$  from the left side, we obtain  $aa^\dagger = aa^\#$ . Since  $aa^\dagger$  is symmetric, we conclude that  $aa^\#$  is symmetric. From Theorem 1.2, we get that  $a$  is EP.

(v) Applying the involution to  $a^* a^n = a^n a^\#$ , we get

$$(a^*)^n (a^*)^* = (a^n a^\#)^* = (a^\# a^n)^* = (a^*)^n (a^*)^\#.$$

Hence,  $a^*$  satisfies the equality (iv), so  $a$  is EP and a partial isometry.

(vi) The assumption  $a^* (a^\dagger)^n = a^\dagger (a^\#)^n$  implies

$$\begin{aligned} aa^\# &= a^{n+1} (a^\#)^{n+1} = a^n a a^\dagger a (a^\#)^{n+1} = a^{n+1} (a^\dagger (a^\#)^n) = a^{n+1} a^* (a^\dagger)^n \\ &= a^{n+1} (a^* (a^\dagger)^n) a a^\dagger = a^{n+1} a^\dagger (a^\#)^n a a^\dagger = a^n a a^\dagger a (a^\#)^n a^\dagger \\ &= a^{n+1} (a^\#)^n a^\dagger = a a^\dagger. \end{aligned}$$

Thus, the element  $aa^\#$  is symmetric and  $a$  is EP, by Theorem 1.2. By  $a^\# = a^\dagger$  and (vi), we have  $a^* (a^\#)^n = a^\dagger (a^\#)^n$ , i.e. the condition (iii) of Theorem 2.1 is satisfied. So,  $a$  is a partial isometry.

(vii) Applying the involution to  $(a^\dagger)^n a^* = (a^\#)^n a^\dagger$ , we obtain

$$(a^*)^* [(a^*)^\dagger]^n = (a^*)^\dagger [(a^*)^\#]^n.$$

Hence,  $a^*$  satisfies the condition (vi).

(viii) The condition  $(a^\dagger)^n a^* = a^\dagger (a^\#)^n$  implies

$$\begin{aligned} aa^\# &= a^n aa^\dagger a (a^\#)^{n+1} = a^{n+1} (a^\dagger (a^\#)^n) = a^{n+1} (a^\dagger)^n a^* \\ &= a^{n+1} ((a^\dagger)^n a^*) aa^\dagger = a^{n+1} a^\dagger (a^\#)^n aa^\dagger = a^n aa^\dagger a (a^\#)^n a^\dagger \\ &= a^{n+1} (a^\#)^n a^\dagger = aa^\dagger. \end{aligned}$$

Therefore,  $aa^\#$  is symmetric, and by Theorem 1.2  $a$  is EP. From  $a^\dagger = a^\#$  and (viii) we obtain  $(a^\dagger)^n a^* = (a^\#)^n a^\dagger$ , i.e. the equality (vii) holds.

(ix) Applying the involution to  $a^* (a^\dagger)^n = (a^\#)^n a^\dagger$ , we have

$$[(a^*)^\dagger]^n (a^*)^* = (a^*)^\dagger [(a^*)^\#]^n.$$

Thus, the element  $a^*$  satisfies the condition (viii).

(x) From the assumption  $a^* (a^\#)^n = (a^\#)^n a^\dagger$ , we get

$$\begin{aligned} a^* a^n &= (a^* (a^\#)^n) a^{2n} = (a^\#)^n a^\dagger a^{2n} \\ &= (a^\#)^{n+1} aa^\dagger a a^{2n-1} = (a^\#)^{n+1} a^{2n} = a^n a^\#. \end{aligned}$$

So, the statement (v) holds.

(xi) Assume that  $a^* (a^\dagger)^n = (a^\#)^{n+1}$ . Now, we see

$$a^* (a^\dagger)^n = (a^* (a^\dagger)^n) aa^\dagger = (a^\#)^{n+1} aa^\dagger = (a^\#)^n a^\dagger,$$

that is the equality (ix) is satisfied.

(xii) Using the condition  $a^* (a^\#)^n = (a^\dagger)^{n+1}$ , we observe that

$$(a^\#)^n = aa^\dagger a (a^\#)^{n+1} = (a^\dagger)^* (a^* (a^\#)^n)^* = (a^\dagger)^* (a^\dagger)^{n+1},$$

which yields

$$\begin{aligned} aa^\# &= a^n (a^\#)^n = a^n (a^\dagger)^* (a^\dagger)^{n+1} \\ &= a^n ((a^\dagger)^* (a^\dagger)^{n+1}) aa^\dagger = a^n (a^\#)^n aa^\dagger = aa^\dagger. \end{aligned}$$

Hence,  $aa^\#$  is symmetric, so  $a$  is EP by Theorem 1.2. Then, by  $a^\dagger = a^\#$  and (xii), we deduce that  $a^* (a^\#)^n = (a^\#)^n a^\dagger$ , which is the condition (x).

(xiii) Suppose that  $a^* (a^\#)^n = (a^\#)^{n+1}$ . Then, by

$$a^* a^n = (a^* (a^\#)^n) a^{2n} = (a^\#)^{n+1} a^{2n} = a^n a^\#,$$

we see that the equality (v) holds.

(xiv) Multiplying  $aa^*(a^\dagger)^n = (a^\#)^n$  on the left-hand side by  $a^\dagger$ , we have

$$a^*(a^\dagger)^n = a^\dagger(a^\#)^n.$$

So, the condition (vi) is satisfied.

(xv) Multiplying  $aa^*(a^\#)^n = (a^\dagger)^n$  on the left-hand side by  $a^\dagger$ , we obtain the condition (xii).

(xvi) Multiplying  $a^*a^{n+1} = a^n$  on the right-hand side by  $a^\#$ , we get

$$a^*a^n = a^n a^\#.$$

Thus, the statement (v) holds.

(xvii) Applying the involution to  $a^{n+1}a^* = a^n$ , we show that

$$(a^*)^*(a^*)^{n+1} = (a^*)^n$$

which gives that  $a^*$  satisfies the equality (xvi).

(xviii) Multiplying  $a(a^\dagger)^n a^* = (a^\#)^n$  on the left-hand side by  $a^\dagger$ , we have

$$(a^\dagger)^n a^* = a^\dagger(a^\#)^n.$$

Hence,  $a$  satisfies the condition (viii).

(xix) Applying the involution to  $a^*(a^\dagger)^n a = (a^\#)^n$ , we obtain

$$a^*[(a^*)^\dagger]^n (a^*)^* = [(a^*)^\#]^n.$$

Therefore,  $a^*$  satisfies the condition (xviii). □

In the rest of paper we study several equivalent conditions for an element  $a$  in a ring with involution to satisfy  $(a^*)^n = (a^\dagger)^n$ . If  $n = 1$ , then we get some conditions of Theorem 2.1, and for  $m = n = 1$  we obtain (Theorem 2.1 and some equalities of Theorem 2.2 in [11]).

**Theorem 2.3.** *Suppose that  $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ , and let  $m, n \in \mathbb{N}$ . Then  $(a^*)^n = (a^\dagger)^n$  if and only if one of the following equivalent conditions holds:*

- (i)  $a^m (a^*)^n = a^m (a^\dagger)^n$ ;
- (ii)  $(a^*)^n a^m = (a^\dagger)^n a^m$ ;
- (iii)  $(a^*)^n (a^\#)^m = (a^\dagger)^n (a^\#)^m$ ;



$$(iv) (a^\#)^m(a^*)^n = (a^\#)^m(a^\dagger)^n.$$

*Proof.* If  $(a^*)^n = (a^\dagger)^n$ , then it is obvious that conditions (i)-(iv) hold.

Conversely, we will show that either the condition  $(a^*)^n = (a^\dagger)^n$  holds, or one of the preceding already established condition of this theorem is satisfied.

(i) By the assumption  $a^m(a^*)^n = a^m(a^\dagger)^n$ , we observe

$$\begin{aligned} (a^*)^n &= a^\dagger a (a^*)^n = a^\dagger (a^\#)^{m-1} (a^m (a^*)^n) \\ &= a^\dagger (a^\#)^{m-1} a^m (a^\dagger)^n = a^\dagger a (a^\dagger)^n = (a^\dagger)^n. \end{aligned}$$

(ii) Applying the involution to  $(a^*)^n a^m = (a^\dagger)^n a^m$ , we see that  $a^*$  satisfies the equality (i). So,  $[(a^*)^*]^n = [(a^*)^\dagger]^n$  and, applying involution to this equality, we obtain  $(a^*)^n = (a^\dagger)^n$ .

(iii) Assume that  $(a^*)^n (a^\#)^m = (a^\dagger)^n (a^\#)^m$ . Then

$$\begin{aligned} (a^*)^n &= (a^*)^n a a^\dagger = ((a^*)^n (a^\#)^m) a^{m+1} a^\dagger \\ &= (a^\dagger)^n (a^\#)^m a^{m+1} a^\dagger = (a^\dagger)^n a a^\dagger = (a^\dagger)^n. \end{aligned}$$

(iv) Applying the involution to  $(a^\#)^m (a^*)^n = (a^\#)^m (a^\dagger)^n$ , we show that the condition (iii) holds for  $a^*$ .  $\square$

Next, some necessary and sufficient conditions for an element  $a$  in a ring with involution to satisfy  $(a^*)^n = (a^\dagger)^n$  and to be EP are given. If  $n = 1$ , then we obtain condition (iv) and (v) of Theorem 2.2. If  $m = n = 1$ , then we obtain three statements of (Theorem 2.3 in [11]).

**Theorem 2.4.** *Suppose that  $a \in \mathcal{R}^\dagger \cap \mathcal{R}^\#$ , and let  $m, n \in \mathbb{N}$ . Then  $(a^*)^n = (a^\dagger)^n$  and  $a$  is EP, if and only if one of the following equivalent conditions holds:*

- (i)  $(a^*)^n = (a^\#)^n$ ;
- (ii)  $a^m (a^*)^n = a^m (a^\#)^n$ ;
- (iii)  $(a^*)^n a^m = (a^\#)^n a^m$ .

*Proof.* If  $(a^*)^n = (a^\dagger)^n$  and  $a$  is EP, then  $a^\dagger = a^\#$  and  $(a^*)^n = (a^\#)^n$ . So, conditions (i)-(iii) hold.

(i) The equality  $(a^*)^n = (a^\#)^n$  implies

$$a^\# a = (a^\#)^n a^n = (a^*)^n a^n.$$

Since  $(a^*)^n a^n$  is symmetric, then  $a^\# a$  is symmetric too and, by Theorem 1.2,  $a$  is EP. Using  $a^\dagger = a^\#$  and (i), we get  $(a^*)^n = (a^\dagger)^n$ .

(ii) From the assumption  $a^m (a^*)^n = a^m (a^\#)^n$ , we obtain

$$(a^*)^n = a^\dagger a (a^*)^n = a^\dagger (a^\#)^{m-1} (a^m (a^*)^n) = a^\dagger (a^\#)^{m-1} a^m (a^\#)^n = a^\dagger a (a^\#)^n$$

and then

$$a^n (a^*)^n = a^n a^\dagger a (a^\#)^n = a^n (a^\#)^n = a a^\#,$$

which implies that  $a a^\#$  is symmetric, and  $a$  is EP by Theorem 1.2. Since  $(a^*)^n = a^\dagger a (a^\#)^n$  and  $a^\dagger = a^\#$ , obviously,  $(a^*)^n = (a^\dagger)^n$ .

(iii) Applying the involution to (iii), we observe that  $a^*$  satisfies (ii) implying  $[(a^*)^*]^n = [(a^*)^\dagger]^n$  and  $a^*$  is EP. Consequently, the element  $a$  is EP and  $(a^*)^n = (a^\dagger)^n$ .  $\square$

### 3 Conclusions

In this paper we studied equations involving an element in a ring with involution, its adjoint, Moore-Penrose and group inverse. We applied a purely algebraic technique to prove a number of new equivalent characterizations of partial isometries and EP elements. Some well-known results for complex matrices and elements in rings with involution are obtained as consequences.

**Acknowledgement.** We are grateful to the referees for their helpful comments concerning the paper.

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