Representation for the generalized Drazin inverse of block matrices in Banach algebras

Dijana Mosić and Dragan S. Djordjević*

Abstract

Several representations of the generalized Drazin inverse of a block matrix with a group invertible generalized Schur complement in Banach algebra are presented.

Key words and phrases: generalized Drazin inverse, Schur complement, block matrix.

2010 *Mathematics subject classification*: 46H05, 47A05, 15A09.

1 Introduction

The Drazin inverse has applications in a number of areas such as control theory, Markov chains, singular differential and difference equations, iterative methods in numerical linear algebra, etc. Representations for the Drazin inverse of block matrices under certain conditions where given in the literature [1, 3, 4, 5, 6, 7, 9, 14, 17].

In this paper, we present formulas for the generalized Drazin inverse of block matrix with generalized Schur complement being group invertible in Banach algebra. Moreover, necessary and sufficient conditions for the existence as well as the expressions for the group inverse of triangular matrices are obtained.

Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1. For $a \in \mathcal{A}$, we use $\sigma(a)$ and $\rho(a)$, respectively, to denote the spectrum and the resolvent set of a. The sets of all nilpotent and quasinilpotent elements ($\sigma(a) = \{0\}$) of $\mathcal{A}$ will be denoted by $\mathcal{A}^{nil}$ and $\mathcal{A}^{qnil}$, respectively.

The generalized Drazin inverse of $a \in \mathcal{A}$ (or Koliha–Drazin inverse of a) is the element $b \in \mathcal{A}$ which satisfies

$$bab = b, \qquad ab = ba, \qquad a - a^2 b \in \mathcal{A}^{qnil}.$$

*The authors are supported by the Ministry of Education and Science, Republic of Serbia, grant no. 174007.

If the generalized Drazin inverse of a exists, it is unique and denoted by a^{d}, and a is generalized Drazin invertible (see [11]). Recall that $p = 1 - aa^{\mathrm{d}}$ is the *spectral idempotent* of a corresponding to the set $\{0\}$, and it will be denoted by a^{π}. The set of all generalized Drazin invertible elements of $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathrm{d}}$. The Drazin inverse is a special case of the generalized Drazin inverse for which $a - a^2b \in \mathcal{A}^{nil}$ instead of $a - a^2b \in \mathcal{A}^{qnil}$. Obviously, if a is Drazin invertible, then it is generalized Drazin invertible. The group inverse is the Drazin inverse for which the condition $a - a^2b \in \mathcal{A}^{nil}$ is replaced with $a = aba$. We use $a^{\#}$ to denote the group inverse of a, and we use $\mathcal{A}^{\#}$ to denote the set of all group invertible elements of $\mathcal{A}$.

Let $p = p^2 \in \mathcal{A}$ be an idempotent. Then we can represent element $a \in \mathcal{A}$ as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$.

We use the following lemmas.

Lemma 1.1. *Let $b \in \mathcal{A}^{\mathrm{d}}$ and $a \in \mathcal{A}^{qnil}$.*

(i) [2, Corollary 3.4] *If $ab = 0$, then $a+b \in \mathcal{A}^{\mathrm{d}}$ and $(a+b)^{\mathrm{d}} = \sum_{n=0}^{\infty} (b^{\mathrm{d}})^{n+1} a^n$.*

(ii) *If $ba = 0$, then $a + b \in \mathcal{A}^{\mathrm{d}}$ and $(a+b)^{\mathrm{d}} = \sum_{n=0}^{\infty} a^n (b^{\mathrm{d}})^{n+1}$.*

Specializing [2, Corollary 3.4] (with multiplication reversed) to bounded linear operators Castro–González et al. [2] recovered [8, Theorem 2.2] which is a spacial case of Lemma 1.1(ii).

By the following lemma, Castro–González et al. [2] recovered [10, Theorem 2.1] for matrices and [8, Theorem 2.3] for bounded linear operators.

Lemma 1.2. [2, Example 4.5] *Let $a, b \in \mathcal{A}^{\mathrm{d}}$ and let $ab = 0$. Then*

$$(a+b)^{\mathrm{d}} = \sum_{n=0}^{\infty} (b^{\mathrm{d}})^{n+1} a^n a^{\pi} + \sum_{n=0}^{\infty} b^{\pi} b^n (a^{\mathrm{d}})^{n+1}.$$

The following result is well-known for complex matrices (see [16]) and it is proved for elements of Banach algebra in [12].

Lemma 1.3. [12, Lemma 2.2] *Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}$ relative to the idempotent $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^{\mathrm{d}}$ and let $w = aa^{\mathrm{d}} + a^{\mathrm{d}}bca^{\mathrm{d}}$ be such that $aw \in (p\mathcal{A}p)^{\mathrm{d}}$.*

If $ca^{\pi} = 0$, $a^{\pi}b = 0$ and the generalized Schur complement $s = d - ca^{\mathrm{d}}b$ is equal to 0, then

$$x^{\mathrm{d}} = \begin{bmatrix} p & 0 \\ ca^{\mathrm{d}} & 0 \end{bmatrix} \begin{bmatrix} [(aw)^{\mathrm{d}}]^2 a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & a^{\mathrm{d}}b \\ 0 & 0 \end{bmatrix}. \tag{1}$$

Let

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A} \tag{2}$$

relative to the idempotent $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^{\mathrm{d}}$ and let the generalized Schur complement $s = d - ca^{\mathrm{d}}b \in ((1-p)\mathcal{A}(1-p))^{\#}$. The generalized Schur complement s plays an important role in the representations for x^{d} in many cases [9, 14, 15, 17].

Hartwig et al. [9] presented representations for the Drazin inverse of a 2×2 block matix under conditions which involve $W = AA^D + A^DBCA^D$ and that the generalized Schur complement is equal to 0. Li [13] investigated a representation for the Drazin inverse of block matrices with a singular and group invertible generalized Schur complement, recovering the formula (1) for complex matrices [16].

We investigate representations of the generalized Drazin inverse of a block matrix x in (2) with a group invertible generalized Schur complement $s = d - ca^{\mathrm{d}}b$ under different conditions. Thus, we extend some results from [13, 16] to more general settings. Also, we obtain equivalent condition for the existence and representations for the group inverse of triangular matrices in Banach algebra.

The paper's aim is to further weaken the conditions on the elements needed to produce explicit formulae for the generalized Drazin inverse of x compared to those known from the literature. Such formulae are very complicated, but the main goal is to establish that x has the generalized Drazin inverse, and the formulae are the means to produce that result.

2 Results

In this section, we assume that

(i) the element x is defined as in (2) relative to the idempotent $p \in \mathcal{A}$,

(ii) $a \in (p\mathcal{A}p)^{\mathrm{d}}$,

(iii) $s = d - ca^{\mathrm{d}}b$ and $s \in ((1-p)\mathcal{A}(1-p))^{\#}$,

(iv) $w = aa^{\mathrm{d}} + a^{\mathrm{d}}bs^{\pi}ca^{\mathrm{d}}$.

First, we give a formula for the generalized Drazin inverse of block matrix x in (2) in terms of $w = aa^{\mathrm{d}} + a^{\mathrm{d}}bs^{\pi}ca^{\mathrm{d}}$ and the group invertible Schur complement s.

Theorem 2.1. *If $aw \in (p\mathcal{A}p)^{\mathrm{d}}$,*

$$a^{\pi}b = 0, \qquad bs^{\pi}ca^{\pi} = 0, \qquad wbss^{\#} = 0, \qquad ss^{\#}ca^{\mathrm{d}}bss^{\#} = 0, \tag{3}$$

then $x \in \mathcal{A}^{\mathrm{d}}$ and

$$\begin{aligned} x^{\mathrm{d}} &= \left(1 + \begin{bmatrix} 0 & bs^{\#} \\ 0 & ca^{\mathrm{d}}bs^{\#} \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 0 \\ -s^{\#}cw[(aw)^{\mathrm{d}}]^2a & s^{\#} - s^{\#}cw[(aw)^{\mathrm{d}}]^2bs^{\pi} \end{bmatrix}\right. \\ &+ \left.\sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ (s^{\#})^{n+1}ca^{n-1}a^{\pi} + v_na & v_nbs^{\pi} \end{bmatrix}\right) \\ &+ \begin{bmatrix} p & -bs^{\#} \\ 0 & (1-p) - ds^{\#} \end{bmatrix} r, \end{aligned} \tag{4}$$

where $v_n = (s^{\#})^{n+1}ca^{\mathrm{d}}(aw)^{n-1}(aw)^{\pi}$, $(n = 1, 2, \dots)$, and

$$r = \begin{bmatrix} [(aw)^{\mathrm{d}}]^2a & [(aw)^{\mathrm{d}}]^2bs^{\pi} \\ ca^{\mathrm{d}}[(aw)^{\mathrm{d}}]^2a & ca^{\mathrm{d}}[(aw)^{\mathrm{d}}]^2bs^{\pi} \end{bmatrix}.$$

Proof. Using the equalities $aa^{\mathrm{d}} + a^{\pi} = p$, $ss^{\#} + s^{\pi} = 1 - p$ and $ss^{\pi} = 0$, note that

$$x = \begin{bmatrix} a^2a^{\mathrm{d}} & bs^{\pi} \\ caa^{\mathrm{d}} & ca^{\mathrm{d}}bs^{\pi} \end{bmatrix} + \begin{bmatrix} aa^{\pi} & bss^{\#} \\ ca^{\pi} & dss^{\#} \end{bmatrix} := y + z.$$

By $a^{\mathrm{d}}a^{\pi} = 0$ and (3), we obtain

$$yz = \begin{bmatrix} bs^{\pi}ca^{\pi} & awbss^{\#} \\ ca^{\mathrm{d}}bs^{\pi}ca^{\pi} & cwbss^{\#} \end{bmatrix} = 0.$$

To check that $y \in \mathcal{A}^{\mathrm{d}}$, set $A_y \equiv a^2a^{\mathrm{d}}$, $B_y \equiv bs^{\pi}$, $C_y \equiv caa^{\mathrm{d}}$ and $D_y \equiv ca^{\mathrm{d}}bs^{\pi}$. Since $(a^2a^{\mathrm{d}})^{\#} = a^{\mathrm{d}}$, $A_y \in (p\mathcal{A}p)^{\#}$ and $S_y \equiv D_y - C_yA_y^{\#}B_y = 0$. Also, $A_y^{\pi}B_y = a^{\pi}bs^{\pi} = 0$, $C_yA_y^{\pi} = 0$ and $W_y = A_yA_y^{\#} + A_y^{\#}B_yC_yA_y^{\#} = w$. Applying Lemma 1.3, observe that $y \in \mathcal{A}^{\mathrm{d}}$ and

$$y^{\mathrm{d}} = \begin{bmatrix} p & 0 \\ ca^{\mathrm{d}} & 0 \end{bmatrix} \begin{bmatrix} [(aw)^{\mathrm{d}}]^2a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & a^{\mathrm{d}}bs^{\pi} \\ 0 & 0 \end{bmatrix} = r.$$

In order to prove that $z \in \mathcal{A}^{\mathrm{d}}$, we write

$$\begin{aligned} z &= \begin{bmatrix} aa^\pi & 0 \\ ca^\pi & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 0 & bss^\# \\ 0 & ca^{\mathrm{d}}bss^\# \end{bmatrix} \\ &:= z_1 + z_2 + z_3. \end{aligned}$$

Recall that, for $u = \begin{bmatrix} m & t \\ 0 & n \end{bmatrix}$,

$$\lambda \in \rho_{p\mathcal{A}p}(m) \cap \rho_{(1-p)\mathcal{A}(1-p)}(n) \ \Rightarrow \ \lambda \in \rho(u),$$

i.e.

$$\sigma(u) \subseteq \sigma_{p\mathcal{A}p}(m) \cup \sigma_{(1-p)\mathcal{A}(1-p)}(n).$$

From $aa^\pi \in (p\mathcal{A}p)^{qnil}$ and $s \in \mathcal{A}^\#$, we conclude that $z_1 \in \mathcal{A}^{qnil}$, $z_2 \in \mathcal{A}^\#$ and $z_2^\# = \begin{bmatrix} 0 & 0 \\ 0 & s^\# \end{bmatrix}$. Using Lemma 1.1(i), by $z_1 z_2 = 0$, we get $z_1 + z_2 \in \mathcal{A}^{\mathrm{d}}$ and $(z_1+z_2)^{\mathrm{d}} = \sum\limits_{n=0}^{\infty} (z_2^\#)^{n+1} z_1^n$. Further, $z_3^2 = 0$, i.e. $z_3 \in \mathcal{A}^{nil}$ and $(z_1+z_2)z_3 = 0$. By Lemma 1.1(ii), $z \in \mathcal{A}^{\mathrm{d}}$ and $z^{\mathrm{d}} = (z_1+z_2)^{\mathrm{d}} + z_3[(z_1+z_2)^{\mathrm{d}}]^2$.

Therefore, by Lemma 1.2, we deduce that $x \in \mathcal{A}^{\mathrm{d}}$ and

$$\begin{aligned} x^{\mathrm{d}} &= \sum_{n=0}^{\infty} (z^{\mathrm{d}})^{n+1} y^n y^\pi + \sum_{n=0}^{\infty} z^\pi z^n (y^{\mathrm{d}})^{n+1} \\ &= \sum_{n=0}^{\infty} (1 + z_3(z_1+z_2)^{\mathrm{d}})[(z_1+z_2)^{\mathrm{d}}]^{n+1} y^n y^\pi + \sum_{n=0}^{\infty} z^\pi z^n (y^{\mathrm{d}})^{n+1} \\ &:= X_1 + X_2, \end{aligned} \tag{5}$$

where

$$X_1 = (1 + z_3(z_1+z_2)^{\mathrm{d}}) x_1 y^\pi, \qquad x_1 = \sum_{n=0}^{\infty} [(z_1+z_2)^{\mathrm{d}}]^{n+1} y^n,$$

$$X_2 = z^\pi y^{\mathrm{d}} + \sum_{n=1}^{\infty} z^\pi z^n (y^{\mathrm{d}})^{n+1}.$$

Since $z_1 y = 0$, then $(z_1+z_2)^{\mathrm{d}} y = \begin{bmatrix} 0 & 0 \\ 0 & s^{\#} \end{bmatrix} y = z_2^{\#} y$ and

$$\begin{aligned} x_1 &= (z_1+z_2)^{\mathrm{d}} + \sum_{n=1}^{\infty} [(z_1+z_2)^{\mathrm{d}}]^n z_2^{\#} y^n \\ &= (z_1+z_2)^{\mathrm{d}} + \sum_{n=1}^{\infty} \left(z_2^{\#} + \sum_{k=1}^{\infty} (z_2^{\#})^{k+1} z_1^k \right)^n z_2^{\#} y^n \\ &= (z_1+z_2)^{\mathrm{d}} + \sum_{n=1}^{\infty} \left((z_2^{\#})^n + (z_2^{\#})^{n-1} \sum_{k=1}^{\infty} (z_2^{\#})^{k+1} z_1^k \right) z_2^{\#} y^n \\ &= \sum_{n=0}^{\infty} (z_2^{\#})^{n+1} z_1^n + \sum_{n=1}^{\infty} (z_2^{\#})^{n+1} y^n. \end{aligned}$$

Now, we have

$$\begin{aligned} X_1 &= \left(1 + \begin{bmatrix} 0 & b \\ 0 & ca^{\mathrm{d}}b \end{bmatrix} \sum_{k=0}^{\infty} (z_2^{\#})^{k+1} z_1^k \right) \left(\sum_{n=0}^{\infty} (z_2^{\#})^{n+1} z_1^n + \sum_{n=1}^{\infty} (z_2^{\#})^{n+1} y^n \right) y^{\pi} \\ &= \left(1 + \begin{bmatrix} 0 & b \\ 0 & ca^{\mathrm{d}}b \end{bmatrix} \left[z_2^{\#} + \sum_{k=1}^{\infty} (z_2^{\#})^{k+1} z_1^k \right] \right) z_2^{\#} \\ &\times \left(\sum_{n=0}^{\infty} (z_2^{\#})^n z_1^n + \sum_{n=1}^{\infty} (z_2^{\#})^n y^n \right) y^{\pi} \\ &= \left(1 + \begin{bmatrix} 0 & bs^{\#} \\ 0 & ca^{\mathrm{d}}bs^{\#} \end{bmatrix} \right) \left(z_2^{\#} y^{\pi} + \sum_{n=1}^{\infty} (z_2^{\#})^{n+1} z_1^n y^{\pi} + \sum_{n=1}^{\infty} (z_2^{\#})^{n+1} y^n y^{\pi} \right) \\ &= \left(1 + \begin{bmatrix} 0 & bs^{\#} \\ 0 & ca^{\mathrm{d}}bs^{\#} \end{bmatrix} \right) \left(z_2^{\#} y^{\pi} + \sum_{n=1}^{\infty} (z_2^{\#})^{n+1} z_1^n + \sum_{n=1}^{\infty} (z_2^{\#})^{n+1} y^n y^{\pi} \right). \quad (6) \end{aligned}$$

Observe that $aa^{\mathrm{d}}(aw) = aw = (aw)aa^{\mathrm{d}}$,

$$y^{\pi} = 1 - yy^{\mathrm{d}} = \begin{bmatrix} p - (aw)^{\mathrm{d}}a & -(aw)^{\mathrm{d}}bs^{\pi} \\ -cw[(aw)^{\mathrm{d}}]^2 a & (1-p) - cw[(aw)^{\mathrm{d}}]^2 bs^{\pi} \end{bmatrix}$$

and

$$y^n y^{\pi} = \begin{bmatrix} aa^{\mathrm{d}}(aw)^{n-1}(aw)^{\pi}a & (aw)^{n-1}(aw)^{\pi}bs^{\pi} \\ ca^{\mathrm{d}}(aw)^{n-1}(aw)^{\pi}a & ca^{\mathrm{d}}(aw)^{n-1}(aw)^{\pi}bs^{\pi} \end{bmatrix} \qquad (n = 1, 2, \ldots).$$

Hence, these equalities and (6) give

$$\begin{aligned}
X_1 &= \left(1+\begin{bmatrix} 0 & bs^{\#} \\ 0 & ca^{\mathrm{d}}bs^{\#} \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 0 \\ -s^{\#}cw[(aw)^{\mathrm{d}}]^2a & s^{\#}-s^{\#}cw[(aw)^{\mathrm{d}}]^2bs^{\pi} \end{bmatrix}\right. \\
&+ \sum_{n=1}^{\infty}\begin{bmatrix} 0 & 0 \\ 0 & (s^{\#})^{n+1} \end{bmatrix}\begin{bmatrix} a^na^{\pi} & 0 \\ ca^{n-1}a^{\pi} & 0 \end{bmatrix} \\
&+ \left.\sum_{n=1}^{\infty}\begin{bmatrix} 0 & 0 \\ (s^{\#})^{n+1}ca^{\mathrm{d}}(aw)^{n-1}(aw)^{\pi}a & (s^{\#})^{n+1}ca^{\mathrm{d}}(aw)^{n-1}(aw)^{\pi}bs^{\pi} \end{bmatrix}\right) \\
&= \left(1+\begin{bmatrix} 0 & bs^{\#} \\ 0 & ca^{\mathrm{d}}bs^{\#} \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 0 \\ -s^{\#}cw[(aw)^{\mathrm{d}}]^2a & s^{\#}-s^{\#}cw[(aw)^{\mathrm{d}}]^2bs^{\pi} \end{bmatrix}\right. \\
&+ \left.\sum_{n=1}^{\infty}\begin{bmatrix} 0 & 0 \\ (s^{\#})^{n+1}ca^{n-1}a^{\pi}+v_na & v_nbs^{\pi} \end{bmatrix}\right). \qquad (7)
\end{aligned}$$

From

$$\begin{aligned}
zz^{\mathrm{d}}y &= [(z_1+z_2)(z_1+z_2)^{\mathrm{d}}+z_3(z_1+z_2)^{\mathrm{d}}]y = (z_1+z_2)z_2^{\#}y+z_3z_2^{\#}y \\
&= (z_2+z_3)z_2^{\#}y,
\end{aligned}$$

we get $zz^{\mathrm{d}}y^{\mathrm{d}} = (z_2+z_3)z_2^{\#}y^{\mathrm{d}}$ and $z^{\pi}y^{\mathrm{d}} = \begin{bmatrix} p & -bs^{\#} \\ 0 & (1-p)-ds^{\#} \end{bmatrix}r$. Thus, by $z\begin{bmatrix} p & -bs^{\#} \\ 0 & (1-p)-ds^{\#} \end{bmatrix} = z_1$ and $a^{\pi}(aw)^{\mathrm{d}} = a^{\pi}aa^{\mathrm{d}}aw[(aw)^{\mathrm{d}}]^2 = 0$, notice that $zz^{\pi}y^{\mathrm{d}} = z\begin{bmatrix} p & -bs^{\#} \\ 0 & (1-p)-ds^{\#} \end{bmatrix}r = z_1r = 0$ and

$$X_2 = z^{\pi}y^{\mathrm{d}} + \sum_{n=1}^{\infty} z^nz^{\pi}(y^{\mathrm{d}})^{n+1} = z^{\pi}r. \qquad (8)$$

So, (5), (7) and (8) imply (4). $\square$

Our conditions $a^{\pi}b = 0$, $bs^{\pi}ca^{\pi} = 0$, $wbss^{\#} = 0$, $ss^{\#}ca^{\mathrm{d}}bss^{\#} = 0$ in Theorem 2.1 can be formulated geometrically as

$$b\mathcal{A} \subset a\mathcal{A}, \qquad s^{\pi}ca^{\pi}\mathcal{A} \subset b^{\circ}, \qquad bss^{\#}\mathcal{A} \subset w^{\circ}, \qquad ca^{\mathrm{d}}bss^{\#}\mathcal{A} \subset s^{\circ},$$

where $e^{\circ} = \{f \in \mathcal{A} : ef = 0\}$.

If we assume that a is a group invertible and $w = 1$ in Theorem 2.1, we obtain the following result as a consequence.

Corollary 2.1. *Let $a \in (p\mathcal{A}p)^{\#}$ and let $w = aa^{\#} + a^{\#}bca^{\#} = 1$. If $a^{\pi}b = 0$, $bca^{\pi} = 0$ and $bss^{\#} = 0$, then $x \in \mathcal{A}^{\mathrm{d}}$ and*

$$x^{\mathrm{d}} = \begin{bmatrix} a^{\#} & (a^{\#})^2 b \\ -s^{\#}ca^{\#} + (s^{\#})^2 ca^{\pi} + s^{\pi}c(a^{\#})^2 & s^{\#} - s^{\#}c(a^{\#})^2 b + s^{\pi}c(a^{\#})^3 b \end{bmatrix}.$$

In the following theorem, the other formula for the generalized Drazin inverse of block matrix is presented.

Theorem 2.2. *If $aw \in (p\mathcal{A}p)^{\mathrm{d}}$,*

$$ca^{\pi} = 0, \qquad a^{\pi}bs^{\pi}c = 0, \qquad ss^{\#}cw = 0, \qquad ss^{\#}ca^{\mathrm{d}}bss^{\#} = 0,$$

then $x \in \mathcal{A}^{\mathrm{d}}$ and

$$\begin{aligned} x^{\mathrm{d}} &= \left(\begin{bmatrix} 0 & -(aw)^{\mathrm{d}}bs^{\#} \\ 0 & s^{\#} - s^{\pi}ca^{\mathrm{d}}(aw)^{\mathrm{d}}bs^{\#} \end{bmatrix} \right. \\ &+ \left. \sum_{n=1}^{\infty} \begin{bmatrix} 0 & a^{n-1}a^{\pi}b(s^{\#})^{n+1} + aa^{\mathrm{d}}(aw)^{n-1}(aw)^{\pi}b(s^{\#})^{n+1} \\ 0 & s^{\pi}ca^{\mathrm{d}}(aw)^{n-1}(aw)^{\pi}b(s^{\#})^{n+1} \end{bmatrix} \right) \\ &\times \left(1 + \begin{bmatrix} 0 & 0 \\ s^{\#}c & s^{\#}ca^{\mathrm{d}}b \end{bmatrix} \right) + t \begin{bmatrix} p & 0 \\ -s^{\#}c & (1-p) - s^{\#}d \end{bmatrix}, \end{aligned} \tag{9}$$

where

$$t = \begin{bmatrix} [(aw)^{\mathrm{d}}]^2 a & [(aw)^{\mathrm{d}}]^2 b \\ s^{\pi}ca^{\mathrm{d}}[(aw)^{\mathrm{d}}]^2 a & s^{\pi}ca^{\mathrm{d}}[(aw)^{\mathrm{d}}]^2 b \end{bmatrix}.$$

Proof. If we write

$$x = \begin{bmatrix} a^2a^{\mathrm{d}} & aa^{\mathrm{d}}b \\ s^{\pi}c & s^{\pi}ca^{\mathrm{d}}b \end{bmatrix} + \begin{bmatrix} aa^{\pi} & a^{\pi}b \\ ss^{\#}c & ss^{\#}d \end{bmatrix} := y + z,$$

then $zy = 0$. Using Lemma 1.3, we conclude that $y \in \mathcal{A}^{\mathrm{d}}$ and $y^{\mathrm{d}} = t$.

To show that $z \in \mathcal{A}^{\mathrm{d}}$, let

$$z = \begin{bmatrix} aa^{\pi} & a^{\pi}b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ ss^{\#}c & ss^{\#}ca^{\mathrm{d}}b \end{bmatrix} := z_1 + z_2 + z_3.$$

Then $z_1 \in \mathcal{A}^{qnil}$, $z_2 \in \mathcal{A}^{\#}$ and $z_2z_1 = 0$. By Lemma 1.1(ii), $z_1 + z_2 \in \mathcal{A}^{\mathrm{d}}$ and $(z_1 + z_2)^{\mathrm{d}} = \sum_{n=0}^{\infty} z_1^n (z_2^{\#})^{n+1}$. From $z_3^2 = 0$, $z_3(z_1 + z_2) = 0$ and Lemma 1.1(i), $z \in \mathcal{A}^{\mathrm{d}}$ and $z^{\mathrm{d}} = (z_1 + z_2)^{\mathrm{d}} + [(z_1 + z_2)^{\mathrm{d}}]^2 z_3$.

Applying Lemma 1.2, observe that $x \in \mathcal{A}^{\mathrm{d}}$ and

$$x^{\mathrm{d}} = \sum_{n=0}^{\infty} (y^{\mathrm{d}})^{n+1} z^n z^{\pi} + \sum_{n=0}^{\infty} y^{\pi} y^n (z^{\mathrm{d}})^{n+1} := X_1 + X_2. \tag{10}$$

The equalities $yz_1 = 0$ and $y(z_1+z_2)^{\mathrm{d}} = yz_2^{\#}$ imply $y^{\mathrm{d}} z^{\pi} = t \begin{bmatrix} p & 0 \\ -s^{\#}c & (1-p)-s^{\#}d \end{bmatrix}$ and $y^{\mathrm{d}} z^{\pi} z = 0$.

$$X_1 = y^{\mathrm{d}} z^{\pi} + \sum_{n=1}^{\infty} (y^{\mathrm{d}})^{n+1} z^{\pi} z^n = t z^{\pi}. \tag{11}$$

Next, we obtain

$$\begin{aligned}
X_2 &= \sum_{n=0}^{\infty} y^{\pi} y^n (z^{\mathrm{d}})^{n+1} = \sum_{n=0}^{\infty} y^{\pi} y^n [(z_1+z_2)^{\mathrm{d}}]^{n+1} (1 + (z_1+z_2)^{\mathrm{d}} z_3) \\
&= \left(y^{\pi}(z_1+z_2)^{\mathrm{d}} + \sum_{n=1}^{\infty} y^{\pi} y^n z_2^{\#} [(z_1+z_2)^{\mathrm{d}}]^n \right) (1 + (z_1+z_2)^{\mathrm{d}} z_3) \\
&= \left[y^{\pi} \sum_{n=0}^{\infty} z_1^n (z_2^{\#})^{n+1} + \sum_{n=1}^{\infty} y^{\pi} y^n z_2^{\#} \left((z_2^{\#})^n + \sum_{k=1}^{\infty} z_1^k (z_2^{\#})^{k+n} \right) \right] \\
&\times \left(1 + z_2^{\#} z_3 + \sum_{n=1}^{\infty} z_1^n (z_2^{\#})^{n+1} z_3 \right) \\
&= \left[y^{\pi} \sum_{n=0}^{\infty} z_1^n (z_2^{\#})^{n+1} + \sum_{n=1}^{\infty} y^{\pi} y^n (z_2^{\#})^{n+1} \right] (1 + z_2^{\#} z_3) \\
&= \left[y^{\pi} z_2^{\#} + \sum_{n=1}^{\infty} z_1^n (z_2^{\#})^{n+1} + \sum_{n=1}^{\infty} y^{\pi} y^n (z_2^{\#})^{n+1} \right] (1 + z_2^{\#} z_3).
\end{aligned}$$

Since $y^{\pi} = \begin{bmatrix} p-(aw)^{\mathrm{d}}a & -(aw)^{\mathrm{d}}b \\ -s^{\pi}ca^{\mathrm{d}}(aw)^{\mathrm{d}}a & (1-p)-s^{\pi}ca^{\mathrm{d}}(aw)^{\mathrm{d}}b \end{bmatrix}$ and

$$y^n y^{\pi} = \begin{bmatrix} aa^{\mathrm{d}}(aw)^{n-1}(aw)^{\pi}a & aa^{\mathrm{d}}(aw)^{n-1}(aw)^{\pi}b \\ s^{\pi}ca^{\mathrm{d}}(aw)^{n-1}(aw)^{\pi}a & s^{\pi}ca^{\mathrm{d}}(aw)^{n-1}(aw)^{\pi}b \end{bmatrix} \qquad (n=1,2,\dots),$$

we get

$$
\begin{aligned}
X_2 &= \left(\begin{bmatrix} 0 & -(aw)^{\mathrm{d}} b s^{\#} \\ 0 & s^{\#} - s^{\pi} c a^{\mathrm{d}} (aw)^{\mathrm{d}} b s^{\#} \end{bmatrix} \right. \\
&+ \left. \sum_{n=1}^{\infty} \begin{bmatrix} 0 & a^{n-1} a^{\pi} b (s^{\#})^{n+1} + a a^{\mathrm{d}} (aw)^{n-1} (aw)^{\pi} b (s^{\#})^{n+1} \\ 0 & s^{\pi} c a^{\mathrm{d}} (aw)^{n-1} (aw)^{\pi} b (s^{\#})^{n+1} \end{bmatrix} \right) \\
&\times \left(1 + \begin{bmatrix} 0 & 0 \\ s^{\#} c & s^{\#} c a^{\mathrm{d}} b \end{bmatrix} \right) \qquad (12)
\end{aligned}
$$

Thus, from (10), (11) and (12), we obtain (9). □

Notice that explicit formulae (4) and (9) for the generalized Drazin inverse of x are complicated, but the conditions on the elements needed to produce these formulae are weaken than those known from the literature and x has the generalized Drazin inverse under these conditions.

In Theorem 2.2, supposing that $a \in (p\mathcal{A}p)^{\#}$ and $w = 1$, the next corollary follows.

Corollary 2.2. *Let $a \in (p\mathcal{A}p)^{\#}$ and let $w = aa^{\#} + a^{\#} b c a^{\#} = 1$. If $ca^{\pi} = 0$, $a^{\pi} bc = 0$ and $ss^{\#} c = 0$, then $x \in \mathcal{A}^{\mathrm{d}}$ and*

$$
x^{\mathrm{d}} = \begin{bmatrix} a^{\#} & -a^{\#} b s^{\#} + a^{\pi} b (s^{\#})^2 + (a^{\#})^2 b s^{\pi} \\ s^{\pi} c (a^{\#})^2 & s^{\#} - c (a^{\#})^2 b s^{\#} + s^{\pi} c (a^{\#})^3 b s^{\pi} \end{bmatrix}.
$$

The following result is a consequence of Theorem 2.1 and Theorem 2.2.

Corollary 2.3. *If $aw \in (p\mathcal{A}p)^{\mathrm{d}}$, $s = 0$ and if one of the following conditions holds:*

(i) $a^{\pi} b = 0$ *and* $ca^{\pi} = 0$,

(ii) $a^{\pi} b = 0$ *and* $bca^{\pi} = 0$,

(iii) $ca^{\pi} = 0$ *and* $a^{\pi} bc = 0$,

then $x \in \mathcal{A}^{\mathrm{d}}$ and

$$
x^{\mathrm{d}} = \begin{bmatrix} [(aw)^{\mathrm{d}}]^2 a & [(aw)^{\mathrm{d}}]^2 b \\ c a^{\mathrm{d}} [(aw)^{\mathrm{d}}]^2 a & c a^{\mathrm{d}} [(aw)^{\mathrm{d}}]^2 b \end{bmatrix}.
$$

Notice that, the preceding corollary recover Lemma 1.3 for elements of Banach algebra and the analogy result for matrices [16].

We will use Theorem 2.1 to find the group inverse of a triangular block matrix. Precisely, for $b = 0$ in Theorem 2.1, we obtain the next result.

Theorem 2.3. *Let* $x = \begin{bmatrix} a & 0 \\ c & s \end{bmatrix} \in \mathcal{A}$ *relative to the idempotent* $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^{\mathrm{d}}$ *and* $s \in ((1-p)\mathcal{A}(1-p))^{\#}$*. Then*

(i) $x \in \mathcal{A}^{\mathrm{d}}$ *and*

$$x^{\mathrm{d}} = \begin{bmatrix} a^{\mathrm{d}} & 0 \\ s^{\pi}c(a^{\mathrm{d}})^2 - s^{\#}ca^{\mathrm{d}} & s^{\#} \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ (s^{\#})^{n+1}ca^{n-1}a^{\pi} & 0 \end{bmatrix}; \quad (13)$$

(ii) $x \in \mathcal{A}^{\#}$ *if and only if* $a \in (p\mathcal{A}p)^{\#}$ *and* $s^{\pi}ca^{\pi} = 0$*. Furthermore, if* $a \in (p\mathcal{A}p)^{\#}$ *and* $s^{\pi}ca^{\pi} = 0$*, then*

$$x^{\#} = \begin{bmatrix} a^{\#} & 0 \\ s^{\pi}c(a^{\#})^2 - s^{\#}ca^{\#} + (s^{\#})^2ca^{\pi} & s^{\#} \end{bmatrix}.$$

Proof. (i) If $b = 0$ in Theorem 2.1, then $s = d$, $w = aa^{\mathrm{d}}$, $aw = a^2a^{\mathrm{d}} \in (p\mathcal{A}p)^{\#}$, $(aw)^{\#} = a^{\mathrm{d}}$ and $a^{\mathrm{d}}(aw)^{\pi} = a^{\mathrm{d}}a^{\pi} = 0$ implying (13).

(ii) By the part (i), note that

$$x^2x^{\mathrm{d}} = \begin{bmatrix} a^2a^{\mathrm{d}} & 0 \\ caa^{\mathrm{d}} + \sum\limits_{n=1}^{\infty} s^2(s^{\#})^{n+1}ca^{n-1}a^{\pi} & s \end{bmatrix}.$$

Consequently, $x^2x^{\mathrm{d}} = x \Leftrightarrow a^2a^{\mathrm{d}} = a$ and $caa^{\mathrm{d}} + \sum\limits_{n=1}^{\infty} s(s^{\#})^n ca^{n-1}a^{\pi} = c$. Hence, $x \in \mathcal{A}^{\#}$ is equivalent to $a \in (p\mathcal{A}p)^{\#}$ and $ss^{\#}ca^{\pi} = ca^{\pi}$. $\square$

In the same manner as in the proof of Theorem 2.3, if $c = 0$ in Theorem 2.2, we verify the following theorem in which necessary and sufficient condition for the existence and representation of the group inverse are considered.

Theorem 2.4. *Let* $x = \begin{bmatrix} a & b \\ 0 & s \end{bmatrix} \in \mathcal{A}$ *relative to the idempotent* $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^{\mathrm{d}}$ *and* $s \in ((1-p)\mathcal{A}(1-p))^{\#}$*. Then*

(i) $x \in \mathcal{A}^{\mathrm{d}}$ *and*

$$x^{\mathrm{d}} = \begin{bmatrix} a^{\mathrm{d}} & (a^{\mathrm{d}})^2bs^{\pi} - a^{\mathrm{d}}bs^{\#} \\ 0 & s^{\#} \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & a^{n-1}a^{\pi}b(s^{\#})^{n+1} \\ 0 & 0 \end{bmatrix};$$

(ii) $x \in \mathcal{A}^{\#}$ *if and only if* $a \in (p\mathcal{A}p)^{\#}$ *and* $a^{\pi}bs^{\pi} = 0$. *Furthermore, if* $a \in (p\mathcal{A}p)^{\#}$ *and* $a^{\pi}bs^{\pi} = 0$*, then*

$$x^{\#} = \begin{bmatrix} a^{\#} & (a^{\#})^2bs^{\pi} - a^{\#}bs^{\#} + a^{\pi}b(s^{\#})^2 \\ 0 & s^{\#} \end{bmatrix}.$$

Observe that the part (i) of Theorem 2.3 and the same part of Theorem 2.4 are the special cases of [2, Theorem 2.3] for Banach algebra elements and [8, Theorem 2.2] for bounded linear operators.

Finally, we give an example to illustrate our results.

Example 2.1. In Banach algebra $\mathcal{A}$, if $x = \begin{bmatrix} p & b \\ 0 & 0 \end{bmatrix} \in \mathcal{A}$ (or $x = \begin{bmatrix} p & 0 \\ c & 0 \end{bmatrix} \in \mathcal{A}$) relative to the idempotent $p \in \mathcal{A}$, then $a^{\mathrm{d}} = a = p$, $a^{\pi} = 0$, $s = 0 = s^{\#}$, $s^{\pi} = 1 - p$ and $w = p = aw = (aw)^{\mathrm{d}}$. Using Theorem 2.1 or Theorem 2.2, we get that $x \in \mathcal{A}^{\mathrm{d}}$ and $x^{\mathrm{d}} = \begin{bmatrix} p & b \\ 0 & 0 \end{bmatrix}$ (or $x^{\mathrm{d}} = \begin{bmatrix} p & 0 \\ c & 0 \end{bmatrix}$).

References

[1] N. Castro–González, E. Dopazo, J. Robles, *Formulas for the Drazin inverse of special block matrices*, Appl. Math. Comput. 174 (2006) 252–270.

[2] N. Castro–González, J.J. Koliha, *New additive results for the g-Drazin inverse*, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004) 1085-1097.

[3] D.S. Cvetkovic-Ilić, Y. Wei, *Representations for the Drazin inverse of bounded operators on Banach space*, Electron. J. Linear Algebra 18 (2009) 613–627.

[4] C. Deng, *A note on the Drazin inverses with Banachiewicz-Schur forms*, Appl. Math. Comput. 213 (2009) 230–234.

[5] C. Deng, D.S. Cvetkovic-Ilić, Y.Wei, *Some results on the generalized Drazin inverse of operator matrices*, Linear and Multilinear Algebra 58 (2010) 503–521.

[6] C. Deng, Y. Wei, *Representations for the Drazin inverse of* 2×2 *block-operator matrix with singular Schur complement*, Linear Algebra and its Applications 435 (2011) 2766–2783.

[7] D.S. Djordjević, P.S. Stanimirović, *On the generalized Drazin inverse and generalized resolvent*, Czechoslovak Math. J. 51 (2001) 617634.

[8] D.S. Djordjević, Y. Wei, *Additive results for the generalized Drazin inverse*, J. Austral. Math. Soc. 73 (2002) 115125.

[9] R.E. Hartwig, X. Li, Y. Wei, *Representations for the Drazin inverse of* 2×2 *block matrix*, SIAM J. Matrix Anal. Appl. 27 (2006) 757-771.

[10] R.E. Hartwig, G. Wang and Y. Wei, *Some additive results on Drazin inverse*, Linear Algebra Appl. 322 (2001) 207217.

[11] J.J. Koliha, *A generalized Drazin inverse*, Glasgow Math. J. 38 (1996), 367–381.

[12] M.Z. Kolundžija, D. Mosić, D.S. Djordjević, *Generalized Drazin inverse of certain block matrices in Banach algebras*, (preprint).

[13] X. Li, *A representation for the Drazin inverse of block matrices with a singular generalized Schur complement*, Appl. Math. Comput. 217(18) (2011) 7531–7536.

[14] X. Li, Y. Wei, *A note on the representations for the Drazin inverse of* 2×2 *block matrix*, Linear Algebra Appl. 423 (2007) 332-338.

[15] M.F. Martínez-Serrano, N. Castro-González, *On the Drazin inverse of block matrices and generalized Schur complement*, Appl. Math. Comput. 215 (2009) 2733-2740.

[16] J. Miao, *Results of the Drazin inverse of block matrices*, J. Shanghai Normal Univ. 18 (1989) 25-31.

[17] Y. Wei, *Expressions for the Drazin inverse of* 2×2 *block matrix*, Linear and Multilinear Algebra 45 (1998) 131-146.

Address:

Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia

E-mail

D. Mosić: `dijana@pmf.ni.ac.rs`
D. S. Djordjević: `dragan@pmf.ni.ac.rs`