Reverse order law for the Moore–Penrose inverse in C^*-algebras

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Abstract

In this paper we study several equivalent conditions related to the reverse order law for the Moore–Penrose inverse in C^*-algebras. We extend some well-known results to more general settings. Then we apply this result to obtain the reverse order rule for the weighted Moore-Penrose inverse in C^*-algebras.

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1 Introduction

Let S be a semigroup with the unit 1. If $a, b \in S$ are invertible, then ab is invertible too and the inverse of the product ab satisfied the reverse order law $(ab)^{-1} = b^{-1}a^{-1}$. This formula cannot trivially be extended to the Moore–Penrose inverse of the product ab. In this paper we investigate necessary and sufficient conditions which are related to the reverse order law for the Moore–Penrose inverse. The reverse order law for the weighted Moore–Penrose inverse in C^*-algebras follows as a corollary.

Let $\mathcal{A}$ be a unital C^*–algebra. An element $a \in \mathcal{A}$ is regular if there exists some $b \in \mathcal{A}$ satisfying $aba = a$. The set of all regular elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^-$. An element $p \in \mathcal{A}$ is idempotent if $p^2 = p$. An element $a \in \mathcal{A}$ is self-adjoint if $a^* = a$. An element $x \in \mathcal{A}$ is positive if $x = x^*$ and $\sigma(x) \subseteq [0, +\infty)$, where the spectrum of element x is denoted by $\sigma(x)$. Notice that, positive elements are self-adjoint. If $x \in \mathcal{A}$, then x^*x is positive element.

The *Moore–Penrose inverse* (or *MP-inverse*) of $a \in \mathcal{A}$ is the element $b \in \mathcal{A}$, if the following equations hold [13]:

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (3)\ (ab)^* = ab, \quad (4)\ (ba)^* = ba.$$

There is at most one b such that above conditions hold (see [13]), and such b is denoted by $a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^\dagger$. If a is invertible, then $a^\dagger$ coincides with the ordinary inverse of a.

Definition 1.1. Let $\mathcal{A}$ be a unital C^*–algebra and e, f two invertible positive elements in $\mathcal{A}$. We say that the element $a \in \mathcal{A}$ has the weighted MP-inverse with weights e, f if there exists $b \in \mathcal{A}$ such that

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (3e)\ (eba)^* = eba, \quad (4f)\ (fab)^* = fab.$$

The unique weighted MP-inverse with weights e, f, will be denoted by $a^\dagger_{e,f}$ if it exists [11]. The set of all weighted MP-invertible elements of $\mathcal{A}$ with weights e, f, will be denoted by $\mathcal{A}^\dagger_{e,f}$.

If $\delta \subset \{1, 2, 3, 4, 3e, 4f\}$ and b satisfies the equations (i) for all $i \in \delta$, then b is an δ–inverse of a. The set of all δ–inverse of a is denote by $a\{\delta\}$. If $b \in a\{1\}$, then a is regular. If $b \in a\{1, 2\}$ and $ab = ba$, then a is *group invertible*.

Let $\mathcal{R}$ be a ring with involution. An element $a \in \mathcal{R}$ is: left *-cancellable if $a^*ax = a^*ay$ implies $ax = ay$; it is right *-cancellable if $xaa^* = yaa^*$ implies $xa = ya$; and it is *-cancellable if it is both left and right *-cancellable. We observe that a is left *-cancellable if and only if a^* is right *-cancellable. In C^*-algebras all elements are *-cancellable. A ring $\mathcal{R}$ is called *-reducing if every element of $\mathcal{R}$ is *-cancellable. This is equivalent to the implication $a^*a = 0 \Rightarrow a = 0$ for all $a \in \mathcal{R}$.

Now we formulate the following result, which is well-known and frequently used in the rest of the paper.

Theorem 1.1. [6, 12] *For any $a \in \mathcal{A}^\dagger$, the following is satisfied:*

(a) $(a^\dagger)^\dagger = a$;

(b) $(a^*)^\dagger = (a^\dagger)^*$;

(c) $(a^*a)^\dagger = a^\dagger(a^\dagger)^*$;

(d) $(aa^*)^\dagger = (a^\dagger)^*a^\dagger$;

(f) $a^* = a^\dagger a a^* = a^* a a^\dagger$;

(g) $a^\dagger = (a^* a)^\dagger a^* = a^* (a a^*)^\dagger$;

(h) $(a^*)^\dagger = a(a^* a)^\dagger = (a a^*)^\dagger a$.

Theorem 1.2. [8] *In a unital C^*–algebra $\mathcal{A}$, $a \in \mathcal{A}$ is MP-invertible if and only if a is regular.*

It is useful to express the weighted MP-inverse in terms of the ordinary MP-inverse.

Theorem 1.3. [11] *Let $\mathcal{A}$ be a unital C^*–algebra and let e, f be positive invertible elements of $\mathcal{A}$. If $a \in \mathcal{A}$ is regular, then the unique weighted MP-inverse $a^\dagger_{e,f}$ exists and*

$$a^\dagger_{e,f} = e^{-1/2}(f^{1/2} a e^{-1/2})^\dagger f^{1/2}.$$

In [7], Greville proved that $(ab)^\dagger = b^\dagger a^\dagger$ holds for complex matrices if and only if $a^\dagger a$ commutes with bb^* and $bb^\dagger$ commutes with aa^*. Bouldin [2, 3] and Izumino [10] generalized this result for closed range operators on Hilbert spaces. Their proofs are based on operator theoretical methods and use properties of ranges of operators and gaps between subspaces. In [11], a proof of the reverse order rule for the Moore-Penrose inverse in the setting of rings with involution is presented, extending the results for Hilbert space operators from [2, 3, 10]. This result is formulated as follows.

Theorem 1.4. [11] *Let $\mathcal{R}$ be a ring with involution, let $a, b \in \mathcal{R}$ be MP–invertible and let $(1 - a^\dagger a)b$ be left *-cancellable. Then the following conditions are equivalent:*

(a) *ab is MP–invertible and $(ab)^\dagger = b^\dagger a^\dagger$;*

(b) *$a^\dagger a b b^* = b b^* a^\dagger a$ and $b b^\dagger a^* a = a^* a b b^\dagger$.*

Necessary and sufficient conditions for the reverse order rule for the weighted MP-inverse for matrices were given by Sun and Wei in [14] in terms of the inclusion of matrix ranges (column spaces). In [11], the result for the reverse order rule for the weighted Moore-Penrose inverse in C^*-algebras is proved, generalizing the matrix results in [14].

Tian [15, 16] studied a group of rank equalities related to the Moore–Penrose inverse of products of two matrices, which implies necessary and sufficient conditions for $(ab)^\dagger = b^\dagger a^\dagger$. The extensions of these results to the

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weighted Moore–Penrose inverse are considered too. The operator analogues of these results for the Moore–Penrose inverse are proved in [4, 5] for linear bounded operators on Hilbert spaces, using the matrix form of operators induced by some natural decomposition of Hilbert spaces.

In this paper we present a purely algebraic proof of some equivalent conditions related to the reverse order law for the Moore–Penrose inverse in C^*–algebras, extending the known results for matrices [15, 16] and Hilbert space operators [4, 5]. We show that neither the rank (in the finite dimensional case) nor the properties of operator matrices (in the infinite dimensional case) are necessary for the proof of the reverse order rule for the Moore–Penrose inverse valid under certain conditions on regular elements. Thus we extend some recent results to more general settings. As a corollary we obtain the reverse order law for the weighted Moore–Penrose inverse.

2 Reverse order law in C^*–algebras

In this section, we present necessary and sufficient conditions for the reverse order law for the Moore–Penrose inverse to hold. The first list of some equivalent statements is given below.

Theorem 2.1. *Let $\mathcal{A}$ be a unital C^*–algebra and let $a, b \in \mathcal{A}^-$. Then the following conditions are equivalent:*

(a) $abb^\dagger a^\dagger ab = ab$;

(b) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$;

(c) $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$;

(d) $a^\dagger abb^\dagger$ *is an idempotent;*

(e) $bb^\dagger a^\dagger a$ *is an idempotent;*

(f) $b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger = b^\dagger a^\dagger$;

(g) $(a^\dagger abb^\dagger)^\dagger = bb^\dagger a^\dagger a$.

Proof. We can easy get that $ab \in \mathcal{A}^- \Leftrightarrow a^\dagger abb^\dagger \in A^-$.

(a) $\Rightarrow$ (c): From $abb^\dagger a^\dagger ab = ab$, we obtain $a^\dagger abb^\dagger (1 - a^\dagger a)b = 0$ and

$$
\begin{aligned}
a^\dagger a(b^\dagger)^*((1 - a^\dagger a)b)^*(1 - a^\dagger a)b &= a^\dagger a(b^\dagger)^* b^*(1 - a^\dagger a)(1 - a^\dagger a)b \\
&= a^\dagger abb^\dagger (1 - a^\dagger a)b \\
&= 0.
\end{aligned}
\tag{1}
$$

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Because all elements are *-cancellable in C^*-algebra, we get

$$a^\dagger a (b^\dagger)^* ((1 - a^\dagger a)b)^* = 0.$$

This equality implies $a^\dagger a b b^\dagger (1 - a^\dagger a) = 0$, i.e.

$$a^\dagger a b b^\dagger = a^\dagger a b b^\dagger a^\dagger a. \tag{2}$$

Now, by (2), we have

$$a^\dagger a b b^\dagger = a^\dagger a b b^\dagger a^\dagger a = (a^\dagger a b b^\dagger a^\dagger a)^* = (a^\dagger a b b^\dagger)^* = b b^\dagger a^\dagger a.$$

Hence, the condition (c) holds.

(c) $\Rightarrow$ (d): By the equality $a^\dagger a b b^\dagger = b b^\dagger a^\dagger a$, we get the condition (d):

$$a^\dagger a (b b^\dagger a^\dagger a) b b^\dagger = a^\dagger a a^\dagger a b b^\dagger b b^\dagger = a^\dagger a b b^\dagger.$$

(d) $\Rightarrow$ (a): Multiplying the assumption

$$a^\dagger a b b^\dagger = a^\dagger a b b^\dagger a^\dagger a b b^\dagger$$

by a from the left side and by b from the right side, we have that $ab = abb^\dagger a^\dagger ab$. Therefore, the condition (a) is satisfied.

(d) $\Leftrightarrow$ (e): Applying the involution to $a^\dagger a b b^\dagger = a^\dagger a b b^\dagger a^\dagger a b b^\dagger$, we get $b b^\dagger a^\dagger a = b b^\dagger a^\dagger a b b^\dagger a^\dagger a$. The opposite implication is analogous.

(b) $\Rightarrow$ (e): The condition $b^\dagger a^\dagger a b b^\dagger a^\dagger = b^\dagger a^\dagger$ implies

$$b(b^\dagger a^\dagger)a = b b^\dagger a^\dagger a b b^\dagger a^\dagger a.$$

(e) $\Rightarrow$ (b): Multiplying the hypothesis

$$b b^\dagger a^\dagger a = b b^\dagger a^\dagger a b b^\dagger a^\dagger a$$

by $b^\dagger$ from the left side and by $a^\dagger$ from the right side, we get that the condition (b) holds: $b^\dagger a^\dagger = b^\dagger a^\dagger a b b^\dagger a^\dagger$.

(d) $\Rightarrow$ (g): Assume that $a^\dagger a b b^\dagger = a^\dagger a b b^\dagger a^\dagger a b b^\dagger$. Then the condition (c) holds, so (g) follows trivially.

(g) $\Rightarrow$ (d): Using the assumption $(a^\dagger a b b^\dagger)^\dagger = b b^\dagger a^\dagger a$, we have

$$a^\dagger a b b^\dagger a^\dagger a b b^\dagger = a^\dagger a b b^\dagger (b b^\dagger a^\dagger a) a^\dagger a b b^\dagger = a^\dagger a b b^\dagger (a^\dagger a b b^\dagger)^\dagger a^\dagger a b b^\dagger = a^\dagger a b b^\dagger.$$

(g) $\Rightarrow$ (f): If $(a^\dagger a b b^\dagger)^\dagger = b b^\dagger a^\dagger a$, then

$$b^\dagger (a^\dagger a b b^\dagger)^\dagger a^\dagger = b^\dagger b b^\dagger a^\dagger a a^\dagger = b^\dagger a^\dagger.$$

(f) $\Rightarrow$ (g): Suppose that $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = b^\dagger a^\dagger$ holds. Then

$$a^\dagger abb^\dagger(bb^\dagger a^\dagger a)a^\dagger abb^\dagger = a^\dagger ab(b^\dagger a^\dagger)abb^\dagger = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger = a^\dagger abb^\dagger,$$

$$\begin{aligned}
bb^\dagger a^\dagger a(a^\dagger abb^\dagger)bb^\dagger a^\dagger a &= b(b^\dagger a^\dagger)ab(b^\dagger a^\dagger)a = bb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger a \\
&= b(b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)a = bb^\dagger a^\dagger a,
\end{aligned}$$

$$(a^\dagger abb^\dagger bb^\dagger a^\dagger a)^* = a^\dagger abb^\dagger bb^\dagger a^\dagger a,$$

$$(bb^\dagger a^\dagger aa^\dagger abb^\dagger)^* = bb^\dagger a^\dagger aa^\dagger abb^\dagger.$$

Hence, $(a^\dagger abb^\dagger)^\dagger = bb^\dagger a^\dagger a$ holds. $\qquad\square$

The second list of equivalent statements follows.

Theorem 2.2. *Let $\mathcal{A}$ be a unital C^*–algebra and let $a, b, ab \in \mathcal{A}^-$. Then the following conditions are equivalent:*

(a) $ab(ab)^\dagger = abb^\dagger a^\dagger$;

(b) $a^* ab = bb^\dagger a^* ab$;

(c) $a^* abb^\dagger = bb^\dagger a^* a$;

(d) $b^\dagger a^\dagger \in (ab)\{1, 2, 3\}$;

(e) $b^\dagger a^\dagger \in (ab)\{1, 3\}$;

(f) $b\{1, 3\} \cdot a\{1, 3\} \subseteq (ab)\{1, 3\}$;

(g) $ab(ab)^\dagger a = abb^\dagger$;

(h) $(abb^\dagger)^\dagger = bb^\dagger a^\dagger$;

(i) $b^\dagger(abb^\dagger)^\dagger = b^\dagger a^\dagger$.

Proof. Observe that $ab \in \mathcal{A}^- \Leftrightarrow abb^\dagger \in A^-$.

(a) $\Rightarrow$ (b): Since $ab(ab)^\dagger = abb^\dagger a^\dagger$ and $ab(ab)^\dagger$ is self-adjoint, then

$$abb^\dagger a^\dagger = (abb^\dagger a^\dagger)^* = (a^\dagger)^* bb^\dagger a^*$$

and

$$a^* ab = a^* ab(ab)^\dagger ab = a^*(abb^\dagger a^\dagger)ab = a^*(a^\dagger)^* bb^\dagger a^* ab = a^\dagger abb^\dagger a^* ab. \quad (3)$$

Using the hypothesis $ab(ab)^\dagger = abb^\dagger a^\dagger$, we get $abb^\dagger a^\dagger ab = ab(ab)^\dagger ab = ab$. This equality implies, by Theorem 2.1 (parts (a) and (c)), $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$. Now, from (3), we obtain

$$a^* ab = (a^\dagger abb^\dagger)a^* ab = bb^\dagger a^\dagger aa^* ab = bb^\dagger a^* ab.$$

(b) $\Rightarrow$ (c): If $a^* ab = bb^\dagger a^* ab$, we have

$$a^* abb^\dagger = bb^\dagger a^* abb^\dagger = (abb^\dagger)^* abb^\dagger. \tag{4}$$

The right hand side of (4) is self-adjoint, which implies

$$a^* abb^\dagger = (a^* abb^\dagger)^* = bb^\dagger a^* a.$$

(c) $\Rightarrow$ (d): From $a^* abb^\dagger = bb^\dagger a^* a$, we obtain

$$abb^\dagger a^\dagger = aa^\dagger abb^\dagger a^\dagger = (a^\dagger)^*(a^* abb^\dagger)a^\dagger = (a^\dagger)^* bb^\dagger a^* aa^\dagger = (a^\dagger)^* bb^\dagger a^*. \tag{5}$$

Then
$$(abb^\dagger a^\dagger)^* = ((a^\dagger)^* bb^\dagger a^*)^* = abb^\dagger a^\dagger. \tag{6}$$

Using (5) and (c), we get

$$(abb^\dagger a^\dagger)ab = (a^\dagger)^*(bb^\dagger a^* a)b = (a^\dagger)^* a^* abb^\dagger b = ab. \tag{7}$$

Also,

$$\begin{aligned}
b^\dagger a^\dagger abb^\dagger a^\dagger &= b^\dagger a^*(a^\dagger)^* bb^\dagger a^\dagger aa^\dagger = b^\dagger a^*((a^\dagger)^* bb^\dagger a^*)(a^\dagger)^* a^\dagger \\
&= b^\dagger(a^* abb^\dagger)a^\dagger(a^\dagger)^* a^\dagger = b^\dagger bb^\dagger a^* aa^\dagger(a^\dagger)^* a^\dagger \\
&= b^\dagger a^*(a^\dagger)^* a^\dagger = b^\dagger a^\dagger. \tag{8}
\end{aligned}$$

By (6), (7) and (8), we conclude that $b^\dagger a^\dagger \in (ab)\{1,2,3\}$.

(d) $\Rightarrow$ (e): This part is obvious.

(e) $\Rightarrow$ (f): Suppose that $a^{(1,3)} \in a\{1,3\}$ and $b^{(1,3)} \in b\{1,3\}$. Notice that

$$\begin{aligned}
aa^{(1,3)} &= (aa^{(1,3)})^* = (a^{(1,3)})^* a^* = (a^{(1,3)})^*(aa^\dagger a)^* \\
&= (a^{(1,3)})^* a^* aa^\dagger = aa^{(1,3)} aa^\dagger = aa^\dagger. \tag{9}
\end{aligned}$$

Since $abb^\dagger a^\dagger ab = ab$, by Theorem 2.1 (parts (a) and (c)), we have $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$. From this equality and (9), we get

$$abb^{(1,3)} a^{(1,3)} = a(a^\dagger abb^\dagger)a^{(1,3)} = abb^\dagger a^\dagger aa^{(1,3)} = abb^\dagger a^\dagger aa^\dagger = abb^\dagger a^\dagger. \tag{10}$$

Because $b^\dagger a^\dagger \in (ab)\{3\}$, then the element $abb^\dagger a^\dagger$ is self-adjoint and we deduce, from (10), that $abb^{(1,3)}a^{(1,3)}$ is self-adjoint. So, $b^{(1,3)}a^{(1,3)} \in (ab)\{3\}$. Using the equality (10) and the assumption $b^\dagger a^\dagger \in (ab)\{1\}$, we get

$$abb^{(1,3)}a^{(1,3)}ab = abb^\dagger a^\dagger ab = ab,$$

i.e. $b^{(1,3)}a^{(1,3)} \in (ab)\{1\}$. Hence, the condition (f) holds.

(f) $\Rightarrow$ (a): Since $b^\dagger \in b\{1,3\}$ and $a^\dagger \in a\{1,3\}$, by the hypothesis $b\{1,3\} \cdot a\{1,3\} \subseteq (ab)\{1,3\}$, it follows that $b^\dagger a^\dagger \in (ab)\{1,3\}$. This implies

$$
\begin{aligned}
ab(ab)^\dagger &= (ab(ab)^\dagger)^* = (abb^\dagger a^\dagger ab(ab)^\dagger)^* \\
&= (ab(ab)^\dagger)^* (abb^\dagger a^\dagger)^* = ab(ab)^\dagger abb^\dagger a^\dagger \\
&= abb^\dagger a^\dagger.
\end{aligned}
$$

(a) $\Rightarrow$ (g): The equality $ab(ab)^\dagger = abb^\dagger a^\dagger$ gives $abb^\dagger a^\dagger = (a^\dagger)^* bb^\dagger a^*$ and (c), by the previous part of the proof. Now

$$ab(ab)^\dagger a = (abb^\dagger a^\dagger)a = (a^\dagger)^*(bb^\dagger a^* a) = (a^\dagger)^* a^* abb^\dagger = abb^\dagger.$$

(g) $\Rightarrow$ (b): Applying $ab(ab)^\dagger a = abb^\dagger$, we have

$$a^* ab = a^* ab(ab)^\dagger ab = (ab(ab)^\dagger a)^* ab = (abb^\dagger)^* ab = bb^\dagger a^* ab.$$

(d) $\Rightarrow$ (h): Suppose that $b^\dagger a^\dagger \in (ab)\{1,2,3\}$. Then

$$abb^\dagger(bb^\dagger a^\dagger)abb^\dagger = (abb^\dagger a^\dagger ab)b^\dagger = abb^\dagger,$$

$$bb^\dagger a^\dagger(abb^\dagger)bb^\dagger a^\dagger = b(b^\dagger a^\dagger abb^\dagger a^\dagger) = bb^\dagger a^\dagger,$$

$$(abb^\dagger bb^\dagger a^\dagger)^* = (abb^\dagger a^\dagger)^* = abb^\dagger a^\dagger,$$

$$(bb^\dagger a^\dagger abb^\dagger)^* = bb^\dagger a^\dagger abb^\dagger.$$

Hence, $(abb^\dagger)^\dagger = bb^\dagger a^\dagger$.

(h) $\Rightarrow$ (d): The condition $(abb^\dagger)^\dagger = bb^\dagger a^\dagger$ implies

$$abb^\dagger a^\dagger ab = (abb^\dagger bb^\dagger a^\dagger abb^\dagger)b = abb^\dagger b = ab,$$

$$b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger(bb^\dagger a^\dagger abb^\dagger bb^\dagger a^\dagger) = b^\dagger bb^\dagger a^\dagger = b^\dagger a^\dagger,$$

$$(abb^\dagger a^\dagger)^* = (abb^\dagger bb^\dagger a^\dagger)^* = abb^\dagger bb^\dagger a^\dagger = abb^\dagger a^\dagger.$$

Thus, $b^\dagger a^\dagger \in (ab)\{1,2,3\}$.

(h) $\Rightarrow$ (i): Obvious.

(i) $\Rightarrow$ (h): Assume that $b^\dagger(abb^\dagger)^\dagger = b^\dagger a^\dagger$ holds. Now, we have

$$abb^\dagger bb^\dagger a^\dagger abb^\dagger = ab(b^\dagger a^\dagger)abb^\dagger = abb^\dagger(abb^\dagger)^\dagger abb^\dagger = abb^\dagger,$$

$$\begin{aligned}
bb^\dagger a^\dagger abb^\dagger bb^\dagger a^\dagger &= b(b^\dagger a^\dagger)ab(b^\dagger a^\dagger) = bb^\dagger(abb^\dagger)^\dagger abb^\dagger(abb^\dagger)^\dagger \\
&= b(b^\dagger(abb^\dagger)^\dagger) = bb^\dagger a^\dagger,
\end{aligned}$$

$$\begin{aligned}
(abb^\dagger bb^\dagger a^\dagger)^* &= (ab(b^\dagger a^\dagger))^* = (abb^\dagger(abb^\dagger)^\dagger)^* = abb^\dagger(abb^\dagger)^\dagger \\
&= abb^\dagger a^\dagger = abb^\dagger bb^\dagger a^\dagger,
\end{aligned}$$

$$(bb^\dagger a^\dagger abb^\dagger)^* = bb^\dagger a^\dagger abb^\dagger.$$

So, we deduce that $(abb^\dagger)^\dagger = bb^\dagger a^\dagger$ holds. $\qquad\square$

Theorem 2.3. *Let $\mathcal{A}$ be a unital C^*–algebra and let $a, b, ab \in \mathcal{A}^-$. Then the following conditions are equivalent:*

  (a) $(ab)^\dagger ab = b^\dagger a^\dagger ab$;

  (b) $abb^* = abb^* a^\dagger a$;

  (c) $bb^* a^\dagger a = a^\dagger abb^*$;

  (d) $b^\dagger a^\dagger \in (ab)\{1, 2, 4\}$;

  (e) $b^\dagger a^\dagger \in (ab)\{1, 4\}$;

  (f) $b\{1, 4\} \cdot a\{1, 4\} \subseteq (ab)\{1, 4\}$;

  (g) $b(ab)^\dagger ab = a^\dagger ab$;

  (h) $(a^\dagger ab)^\dagger = b^\dagger a^\dagger a$;

  (i) $(a^\dagger ab)^\dagger a^\dagger = b^\dagger a^\dagger$.

Proof. Notice that $ab \in \mathcal{A}^- \Leftrightarrow a^\dagger ab \in A^-$.

  (a) $\Rightarrow$ (b): Since $(ab)^\dagger ab = b^\dagger a^\dagger ab$ and $(ab)^\dagger ab$ is self-adjoint, it follows that

$$b^\dagger a^\dagger ab = (b^\dagger a^\dagger ab)^* = b^* a^\dagger a(b^\dagger)^*.$$

Then

$$abb^* = ab(ab)^\dagger abb^* = ab(b^\dagger a^\dagger ab)b^* = abb^* a^\dagger a(b^\dagger)^* b^* = abb^* a^\dagger abb^\dagger. \quad (11)$$

Using the assumption $(ab)^\dagger ab = b^\dagger a^\dagger ab$, we obtain $abb^\dagger a^\dagger ab = ab(ab)^\dagger ab = ab$. By Theorem 2.1 (parts (a) and (c)), this equality gives $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$. So, from (11), we get

$$abb^* = abb^*(a^\dagger abb^\dagger) = abb^*bb^\dagger a^\dagger a = abb^* a^\dagger a.$$

(b) $\Rightarrow$ (c): By the condition $abb^* = abb^* a^\dagger a$, we have

$$a^\dagger abb^* = a^\dagger abb^* a^\dagger a = a^\dagger ab(a^\dagger ab)^*. \tag{12}$$

The right hand side of (12) is self-adjoint, which gives

$$a^\dagger abb^* = (a^\dagger abb^*)^* = bb^* a^\dagger a.$$

(c) $\Rightarrow$ (d): The hypothesis $bb^* a^\dagger a = a^\dagger abb^*$ gives

$$b^\dagger a^\dagger ab = b^\dagger a^\dagger abb^\dagger b = b^\dagger(a^\dagger abb^*)(b^\dagger)^* = b^\dagger bb^* a^\dagger a(b^\dagger)^* = b^* a^\dagger a(b^\dagger)^*. \tag{13}$$

Now
$$(b^\dagger a^\dagger ab)^* = (b^* a^\dagger a(b^\dagger)^*)^* = b^\dagger a^\dagger ab. \tag{14}$$

From the equalities (13) and (c), we obtain

$$ab(b^\dagger a^\dagger ab) = a(bb^* a^\dagger a)(b^\dagger)^* = aa^\dagger abb^*(b^\dagger)^* = ab \tag{15}$$

and

$$\begin{aligned}
b^\dagger a^\dagger abb^\dagger a^\dagger &= b^\dagger bb^\dagger a^\dagger a(b^\dagger)^* b^* a^\dagger = b^\dagger(b^\dagger)^*(b^* a^\dagger a(b^\dagger)^*)b^* a^\dagger \\
&= b^\dagger(b^\dagger)^* b^\dagger(a^\dagger abb^*)a^\dagger = b^\dagger(b^\dagger)^* b^\dagger bb^* a^\dagger aa^\dagger \\
&= b^\dagger(b^\dagger)^* b^* a^\dagger = b^\dagger a^\dagger.
\end{aligned} \tag{16}$$

Hence, by (14), (15) and (16), it follows that $b^\dagger a^\dagger \in (ab)\{1, 2, 4\}$.

(d) $\Rightarrow$ (e): Obviously.

(e) $\Rightarrow$ (f): Assume that $a^{(1,4)} \in a\{1, 4\}$ and $b^{(1,4)} \in b\{1, 4\}$. Observe that

$$\begin{aligned}
a^{(1,4)}a &= (a^{(1,4)}a)^* = a^*(a^{(1,4)})^* = (aa^\dagger a)^*(a^{(1,4)})^* \\
&= a^\dagger aa^*(a^{(1,4)})^* = a^\dagger aa^{(1,4)}a = a^\dagger a.
\end{aligned} \tag{17}$$

It is well known that the hypothesis $b^\dagger a^\dagger \in (ab)\{1\}$ implies $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$. By this and (17), we obtain

$$b^{(1,4)}a^{(1,4)}ab = b^{(1,4)}(a^\dagger abb^\dagger)b = b^{(1,4)}bb^\dagger a^\dagger ab = b^\dagger bb^\dagger a^\dagger ab = b^\dagger a^\dagger ab. \tag{18}$$

Now the condition $b^\dagger a^\dagger \in (ab)\{4\}$ implies that $b^{(1,4)}a^{(1,4)} \in (ab)\{4\}$. From the equality (18) and the assumption $b^\dagger a^\dagger \in (ab)\{1\}$, we get

$$abb^{(1,4)}a^{(1,4)}ab = abb^\dagger a^\dagger ab = ab.$$

Thus, $b^{(1,4)}a^{(1,4)} \in (ab)\{1\}$ and the condition (f) is satisfied.

(f) $\Rightarrow$ (a): Because $b^\dagger \in b\{1,4\}$ and $a^\dagger \in a\{1,4\}$, from the assumption $b\{1,4\} \cdot a\{1,4\} \subseteq (ab)\{1,4\}$, we get $b^\dagger a^\dagger \in (ab)\{1,4\}$. Then

$$
\begin{aligned}
(ab)^\dagger ab &= ((ab)^\dagger ab)^* = ((ab)^\dagger abb^\dagger a^\dagger ab)^* \\
&= (b^\dagger a^\dagger ab)^*((ab)^\dagger ab)^* = b^\dagger a^\dagger ab(ab)^\dagger ab \\
&= b^\dagger a^\dagger ab.
\end{aligned}
$$

(a) $\Rightarrow$ (g): The equality $(ab)^\dagger ab = b^\dagger a^\dagger ab$ implies $b^\dagger a^\dagger ab = b^* a^\dagger a(b^\dagger)^*$ and (c), by the previous part of the proof. Then

$$b(ab)^\dagger ab = b(b^\dagger a^\dagger ab) = (bb^* a^\dagger a)(b^\dagger)^* = a^\dagger abb^*(b^\dagger)^* = a^\dagger ab.$$

(g) $\Rightarrow$ (b): From $b(ab)^\dagger ab = a^\dagger ab$, we obtian

$$abb^* = ab(ab)^\dagger abb^* = ab(b(ab)^\dagger ab)^* = ab(a^\dagger ab)^* = abb^* a^\dagger a.$$

(d) $\Rightarrow$ (h): Assume that $b^\dagger a^\dagger \in (ab)\{1,2,4\}$. Now

$$a^\dagger abb^\dagger a^\dagger aa^\dagger ab = a^\dagger(abb^\dagger a^\dagger ab) = a^\dagger ab,$$

$$b^\dagger a^\dagger aa^\dagger abb^\dagger a^\dagger a = (b^\dagger a^\dagger abb^\dagger a^\dagger)a = b^\dagger a^\dagger a,$$

$$(a^\dagger abb^\dagger a^\dagger a)^* = a^\dagger abb^\dagger a^\dagger a,$$

$$(b^\dagger a^\dagger aa^\dagger ab)^* = (b^\dagger a^\dagger ab)^* = b^\dagger a^\dagger ab = b^\dagger a^\dagger aa^\dagger ab.$$

Thus, by definition, $(a^\dagger ab)^\dagger = b^\dagger a^\dagger a$.

(h) $\Rightarrow$ (d): If $(a^\dagger ab)^\dagger = b^\dagger a^\dagger a$, then

$$abb^\dagger a^\dagger ab = a(a^\dagger abb^\dagger a^\dagger aa^\dagger ab) = aa^\dagger ab = ab,$$

$$b^\dagger a^\dagger abb^\dagger a^\dagger = (b^\dagger a^\dagger aa^\dagger abb^\dagger a^\dagger a)a^\dagger = b^\dagger a^\dagger aa^\dagger = b^\dagger a^\dagger,$$

$$(b^\dagger a^\dagger ab)^* = (b^\dagger a^\dagger aa^\dagger ab)^* = b^\dagger a^\dagger aa^\dagger ab = b^\dagger a^\dagger ab.$$

Therefore, $b^\dagger a^\dagger \in (ab)\{1,2,4\}$.

(h) $\Rightarrow$ (i): Obvious.

(i) $\Rightarrow$ (h): Applying the equality $(a^\dagger ab)^\dagger a^\dagger = b^\dagger a^\dagger$, we have

$$a^\dagger abb^\dagger a^\dagger aa^\dagger ab = a^\dagger ab(b^\dagger a^\dagger)ab = a^\dagger ab(a^\dagger ab)^\dagger a^\dagger ab = a^\dagger ab,$$

$$b^\dagger a^\dagger a a^\dagger a b b^\dagger a^\dagger a = (b^\dagger a^\dagger)ab(b^\dagger a^\dagger)a = (a^\dagger ab)^\dagger a^\dagger ab(a^\dagger ab)^\dagger a^\dagger a$$
$$= ((a^\dagger ab)^\dagger a^\dagger)a = b^\dagger a^\dagger a,$$

$$(a^\dagger ab b^\dagger a^\dagger a)^* = a^\dagger ab b^\dagger a^\dagger a,$$

$$(b^\dagger a^\dagger a a^\dagger ab)^* = ((b^\dagger a^\dagger)ab)^* = ((a^\dagger ab)^\dagger a^\dagger ab)^* = ((a^\dagger ab)^\dagger a^\dagger)ab$$
$$= b^\dagger a^\dagger ab = b^\dagger a^\dagger a a^\dagger ab.$$

Hence, $(a^\dagger ab)^\dagger = b^\dagger a^\dagger a$. $\qquad\square$

The combination of Theorem 2.2 and Theorem 2.3 yields a group of equivalent conditions for $(ab)^\dagger = b^\dagger a^\dagger$ to hold. Notice that the equivalences (c) and (d) of the following theorem appear in [9, Lemma 5].

Theorem 2.4. *Let $\mathcal{A}$ be a unital C^*–algebra and let $a, b, ab \in \mathcal{A}^-$. Then the following conditions are equivalent:*

(a) $(ab)^\dagger = b^\dagger a^\dagger$;

(b) $ab(ab)^\dagger = abb^\dagger a^\dagger$ *and* $(ab)^\dagger ab = b^\dagger a^\dagger ab$;

(c) $a^*ab = bb^\dagger a^*ab$ *and* $abb^* = abb^* a^\dagger a$;

(d) $a^*abb^\dagger = bb^\dagger a^*a$ *and* $bb^* a^\dagger a = a^\dagger abb^*$;

(e) $b^\dagger a^\dagger \in (ab)\{1,3,4\}$;

(f) $b\{1,3\} \cdot a\{1,3\} \subseteq (ab)\{1,3\}$ *and* $b\{1,4\} \cdot a\{1,4\} \subseteq (ab)\{1,4\}$;

(g) $ab(ab)^\dagger a = abb^\dagger$ *and* $b(ab)^\dagger ab = a^\dagger ab$;

(h) $(abb^\dagger)^\dagger = bb^\dagger a^\dagger$ *and* $(a^\dagger ab)^\dagger = b^\dagger a^\dagger a$;

(i) $b^\dagger (abb^\dagger)^\dagger = b^\dagger a^\dagger$ *and* $(a^\dagger ab)^\dagger a^\dagger = b^\dagger a^\dagger$.

Remark. The preceding theorems hold in rings with involution assuming that $(1 - a^\dagger a)b$ is left *-cancellable. This hypothesis is automatically satisfied in C^* and *-reducing rings. Hence we recover the results in [4, 5] for Hilbert space operators. The results of Tian [16] are obtained as a special case of our results.

In the following theorem, we prove another group of equivalent conditions for $(ab)^\dagger = b^\dagger a^\dagger$ to be satisfied.

Theorem 2.5. *Let $\mathcal{A}$ be a unital C^*–algebra and let $a, b, ab \in \mathcal{A}^-$. Then $(ab)^\dagger = b^\dagger a^\dagger$ if and only if $(ab)^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger$ and any one of the following equivalent conditions holds:*

(a) $abb^\dagger a^\dagger ab = ab$;

(b) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$;

(c) $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$;

(d) $a^\dagger abb^\dagger$ *is an idempotent;*

(e) $bb^\dagger a^\dagger a$ *is an idempotent;*

(f) $b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger = b^\dagger a^\dagger$;

(g) $(a^\dagger abb^\dagger)^\dagger = bb^\dagger a^\dagger a$.

Proof. $\implies$: From the equality $(ab)^\dagger = b^\dagger a^\dagger$, we get $abb^\dagger a^\dagger ab = ab$. Thus, by Theorem 2.1, the conditions (a)-(g) are satisfied and $(ab)^\dagger = b^\dagger a^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger$.

$\impliedby$: Conversely, the conditions (a)-(g) imply $b^\dagger a^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger$. Now the hypothesis $(ab)^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger$ gives $(ab)^\dagger = b^\dagger a^\dagger$. $\square$

The condition $(ab)^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger$ in Theorem 2.5 can be replaced by some equivalent conditions, as it can be seen in the following theorem.

Theorem 2.6. *Let $\mathcal{A}$ be a unital C^*–algebra and let $a, b, ab \in \mathcal{A}^-$. Then the following statements are equivalent:*

(a) $(ab)^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger$;

(b) $(a^\dagger abb^\dagger)^\dagger = b(ab)^\dagger a$;

(c) $(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger = b^\dagger (abb^\dagger)^\dagger$;

(d) $(a^\dagger ab)^\dagger = (ab)^\dagger a$ *and* $(abb^\dagger)^\dagger = b(ab)^\dagger$;

(e) $(a^\dagger ab)^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger$ *and* $(abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger$;

(f) $(ab)^\dagger = b^* (a^* abb^*)^\dagger a^*$;

(g) $(a^* abb^*)^\dagger = (b^*)^\dagger (ab)^\dagger (a^*)^\dagger$.

Proof. We can easily see that $ab \in \mathcal{A}^- \Leftrightarrow a^*abb^* \in A^-$.

(a) $\Rightarrow$ (e): By the hypothesis $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$, we have

$$a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab = a^\dagger ab(ab)^\dagger ab = a^\dagger ab,$$

$$b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger,$$

$$(a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger)^* = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger,$$

$$(b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab)^* = ((ab)^\dagger ab)^* = (ab)^\dagger ab = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab.$$

So, $(a^\dagger ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger$.

Similarly

$$abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger = ab(ab)^\dagger abb^\dagger = abb^\dagger,$$

$$(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger,$$

$$(abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)^* = (ab(ab)^\dagger)^* = ab(ab)^\dagger = abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger,$$

$$((a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger)^* = (a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger.$$

Thus, $(abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger$.

(e) $\Rightarrow$ (a): The conditions $(a^\dagger ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger$ and $(abb^\dagger)^\dagger = (a^\dagger abb^\dagger)^\dagger a^\dagger$ imply

$$abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab = aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger b = aa^\dagger abb^\dagger b = ab,$$

$$b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger,$$

$$(abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)^* = abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger,$$

$$(b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab)^* = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab,$$

i.e. $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$.

(b) $\Rightarrow$ (a): From the equality $(a^\dagger abb^\dagger)^\dagger = b(ab)^\dagger a$, we get

$$abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger ab = abb^\dagger b(ab)^\dagger aa^\dagger ab = ab(ab)^\dagger ab = ab,$$

$$
\begin{aligned}
b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger &= b^\dagger b(ab)^\dagger aa^\dagger abb^\dagger b(ab)^\dagger aa^\dagger \\
&= b^\dagger b(ab)^\dagger ab(ab)^\dagger aa^\dagger \\
&= b^\dagger b(ab)^\dagger aa^\dagger \\
&= b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger,
\end{aligned}
$$

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$$
\begin{aligned}
(abb^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger})^{*} &= (abb^{\dagger}b(ab)^{\dagger}aa^{\dagger})^{*} = (ab(ab)^{\dagger}aa^{\dagger})^{*} \\
&= aa^{\dagger}ab(ab)^{\dagger} = ab(ab)^{\dagger} = (ab(ab)^{\dagger})^{*} \\
&= (aa^{\dagger}ab(ab)^{\dagger})^{*} = ab(ab)^{\dagger}aa^{\dagger} \\
&= abb^{\dagger}b(ab)^{\dagger}aa^{\dagger} = abb^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger},
\end{aligned}
$$

$$
\begin{aligned}
(b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}ab)^{*} &= (b^{\dagger}b(ab)^{\dagger}aa^{\dagger}ab)^{*} = (b^{\dagger}b(ab)^{\dagger}ab)^{*} \\
&= (ab)^{\dagger}abb^{\dagger}b = (ab)^{\dagger}ab = ((ab)^{\dagger}ab)^{*} \\
&= ((ab)^{\dagger}abb^{\dagger}b)^{*} = b^{\dagger}b(ab)^{\dagger}ab \\
&= b^{\dagger}b(ab)^{\dagger}aa^{\dagger}ab = b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}ab.
\end{aligned}
$$

Therefore, $(ab)^{\dagger} = b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}$.

(a) $\Rightarrow$ (c): Assume that $(ab)^{\dagger} = b^{\dagger}(a^{\dagger}abb^{\dagger})^{\dagger}a^{\dagger}$ holds. Then the condition (e) holds and $(ab)^{\dagger} = (a^{\dagger}ab)^{\dagger}a^{\dagger} = b^{\dagger}(abb^{\dagger})^{\dagger}$.

(c) $\Rightarrow$ (d): The hypothesis $(ab)^{\dagger} = (a^{\dagger}ab)^{\dagger}a^{\dagger}$ implies

$$
a^{\dagger}ab(ab)^{\dagger}aa^{\dagger}ab = a^{\dagger}ab(ab)^{\dagger}ab = a^{\dagger}ab,
$$

$$
(ab)^{\dagger}aa^{\dagger}ab(ab)^{\dagger}a = (ab)^{\dagger}ab(ab)^{\dagger}a = (ab)^{\dagger}a,
$$

$$
\begin{aligned}
(a^{\dagger}ab(ab)^{\dagger}a)^{*} &= (a^{\dagger}ab(a^{\dagger}ab)^{\dagger}a^{\dagger}a)^{*} = a^{\dagger}aa^{\dagger}ab(a^{\dagger}ab)^{\dagger} \\
&= a^{\dagger}ab(a^{\dagger}ab)^{\dagger} = (a^{\dagger}ab(a^{\dagger}ab)^{\dagger})^{*} \\
&= (a^{\dagger}aa^{\dagger}ab(a^{\dagger}ab)^{\dagger})^{*} = a^{\dagger}ab(a^{\dagger}ab)^{\dagger}a^{\dagger}a \\
&= a^{\dagger}ab(ab)^{\dagger}a,
\end{aligned}
$$

$$
((ab)^{\dagger}aa^{\dagger}ab)^{*} = ((ab)^{\dagger}ab)^{*} = (ab)^{\dagger}ab = (ab)^{\dagger}aa^{\dagger}ab.
$$

Thus, $(a^{\dagger}ab)^{\dagger} = (ab)^{\dagger}a$. By an analogy, from $(ab)^{\dagger} = b^{\dagger}(abb^{\dagger})^{\dagger}$, we obtain

$$
abb^{\dagger}b(ab)^{\dagger}abb^{\dagger} = ab(ab)^{\dagger}abb^{\dagger} = abb^{\dagger},
$$

$$
b(ab)^{\dagger}abb^{\dagger}b(ab)^{\dagger} = b(ab)^{\dagger}ab(ab)^{\dagger} = b(ab)^{\dagger},
$$

$$
(abb^{\dagger}b(ab)^{\dagger})^{*} = (ab(ab)^{\dagger})^{*} = ab(ab)^{\dagger} = abb^{\dagger}b(ab)^{\dagger},
$$

$$
\begin{aligned}
(b(ab)^{\dagger}abb^{\dagger})^{*} &= (bb^{\dagger}(abb^{\dagger})^{\dagger}abb^{\dagger})^{*} = (abb^{\dagger})^{\dagger}abb^{\dagger}bb^{\dagger} \\
&= (abb^{\dagger})^{\dagger}abb^{\dagger} = ((abb^{\dagger})^{\dagger}abb^{\dagger})^{*} \\
&= ((abb^{\dagger})^{\dagger}abb^{\dagger}bb^{\dagger})^{*} = bb^{\dagger}(abb^{\dagger})^{\dagger}abb^{\dagger} \\
&= b(ab)^{\dagger}abb^{\dagger}.
\end{aligned}
$$

So, we deduce that $(abb^\dagger)^\dagger = b(ab)^\dagger$.

(d) $\Rightarrow$ (g): Suppose that $(a^\dagger ab)^\dagger = (ab)^\dagger a$ and $(abb^\dagger)^\dagger = b(ab)^\dagger$. Now,

$$
\begin{aligned}
a^* abb^* (b^*)^\dagger (ab)^\dagger (a^*)^\dagger a^* abb^* &= a^* abb^\dagger b(ab)^\dagger aa^\dagger abb^* \\
&= a^* ab(ab)^\dagger abb^* \\
&= a^* abb^*,
\end{aligned}
$$

$$
\begin{aligned}
(b^*)^\dagger (ab)^\dagger (a^*)^\dagger a^* abb^* (b^*)^\dagger (ab)^\dagger (a^*)^\dagger &= (b^*)^\dagger (ab)^\dagger aa^\dagger abb^\dagger b(ab)^\dagger (a^*)^\dagger \\
&= (b^*)^\dagger (ab)^\dagger ab(ab)^\dagger (a^*)^\dagger \\
&= (b^*)^\dagger (ab)^\dagger (a^*)^\dagger,
\end{aligned}
$$

$$
\begin{aligned}
(a^* abb^* (b^*)^\dagger (ab)^\dagger (a^*)^\dagger)^* &= (a^* abb^\dagger b(ab)^\dagger (a^*)^\dagger)^* = (a^* ab(ab)^\dagger (a^*)^\dagger)^* \\
&= a^\dagger ab(ab)^\dagger a = (a^\dagger ab(ab)^\dagger a)^* \\
&= a^* ab(ab)^\dagger (a^*)^\dagger = a^* abb^\dagger b(ab)^\dagger (a^*)^\dagger \\
&= a^* abb^* (b^*)^\dagger (ab)^\dagger (a^*)^\dagger,
\end{aligned}
$$

$$
\begin{aligned}
((b^*)^\dagger (ab)^\dagger (a^*)^\dagger a^* abb^*)^* &= ((b^*)^\dagger (ab)^\dagger aa^\dagger abb^*)^* = ((b^*)^\dagger (ab)^\dagger abb^*)^* \\
&= b(ab)^\dagger abb^\dagger = (b(ab)^\dagger abb^\dagger)^* \\
&= (b^*)^\dagger (ab)^\dagger abb^* = (b^*)^\dagger (ab)^\dagger aa^\dagger abb^* \\
&= (b^*)^\dagger (ab)^\dagger (a^*)^\dagger a^* abb^*
\end{aligned}
$$

Therefore, $(a^* abb^*)^\dagger = (b^*)^\dagger (ab)^\dagger (a^*)^\dagger$.

(g) $\Rightarrow$ (f): From the condition $(a^* abb^*)^\dagger = (b^*)^\dagger (ab)^\dagger (a^*)^\dagger$, we get $b^*(a^* abb^*)^\dagger a^* = b^*(b^*)^\dagger (ab)^\dagger (a^*)^\dagger a^* = b^\dagger b(ab)^\dagger aa^\dagger$. Then, in the same way as in the proof of (b) $\Rightarrow$ (a), we conclude $(ab)^\dagger = b^*(a^* abb^*)^\dagger a^*$.

(f) $\Rightarrow$ (b): If $(ab)^\dagger = b^*(a^* abb^*)^\dagger a^*$, then

$$
a^\dagger abb^\dagger b(ab)^\dagger aa^\dagger abb^\dagger = a^\dagger ab(ab)^\dagger abb^\dagger = a^\dagger abb^\dagger,
$$

$$
b(ab)^\dagger aa^\dagger abb^\dagger b(ab)^\dagger a = b(ab)^\dagger ab(ab)^\dagger a = b(ab)^\dagger a,
$$

$$
\begin{aligned}
(a^\dagger abb^\dagger b(ab)^\dagger a)^* &= (a^\dagger ab(ab)^\dagger a)^* = a^* ab(ab)^\dagger (a^\dagger)^* \\
&= a^* abb^* (a^* abb^*)^\dagger a^* (a^\dagger)^* = (a^\dagger aa^* abb^* (a^* abb^*)^\dagger)^* \\
&= (a^* abb^* (a^* abb^*)^\dagger)^* = a^* abb^* (a^* abb^*)^\dagger \\
&= a^\dagger aa^* abb^* (a^* abb^*)^\dagger = (a^* abb^* (a^* abb^*)^\dagger a^* (a^\dagger)^*)^* \\
&= (a^* ab(ab)^\dagger (a^\dagger)^*)^* = a^\dagger ab(ab)^\dagger a \\
&= a^\dagger abb^\dagger b(ab)^\dagger a,
\end{aligned}
$$

$$
\begin{aligned}
(b(ab)^\dagger aa^\dagger abb^\dagger)^* &= (b(ab)^\dagger abb^\dagger)^* = (b^\dagger)^*(ab)^\dagger abb^* \\
&= (b^\dagger)^* b^* (a^* abb^*)^\dagger a^* abb^* = ((a^* abb^*)^\dagger a^* abb^* bb^\dagger)^* \\
&= ((a^* abb^*)^\dagger a^* abb^*)^* = (a^* abb^*)^\dagger a^* abb^* \\
&= (a^* abb^*)^\dagger a^* abb^* bb^\dagger = ((b^\dagger)^* b^* (a^* abb^*)^\dagger a^* abb^*)^* \\
&= ((b^\dagger)^*(ab)^\dagger abb^*)^* = b(ab)^\dagger abb^\dagger \\
&= b(ab)^\dagger aa^\dagger abb^\dagger.
\end{aligned}
$$

Hence, $(a^\dagger abb^\dagger)^\dagger = b(ab)^\dagger a$. $\qquad\qquad\square$

3 Reverse order law for the weighted MP-inverse

We can also consider reverse order law for the weighted Moore-Penrose inverse. Based on the results in Section 2, we now can establish various equivalent conditions related to the weighted MP-inverse of a product of elements in C^*–algebra.

Corollary 3.1. *Let $\mathcal{A}$ be a unital C^*–algebra and let e, f, h be positive invertible elements of $\mathcal{A}$. If $a, b \in \mathcal{A}$ are regular, then the following conditions are equivalent:*

(a) $abb^\dagger_{e,f} a^\dagger_{f,h} ab = ab$;

(b) $b^\dagger_{e,f} a^\dagger_{f,h} abb^\dagger_{e,f} a^\dagger_{f,h} = b^\dagger_{e,f} a^\dagger_{f,h}$;

(c) $a^\dagger_{f,h} abb^\dagger_{e,f} = bb^\dagger_{e,f} a^\dagger_{f,h} a$;

(d) $a^\dagger_{f,h} abb^\dagger_{e,f}$ *is an idempotent;*

(e) $bb^\dagger_{e,f} a^\dagger_{f,h} a$ *is an idempotent;*

(f) $b^\dagger_{e,f} (a^\dagger_{f,h} abb^\dagger_{e,f})^\dagger_{f,f} a^\dagger_{f,h} = b^\dagger_{e,f} a^\dagger_{f,h}$;

(g) $(a^\dagger_{f,h} abb^\dagger_{e,f})^\dagger_{f,f} = bb^\dagger_{e,f} a^\dagger_{f,h} a$.

Proof. (a) $\Leftrightarrow$ (b): Suppose that $a_1 = h^{1/2} a f^{-1/2}$ and $b_1 = f^{1/2} be^{-1/2}$. Then $a_1 b_1 = h^{1/2} abe^{-1/2}$ and a_1, b_1 are regular if and only if a, b are regular, respectively. From Theorem 1.3, we have $a^\dagger_{f,h} = f^{-1/2}(h^{1/2} a f^{-1/2})^\dagger h^{1/2} = f^{-1/2} a_1^\dagger h^{1/2}$ and $b^\dagger_{e,f} = e^{-1/2}(f^{1/2} be^{-1/2})^\dagger f^{1/2} = e^{-1/2} b_1^\dagger f^{1/2}$.

It is easy to verify that the condition $abb^\dagger_{e,f}a^\dagger_{f,h}ab = ab$ holds if and only if $a_1b_1b^\dagger_1a^\dagger_1a_1b_1 = a_1b_1$. By Theorem 2.1, this is necessary and sufficient condition for the equality $b^\dagger_1a^\dagger_1a_1b_1b^\dagger_1a^\dagger_1 = b^\dagger_1a^\dagger_1$ which is equivalent to $b^\dagger_{e,f}a^\dagger_{f,h}abb^\dagger_{e,f}a^\dagger_{f,h} = b^\dagger_{e,f}a^\dagger_{f,h}$.

The rest of the proof follows analogously. We only mention that $a^\dagger_{f,h}abb^\dagger_{e,f}$ is regular $\Leftrightarrow f^{-1/2}a^\dagger_1a_1b_1b^\dagger_1f^{1/2}$ is regular $\Leftrightarrow a^\dagger_1a_1b_1b_1$ is regular and, by Theorem 1.3, $(a^\dagger_{f,h}abb^\dagger_{e,f})^\dagger_{f,f} = (f^{-1/2}a^\dagger_1a_1b_1b^\dagger_1f^{1/2})^\dagger_{f,f} = f^{-1/2}(a^\dagger_1a_1b_1b^\dagger_1)^\dagger f^{1/2}$. $\square$

In order to prove Corollary 3.1 (and other corollaries in this section) in ring with involution we need to assume that $(1 - a^\dagger_{f,h}a)b$ is left $*_{e,f}$-cancellable where the function $*_{e,f}$ is not general an involution. Hence, it is not enough to assume that an element is just cancellable.

Let e, f be positive invertible elements of a unital C^*–algebra $\mathcal{A}$ and define $x^{*_{e,f}} = e^{-1}x^*f$.

Corollary 3.2. *Let $\mathcal{A}$ be a unital C^*–algebra and let e, f, h be positive invertible elements of $\mathcal{A}$. If $a, b, ab \in \mathcal{A}$ are regular, then the following conditions are equivalent:*

(a) $ab(ab)^\dagger_{e,h} = abb^\dagger_{e,f}a^\dagger_{f,h}$;

(b) $a^{*_{f,h}}ab = bb^\dagger_{e,f}a^{*_{f,h}}ab$;

(c) $a^{*_{f,h}}abb^\dagger_{e,f} = bb^\dagger_{e,f}a^{*_{f,h}}a$;

(d) $b^\dagger_{e,f}a^\dagger_{f,h} \in (ab)\{1, 2, 4h\}$;

(e) $b^\dagger_{e,f}a^\dagger_{f,h} \in (ab)\{1, 4h\}$;

(f) $b\{1, 4f\} \cdot a\{1, 4h\} \subseteq (ab)\{1, 4h\}$;

(g) $ab(ab)^\dagger_{e,h}a = abb^\dagger_{e,f}$;

(h) $(abb^\dagger_{e,f})^\dagger_{f,h} = bb^\dagger_{e,f}a^\dagger_{f,h}$;

(i) $b^\dagger_{e,f}(abb^\dagger_{e,f})^\dagger_{f,h} = b^\dagger_{e,f}a^\dagger_{f,h}$.

Proof. (a) $\Leftrightarrow$ (b): Let $a_1 = h^{1/2}af^{-1/2}$ and $b_1 = f^{1/2}be^{-1/2}$ as in the previous corollary. Thus $a_1b_1 = h^{1/2}abe^{-1/2}$ and a_1, b_1, a_1b_1 are regular

if and only if a, b, ab are regular, respectively. By Theorem 1.3, we get $(ab)^\dagger_{e,h} = e^{-1/2}(a_1 b_1)^\dagger h^{1/2}$ and $b^\dagger_{e,f} a^\dagger_{f,h} = e^{-1/2} b_1^\dagger a_1^\dagger h^{1/2}$.

Now, by Theorem 2.2,

$$
\begin{aligned}
ab(ab)^\dagger_{e,h} = abb^\dagger_{e,f} a^\dagger_{f,h} \quad &\Leftrightarrow \quad a_1 b_1 (a_1 b_1)^\dagger = a_1 b_1 b_1^\dagger a_1^\dagger \\
&\Leftrightarrow \quad a_1^* a_1 b_1 = b_1 b_1^\dagger a_1^* a_1 b_1 \\
&\Leftrightarrow \quad a^{*f,h} ab = bb^\dagger_{e,f} a^{*f,h} ab.
\end{aligned}
$$

The rest of the proof follows analogously. $\qquad\square$

Corollary 3.3. *Let $\mathcal{A}$ be a unital C^*–algebra and let e, f, h be positive invertible elements of $\mathcal{A}$. If $a, b, ab \in \mathcal{A}$ are regular, then the following conditions are equivalent:*

(a) $(ab)^\dagger_{e,h} ab = b^\dagger_{e,f} a^\dagger_{f,h} ab$;

(b) $abb^{*e,f} = abb^{*e,f} a^\dagger_{f,h} a$;

(c) $bb^{*e,f} a^\dagger_{f,h} a = a^\dagger_{f,h} abb^{*e,f}$;

(d) $b^\dagger_{e,f} a^\dagger_{f,h} \in (ab)\{1, 2, 3e\}$;

(e) $b^\dagger_{e,f} a^\dagger_{f,h} \in (ab)\{1, 3e\}$;

(f) $b\{1, 3e\} \cdot a\{1, 3f\} \subseteq (ab)\{1, 3e\}$;

(g) $b(ab)^\dagger_{e,h} ab = a^\dagger_{f,h} ab$;

(h) $(a^\dagger_{f,h} ab)^\dagger_{e,f} = b^\dagger_{e,f} a^\dagger_{f,h} a$;

(i) $(a^\dagger_{f,h} ab)^\dagger_{e,f} a^\dagger_{f,h} = b^\dagger_{e,f} a^\dagger_{f,h}$.

Proof. Let a_1 and b_1 be the same as in the previous corollary. For conditions (a)–(g) we can find the equivalent expressions in terms of a_1 and b_1, and apply Theorem 2.3 to finish the proof. $\qquad\square$

The combination of Corollary 3.2 and Corollary 3.3 gives a list of equivalent conditions for $(ab)^\dagger_{e,h} = b^\dagger_{e,f} a^\dagger_{f,h}$ to hold.

Corollary 3.4. *Let $\mathcal{A}$ be a unital C^*–algebra and let e, f, h be positive invertible elements of $\mathcal{A}$. If $a, b, ab \in \mathcal{A}$ are regular, then the following conditions are equivalent:*

(a) $(ab)^\dagger_{e,h} = b^\dagger_{e,f} a^\dagger_{f,h}$;

(b) $ab(ab)^\dagger_{e,h} = abb^\dagger_{e,f} a^\dagger_{f,h}$ and $(ab)^\dagger_{e,h} ab = b^\dagger_{e,f} a^\dagger_{f,h} ab$;

(c) $a^{*f,h} ab = bb^\dagger_{e,f} a^{*f,h} ab$ and $abb^{*e,f} = abb^{*e,f} a^\dagger_{f,h} a$;

(d) $a^{*f,h} abb^\dagger_{e,f} = bb^\dagger_{e,f} a^{*f,h} a$ and $bb^{*e,f} a^\dagger_{f,h} a = a^\dagger_{f,h} abb^{*e,f}$;

(e) $b^\dagger_{e,f} a^\dagger_{f,h} \in (ab)\{1, 3e, 4h\}$;

(f) $b\{1, 3e\} \cdot a\{1, 3f\} \subseteq (ab)\{1, 3e\}$ and $b\{1, 4f\} \cdot a\{1, 4h\} \subseteq (ab)\{1, 4h\}$;

(g) $ab(ab)^\dagger_{e,h} a = abb^\dagger_{e,f}$ and $b(ab)^\dagger_{e,h} ab = a^\dagger_{f,h} ab$;

(h) $(abb^\dagger_{e,f})^\dagger_{f,h} = bb^\dagger_{e,f} a^\dagger_{f,h}$ and $(a^\dagger_{f,h} ab)^\dagger_{e,f} = b^\dagger_{e,f} a^\dagger_{f,h} a$;

(i) $b^\dagger_{e,f} (abb^\dagger_{e,f})^\dagger_{f,h} = b^\dagger_{e,f} a^\dagger_{f,h}$ and $(a^\dagger_{f,h} ab)^\dagger_{e,f} a^\dagger_{f,h} = b^\dagger_{e,f} a^\dagger_{f,h}$.

Corollary 3.5. *Let $\mathcal{A}$ be a unital C^*–algebra and let e, f, h be positive invertible elements of $\mathcal{A}$. If $a, b, ab \in \mathcal{A}$ are regular, then the following statements are equivalent:*

(a) $(ab)^\dagger_{e,h} = b^\dagger_{e,f} (a^\dagger_{f,h} abb^\dagger_{e,f})^\dagger_{f,f} a^\dagger_{f,h}$;

(b) $(a^\dagger_{f,h} abb^\dagger_{e,f})^\dagger_{f,f} = b(ab)^\dagger_{e,h} a$;

(c) $(ab)^\dagger_{e,h} = (a^\dagger_{f,h} ab)^\dagger_{e,f} a^\dagger_{f,h} = b^\dagger_{e,f} (abb^\dagger_{e,f})^\dagger_{f,h}$;

(d) $(a^\dagger_{f,h} ab)^\dagger_{e,f} = (ab)^\dagger_{e,h} a$ and $(abb^\dagger_{e,f})^\dagger_{f,h} = b(ab)^\dagger_{e,h}$;

(e) $(a^\dagger_{f,h} ab)^\dagger_{e,f} = b^\dagger_{e,f} (a^\dagger_{f,h} abb^\dagger_{e,f})^\dagger_{f,f}$ and $(abb^\dagger_{e,f})^\dagger_{f,h} = (a^\dagger_{f,h} abb^\dagger_{e,f})^\dagger_{f,f} a^\dagger_{f,h}$;

(f) $(ab)^\dagger_{e,h} = b^{*e,f} (a^{*f,h} abb^{*e,f})^\dagger_{f,f} a^{*f,h}$;

(g) $(a^{*f,h} abb^{*e,f})^\dagger_{f,f} = (b^\dagger)^{*f,e} (ab)^\dagger_{e,h} (a^\dagger)^{*h,f}$.

Proof. Let a_1 and b_1 be the same as in Corollary 3.1. Applying Theorem 2.6 to the equivalent expressions, in terms of a_1 and b_1, for conditions (a)–(g) we prove this corollary. $\square$

In the result result we present again some equivalent conditions for the reverse order rule for the weighted Moore-Penrose inverse.

Corollary 3.6. *Let $\mathcal{A}$ be a unital C^*–algebra and let e, f, h be positive invertible elements of $\mathcal{A}$. If $a, b, ab \in \mathcal{A}$ are regular, then $(ab)^{\dagger}_{e,h} = b^{\dagger}_{e,f} a^{\dagger}_{f,h}$ if and only if $(ab)^{\dagger}_{e,h} = b^{\dagger}_{e,f} (a^{\dagger}_{f,h} abb^{\dagger}_{e,f})^{\dagger}_{f,f} a^{\dagger}_{f,h}$ and any one of the following equivalent conditions holds::*

(a) $abb^{\dagger}_{e,f} a^{\dagger}_{f,h} ab = ab$;

(b) $b^{\dagger}_{e,f} a^{\dagger}_{f,h} abb^{\dagger}_{e,f} a^{\dagger}_{f,h} = b^{\dagger}_{e,f} a^{\dagger}_{f,h}$;

(c) $a^{\dagger}_{f,h} abb^{\dagger}_{e,f} = bb^{\dagger}_{e,f} a^{\dagger}_{f,h} a$;

(d) $a^{\dagger}_{f,h} abb^{\dagger}_{e,f}$ *is an idempotent;*

(e) $bb^{\dagger}_{e,f} a^{\dagger}_{f,h} a$ *is an idempotent;*

(f) $b^{\dagger}_{e,f} (a^{\dagger}_{f,h} abb^{\dagger}_{e,f})^{\dagger}_{f,f} a^{\dagger}_{f,h} = b^{\dagger}_{e,f} a^{\dagger}_{f,h}$;

(g) $(a^{\dagger}_{f,h} abb^{\dagger}_{e,f})^{\dagger}_{f,f} = bb^{\dagger}_{e,f} a^{\dagger}_{f,h} a$.

Proof. This proof follows from Corollary 3.1 in the same way as in the proof of Theorem 2.5. $\qquad\square$

Notice that in Corollary 3.6 the condition $(ab)^{\dagger}_{e,h} = b^{\dagger}_{e,f} (a^{\dagger}_{f,h} abb^{\dagger}_{e,f})^{\dagger}_{f,f} a^{\dagger}_{f,h}$ can be replaced with some equivalent conditions from Corollary 3.5.

Arghiriade [1] proved that $(ab)^{\dagger} = b^{\dagger} a^{\dagger}$ holds if and only if $a^* abb^*$ is EP, i.e. $a^* abb^*$ commutes with its Moore-Penrose inverse. The conjecture is: Is there a similarly result involving weighted MP inverse?

4 Conclusions

In this paper we consider a number of necessary and sufficient conditions related to the reverse order law for the Moore–Penrose inverse in C^*–algebras. Applying this result we obtain the equivalent conditions for the reverse order rule for the weighted Moore-Penrose inverse of elements in C^*-algebras. All of these results are already known for complex matrices and some of them for closed range linear bounded operators on Hilbert spaces. However, we used the different technique in proving the results. In the theory of complex matrices various authors used the matrix rank to prove the equivalent conditions related to the reverse order law. In the case of linear bounded operators on Hilbert spaces, it seems that the method of operator matrices is very useful. In this paper we applied a purely algebraic technique. It

could be interesting to extend this work to the Moore–Penrose inverse and the weighted Moore-Penrose inverse of a triple product.

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