4. Nehomogen linearan sistem DJ

HOMOGENA LINEARNA VEKTORSKA DJ:

$$y'(x) = A(x)\, y(x). \tag{1}$$

NEHOMOGENA LINEARNA VEKTORSKA DJ:

$$y'(x) = A(x)\, y(x) + g(x). \tag{2}$$

$A(x)$ kvadratna matrična funkcija, $A \in C(a, b)$.

Opšte rešenje nehomogenog linearnog sistema DJ (2) je zbir opšteg rešenja odgovarajućeg homogenog sistema DJ (1) i bilo kog partikularnog rešenja nehomogenog sistema DJ (2).

$$Y(x) = Y_h(x) + Y_p(x) = \sum_{i=1}^{n} c_i \varphi_i(x) + Y_p(x)$$

LAGRANŽOVA METODA VARIJACIJE KONSTANATA

Rešenje nehomogenog sistema (2) tražimo u obliku

$$\varphi(x) = \sum_{i=1}^{n} c_i(x) \varphi_i(x), \tag{3}$$

$$\begin{aligned} \sum_{i=1}^{n} c_i'(x)\varphi_i(x) + \sum_{i=1}^{n} c_i(x)\varphi_i'(x) &= A(x) \sum_{i=1}^{n} c_i(x)\varphi_i(x) + g(x). \\ &= \sum_{i=1}^{n} c_i(x) A(x) \varphi_i(x) + g(x). \end{aligned}$$

$$\varphi_i'(x) \equiv A(x)\varphi_i(x),\ i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} c_i'(x)\varphi_i(x) = g(x),\ x \in (a, b). \tag{4}$$

$$\begin{aligned} &\varphi_{11}(x)c_1'(x) + \varphi_{12}(x)c_2'(x) + \cdots + \varphi_{1n}(x)c_n'(x) = g_1(x), \\ &\varphi_{21}(x)c_1'(x) + \varphi_{22}(x)c_2'(x) + \cdots + \varphi_{2n}(x)c_n'(x) = g_2(x), \\ &\vdots \\ &\varphi_{n1}(x)c_1'(x) + \varphi_{n2}(x)c_2'(x) + \cdots + \varphi_{nn}(x)c_n'(x) = g_n(x). \end{aligned}$$

$\Delta(x) = W(x) \neq 0 \;\longrightarrow\;$ sistem (4) ima jedinstveno rešenje

$$c_i'(x) = \psi_i(x), \quad i = 1, 2, \ldots, n$$

$$c_i(x) = \int_{x_0}^{x} \psi_i(u)\, du + c_i, \quad i = 1, 2, \ldots, n, \;\; x_0 \in (a, b).$$

Opšte rešenje nehomogenog sistema DJ (2):

$$\varphi(x) = \sum_{i=1}^{n} c_i \varphi_i(x) + \sum_{i=1}^{n} \varphi_i(x) \int_{x_0}^{x} \psi_i(u)\, du.$$

Sistem (4) iz koga odredjujemo $c_i'(x)$ je oblika

$$c_1'(x) \begin{pmatrix} \varphi_{11}(x) \\ \varphi_{21}(x) \\ \vdots \\ \varphi_{n1}(x) \end{pmatrix} + c_2'(x) \begin{pmatrix} \varphi_{12}(x) \\ \varphi_{22}(x) \\ \vdots \\ \varphi_{n2}(x) \end{pmatrix} + \cdots + c_n'(x) \begin{pmatrix} \varphi_{1n}(x) \\ \varphi_{2n}(x) \\ \vdots \\ \varphi_{nn}(x) \end{pmatrix} = \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{pmatrix},$$

odnosno

$$\begin{pmatrix} \varphi_{11}(x) & \varphi_{12}(x) & \ldots & \varphi_{1n}(x) \\ \varphi_{21}(x) & \varphi_{22}(x) & \ldots & \varphi_{2n}(x) \\ & & \vdots & \\ \varphi_{n1}(x) & \varphi_{n2}(x) & \ldots & \varphi_{nn}(x) \end{pmatrix} \cdot \begin{pmatrix} c_1'(x) \\ c_2'(x) \\ \vdots \\ c_n'(x) \end{pmatrix} = \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{pmatrix},$$

$$\Downarrow$$

$$\Phi(x)\, c'(x) = g(x) \quad \Longrightarrow \quad c'(x) = \Phi^{-1}(x)\, g(x).$$

$$c(x) = c + \int_{x_0}^{x} \Phi^{-1}(s)\, g(s)\, ds, \quad c \in \mathbb{R}^n \text{ proizvoljan konstantni vektor}, \; x_0 \in (a, b)$$

OPŠTE REŠENJE NEHOMOGENOG LINEARNOG SISTEMA DJ (2):

$$Y(x) = \Phi(x)\, c(x) = \Phi(x) c + \Phi(x) \int_{x_0}^{x} \Phi^{-1}(s)\, g(s)\, ds\,, \quad x, x_0 \in (a, b).$$

Teorema 1 *Neka je $\Phi(x)$ fundamentalna matrica homogenog linearnog sistema DJ (1). Tada je opšte rešenje nehomogenog linearnog sistema DJ (2) funkcija*

$$\varphi(x) = \Phi(x) \left(c + \int_{x_0}^{x} \Phi^{-1}(s)\, g(s)\, ds \right), \quad x \in (a, b), \tag{5}$$

gde je c proizvoljan konstantni vektor, a $x_0 \in (a, b)$ proizvoljna tačka.

Dokaz: Imajući u vidu da je $\Phi'(x) - A(x)\Phi(x) \equiv 0$, sledi

$$\begin{aligned} y' - A(x)\,y - g(x)\Big|_{y=\varphi(x)} &= \Phi'(x)\left(c + \int_{x_0}^{x} \Phi^{-1}(s)\,g(s)\,ds\right) \\ &+\Phi(x)\Phi^{-1}(x)g(x) - A(x)\Phi(x)\left(c + \int_{x_0}^{x} \Phi^{-1}(s)\,g(s)\,ds\right) - g(x) \\ &= [\Phi'(x) - A(x)\Phi(x)]\left(c + \int_{x_0}^{x} \Phi^{-1}(s)\,g(s)\,ds\right) + g(x) - g(x) \\ &\equiv 0. \end{aligned}$$

Dakle, $\varphi(x)$ je rešenje sistema (2). Šta više,

$$\varphi_p(x) = \Phi(x)\int_{x_0}^{x} \Phi^{-1}(s)\,g(s)\,ds$$

je partikularno rešenje sistema (2). Kako je $\varphi_h(x) = \Phi(x)c$ opšte rešenje homogenog sistema DJ (1), sledi da je $\varphi(x) = \varphi_h(x) + \varphi_p(x)$ opšte rešenje nehomogenog sistema DJ (2). □

Za početni uslov $y(x_0) = y_0$ iz (5) se određuje Košijevo rešenje nalaženjem konstantnog vektora c_0. Kako je $y_0 = \varphi(x_0) = \Phi(x_0)c_0$, to je $c_0 = \Phi^{-1}(x_0)\,y_0$.

Košijevo rešenje nehomogenog linearnog sistema DJ (2:

$$Y_K(x) = \Phi(x)\left(\Phi^{-1}(x_0)\,y_0 + \int_{x_0}^{x} \Phi^{-1}(s)\,g(s)\,ds\right), \quad x, x_0 \in (a,b).$$

Metod Košijeve funkcije

Matrična funkcija

$$K(x,s) = \Phi(x)\,\Phi^{-1}(s), \quad (x,s) \in (a,b)\times(a,b) = \Omega$$

naziva se *Košijeva funkcija homogenog sistema DJ* (1), odnosno matrice $A(x)$. Košijeva funkcija je $K \in C^{(1)}(\Omega)$.

Opšte rešenje nehomogenog linearnog sistema DJ (2:

$$Y(x) = \Phi(x)c + \int_{x_0}^{x} K(x,s)\,g(s)\,ds\,, \quad x, x_0 \in (a,b).$$

Teorema 2 *Za Košijevu funkciju homogenog sistema DJ (1) važi:*

1. $K(x,x)=\mathbb{I}$
2. $K(x,s)K(s,u)=K(x,u)$
3. $K(x,s)^{-1}=K(s,x)$
4. $\dfrac{\partial}{\partial s}K(x,s)=-K(x,s)A(s)$
5. $\dfrac{\partial}{\partial x}K(x,s)=A(x)K(x,s)$
6. $K(x,x_0)$ *je jedinstveno rešenje KP matrične DJ*

$$Z'=A(x)Z,\qquad Z(x_0)=\mathbb{I}.$$

DOKAZ:

$$(2):\quad K(x,s)K(s,u)=\Phi(x)\,\Phi^{-1}(s)\Phi(s)\,\Phi^{-1}(u)=\Phi(x)\,\Phi^{-1}(u)=K(x,u)$$

$$(3):\ \ (K(x,s))^{-1}=(\Phi(x)\,\Phi^{-1}(s))^{-1}=(\Phi^{-1}(s))^{-1}\Phi^{-1}(x)=\Phi(s)\Phi^{-1}(x)=K(s,x)$$

$$\begin{aligned}(4):\quad \frac{\partial}{\partial s}K(x,s) &= \frac{\partial}{\partial s}\Big(\Phi(x)\,\Phi^{-1}(s)\Big)=\Phi(x)\big(\Phi^{-1}(s)\big)'\\ &= -\Phi(x)\Phi^{-1}(s)\Phi'(s)\Phi^{-1}(s)=-K(x,s)A(s)\Phi(s)\Phi^{-1}(s)\\ &= -K(x,s)A(s)\end{aligned}$$

$$\begin{aligned}(5):\quad \frac{\partial}{\partial x}K(x,s) &= \frac{\partial}{\partial x}\Big(\Phi(x)\,\Phi^{-1}(s)\Big)=\Phi'(x)\Phi^{-1}(s)\\ &= A(x)\Phi(x)\Phi^{-1}(s)=A(x)K(x,s)\end{aligned}$$

Tvrđenje (6) sledi iz (5) i (1). □

Košijev problem homogenog ili nehomogenog linearnog sistema DJ je rešen određivanjem Košijeve funkcije.

KOŠIJEVO REŠENJE HOMOGENOG LINEARNOG SISTEMA DJ (1) koje zadovoljava početni uslov $\varphi(x_0)=y_0$ je:

$$\varphi(x)=K(x,x_0)\,y_0,\quad x\in(a,b).$$

KOŠIJEVO REŠENJE NEHOMOGENOG LINEARNOG SISTEMA DJ (2) koje zadovoljava početni uslov $\varphi(x_0)=y_0$ je:

$$\varphi(x)=K(x,x_0)\,y_0+\int_{x_0}^{x}K(x,s)\,g(s)\,ds,\quad x\in(a,b).$$