Visualization of neighbourhoods in some FK spaces

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Abstract. In this extended abstract, we present the graphical representations of some neighbourhoods in certain FK spaces that have recently been studied. These visualizations strongly support the understanding of the topological and geometric structures of the spaces. We emphasize that the graphics in this paper were created by our own software package and its extensions [1–4].

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INTRODUCTION

We recall some well–known facts and definitions. The set ω of all complex sequences $x=(x_k)_{k=1}^{\infty}$ is a Fréchet space, that is, a complete linear metric space with the metric d_ω and the algebraic operations of addition and multiplication by scalars defined by

$$d_\omega(x,y)=\sum_{k=1}^{\infty}\frac{1}{2^k}\cdot\frac{|x_k-y_k|}{1+|x_k-y_k|} \text{ for all } x,y\in\omega, \tag{1}$$

and $x+y=(x_k+y_k)_{k=1}^{\infty}$ and $\lambda x=(\lambda x_k)_{k=1}^{\infty}$ for all $x,y\in\omega$ and all $\lambda\in\mathbb{C}$; also, convergence in (ω,d_ω) and coordinatewise convergence are equivalent, and d_ω is the weakest metric for which this is true ([5, Example 9.3.7]).

An FK space (X,d) is a complete subspace of ω with its metric d stronger than d_ω on X, that is, in which the coordinates $P_n : X\to\mathbb{C}$ are continuous where $P_n(x)=x_n$ $(x=(x_k)_{k=1}^{\infty}\in X)$; a BK space is an FK space with its metric given by a norm.

If X is a normed space then, as usual, X^* denotes the set of all continuous linear functionals on X. If $X\subset\omega$, then the β–dual of X is the set

$$X^\beta=\left\{\sum_{k=1}^{\infty}a_kx_k : \text{converges for all } x\in X\right\}.$$

Example 1 *(a) Obviously (ω,d_ω) is an FK space and the sets ℓ_∞, c and c_0 of all bounded, convergent and null sequences are BK spaces with their natural norms defined by $\|x\|_\infty=\sup_k|x_k|$ in each case* ([5, p. 55]).
(b) Let $p=(p_k)\in\ell_\infty$ be a positive sequence, $H=\|p\|_\infty$ and $M=\max\{1,H\}$. Then, $(\ell(p),d_{(p)})$ is an FK space, where

$$\ell(p)=\left\{\sum_{k=1}^{\infty}|x_k|^{p_k}<\infty\right\} \text{ and } d_{(p)}(x,y)=\left(\sum_{k=1}^{\infty}|x_k-y_k|^{p_k}\right)^{1/M}$$

for all $x,y\in\ell(p)$; if $p_k=p\geq 1$ for all k, then $\ell(p)$ reduces to the familiar BK space ℓ_p with its natural norm defined by $\|x\|_p=(\sum_{k=1}^{\infty}|x_k|^p)^{1/p}$ ([6, Example 3.3 (b), (c)]).

SPACES

Here, we consider the spaces $w_0^p(\Lambda)$ and $w_\infty^p(\Lambda)$ that were defined in [7], their topological properties, and the first and second dual spaces.

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Throughout, let $1 \leq p < \infty$ and q be the conjugate number of p, that is, $q = \infty$ for $p = 1$ and $q = p/(p-1)$ for $1 < p < \infty$. A non–decreasing sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ is exponentially bounded ([7, Lemma 1]) if and only if

$$\left\{ \begin{array}{c} \text{There are reals } s \leq t \text{ such that for some subsequence } (\lambda_{n(\nu)})_{\nu=0}^{\infty} \\ 0 < s \leq \frac{\lambda_{n(\nu)}}{\lambda_{n(\nu+1)}} < 1 \text{ for } \nu = 0, 1, \ldots. \end{array} \right. \tag{2}$$

Also let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be an exponentially bounded sequence, and $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ an associated subsequence with $\lambda_{n(0)} = \lambda_0$. We write $K^{<\nu>}$ $(\nu = 0, 1, \ldots)$ for the set of all integers k with $n(\nu) \leq k \leq n(\nu+1) - 1$, and consider the sets

$$\begin{aligned} w_0^p(\Lambda) &= \left\{ \lim_{\nu \to \infty} \left(\frac{1}{\lambda_{n(\nu+1)}} \sum_{k \in K^{<\nu>}} |x_k|^p \right) = 0 \right\}, \\ w_\infty^p(\Lambda) &= \left\{ \sup_{\nu} \left(\frac{1}{\lambda_{n(\nu+1)}} \sum_{k \in K^{<\nu>}} |x_k|^p \right) < \infty \right\}. \end{aligned}$$

The first result concerns some topological properties of our spaces.

Theorem 2 ([7, Theorem 1 (a), (b)]) *The sets $w_0^p(\Lambda)$ and $w_\infty^p(\Lambda)$ are BK spaces with respect to the natural norms defined by*

$$\|x\|_\Lambda = \sup_{\nu} \left(\frac{1}{\lambda_{n(\nu+1)}} \sum_{k \in K^{<\nu>}} |x_k|^p \right)^{1/p}.$$

DUALS

Now, we give the β–duals of $w_0^p(\Lambda)$ and $w_\infty^p(\Lambda)$.

We introduce a few notations. Let a be a sequence and X be a normed sequence space. Then, we write $\|a\|_X^* = \sup_{\|x\|=1} |\sum_{k=1}^{\infty} a_k x_k|$ provided the expression on the right exists and is finite, which is the case whenever X is a *BK* space and $a \in X^\beta$ ([5, Theorem 7.2.9]). If Λ is an exponentially bounded sequence with an associated subsequence, then we write $\max_\nu$ and $\sum_\nu$ for the maximum and sum taken over all $k \in K^{<\nu>}$. We denote by $x^{<\nu>} = \sum_\nu x_k e^{(k)}$ $(\nu \in \mathbb{N}_0)$ the ν–block of the sequence x. Finally, we write $\sigma(|x|^p) = (\sigma_\nu(|x|^p))_{\nu=0}^{\infty}$ and $\tau(|x|^p) = (\tau_\nu(|x|^p))_{\nu=0}^{\infty}$ for the sequences with

$$\sigma_\nu(|x|^p) = \left(\frac{1}{\lambda_{n(\nu+1)}} \right)^{1/p} \left\| x^{<\nu>} \right\|_p \text{ and } \tau_\nu(|x|^p) = \lambda_{n(\nu+1)}^{1/p} \left\| x^{<\nu>} \right\|_q$$

for $\nu = 0, 1, \ldots$.

The duals $(w_0^p(\Lambda))^\beta$, $(w_\infty^p(\Lambda))^\beta$ and $(w_0^p(\Lambda))^*$ are given the next theorem.

Theorem 3 ([6, Theorems 5.5, 5.8; Remark 5.6]) *We write*

$$\mathscr{M}_p(\Lambda) = \left\{ \|a\|_{\mathscr{M}_p(\Lambda)} < \infty \right\} \text{ where } \|a\|_{\mathscr{M}_p(\Lambda)} = \|\tau(|a|^p)\|_1.$$

(a) Then, we have $\|\cdot\|_{\mathscr{M}_p(\Lambda)} = \|\cdot\|^*_{w_\infty^p(\Lambda)} = \|\cdot\|^*_{w_0^p(\Lambda)}$ on $\mathscr{M}_p(\Lambda)$; $w_0^p(\Lambda)^*$ of $w_0^p(\Lambda)$ is norm isomorphic to $\mathscr{M}_p(\Lambda)$ with the norm $\|\cdot\|_{\mathscr{M}_p(\Lambda)}$; $(w_\infty^p(\Lambda))^*$ is not given by a sequence space.
(b) We have $(w_\infty^p(\Lambda))^{\beta\beta} = (w_0^p(\Lambda))^{\beta\beta} = w_\infty^p(\Lambda)$, $\|\cdot\|^*_{\mathscr{M}_p(\Lambda)} = \|\cdot\|$ on $(\mathscr{M}_p(\Lambda))^\beta$, and $(\mathscr{M}_p(\Lambda))^*$ is norm isomorphic to $w_\infty^p(\Lambda)$.

VISUALIZATION

Finally, we visualize some neighbourhoods of 0 in $\ell(p)$ and $w_\infty^p(\Lambda)$ and $\mathscr{M}_p(\Lambda)$. This is done in a natural way as follows.

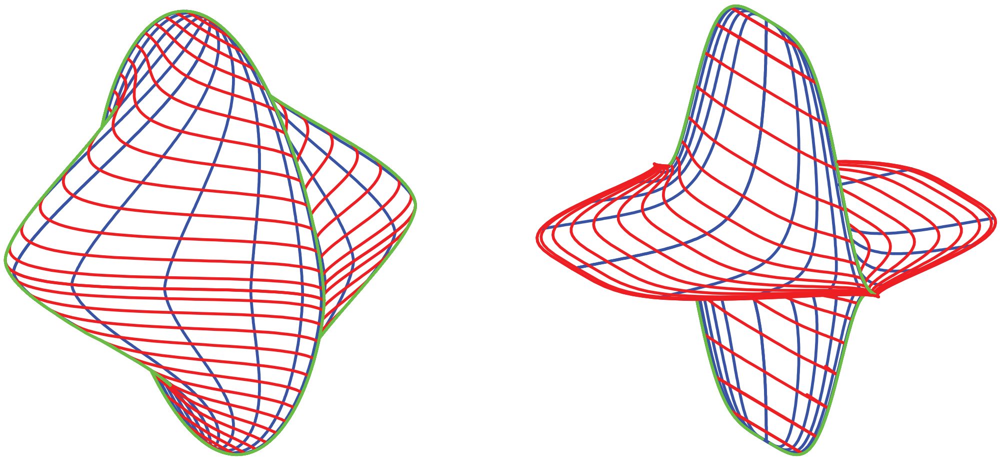

FIGURE 1. $\partial S^3_{d_{(p)}}(0,r)$ for: Left $p = (1/2, 2, 3/2)$; Right $p = (1/2, 4, 1/4)$

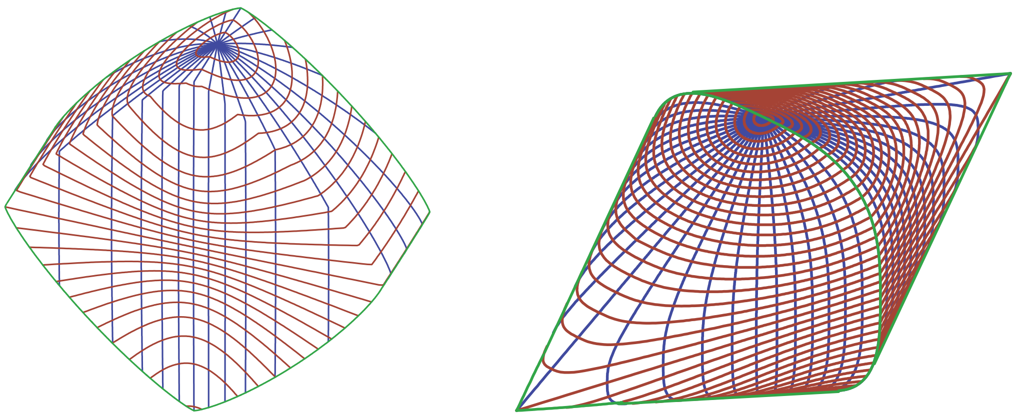

FIGURE 2. Left: neighbourhood in the $w^p_\infty(\Lambda)$–norm for p=1.2, $\lambda_1 = 1$, $\lambda_2 = 6$, $\lambda_3 = 9$; Right: neighbourhood in the dual norm of the $w^p_\infty(\Lambda)$–norm for p=5, $\lambda_1 = 1$, $\lambda_2 = 6$, $\lambda_3 = 9$

Let $P_{n(k)} : \omega \to \mathbb{C}$ for $k = 1,2,3$ be coordinates. We represent open balls $B_r(0) = \{x \in X : d(x,0) < r\}$ in a metric sequence space (X,d) by spheres

$$S^3_d(0,r) = \left\{ (x_{n(1)}, x_{n(2)}, x_{n(3)}) \in \mathbb{C}^3 : d\left(\sum_{k=1}^{3} P_{n(k)}(x) e^{(n(k))}, 0 \right) = r \right\}.$$

More results on neighbourhoods in FK spaces and their graphical representations can be found in [8].

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